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The structure of non-equilibrium shocks in relaxing media

R.W. Lardner* and G.R. Nicklason**

Abstract

The structure of the shock layers that form in high amplitude acoustic waves in relaxing media is analyzed for a general relaxation modulus. The inner and outer expansions are matched to second order and a closed form expression is obtained for Lighthill's shock displacement due to diffusion. The asymptotic behavior of the shock solution at the edges of the shock layer is also analyzed and conditions obtained for the existence of a fully relaxed shock.

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1. Introduction

We shall be concerned with the propagation of large amplitude acoustic waves in media such as relaxing gases, bubbly liquids and viscoelastic or elasto-plastic solids that display stress relaxation. In such media the familiar thermo-viscous dissipation models that lead to Burgers' equation for the wave amplitude [1-3] are often inadequate to explain the observed shock structures. Modified models that are essentially equivalent to an exponential relaxation modulus, sometimes in combination with viscous dissipation, have been proposed and investigated by a number of authors over the years [4-7], the most complete investigation being that of Crighton and Scott [7]. While a few results have been obtained for other relaxation moduli [6,8,9], progress in this direction has been limited by the inability to obtain analytic solutions in the shock layer for anything more general than exponential relaxation. We have found, however, that a closed form expression can be derived for Lighthill's displacement of the shock due to diffusion, and an estimate can be made of the shock width, for an arbitrary relaxation function, even though the shock solution itself can only be obtained numerically. Furthermore the asymptotic behavior of the shock solution at the outer edges of the shock layer can be analyzed and necessary and sufficient conditions obtained for the existence of a fully dispersed shock solution.

In Section 2 the basic equations for plane motions of a relaxing medium are written down and the generalized Burgers' equation is obtained. In Section 3 the inner and outer perturbation solutions are investigated under the assumptions that the stress relaxation function $G(s)$ satisfies the conditions that $\int_0^\infty s^n\, G(s)\, ds$ exists at least for $n = 0$ and 1, and that the relaxation time δ is short compared to the timescale for change in the prescribed boundary values. The outer solution is obtained explicitly to order δ, while certain functional equations are obtained for the first two terms in the inner expansion.

In spite of the fact that the inner solution cannot be obtained explicitly, it is possible to match the inner and outer solutions (Section 4) and hence to derive an expression for Lighthill's displacement. An alternative derivation of this expression using a conserved integral is given in Section 5.

In Sections 6 and 7 the behavior of the inner solution at the edges of the shock layer is investigated. Ahead of the shock (Section 6) the inner solution develops exponentially for all relaxation functions. In the tail of the shock however (Section 7) the inner solution tends exponentially to its outer value only when the relaxation function itself decays exponentially. For a relaxation function that decays algebraically, the tail of the shock layer decays more slowly than exponentially, the decay being directly related to the relaxation function. This result has also been found by Sugimoto and Kakutani [9] for a particular relaxation function (which does not belong to the class investigated here.)

Also in Section 6 some general necessary and sufficient conditions for the existence of a fully dispersed shock solution are obtained.

Finally, in Section 8 we have included a discussion of the width of the shock layer, giving an explicit estimate for the general relaxation function.

2. The physical model

We consider a continuous medium which in its reference state occupies the half-space $x > 0$ and which undergoes plane motions in the x-direction caused by some prescribed motion of the face $x = 0$. We denote by $u(x,t)$ the displacement at time t of the particle whose reference position is x. Then $v = u_t$ is the particle velocity and $e = u_x$ can be used as a linearized measure of the longitudinal strain.

The Lagrangian equation of motion is

$$\rho_0 u_{tt} = \sigma_x \tag{1}$$

where ρ_0 is the density in the reference state, assumed to be constant, and σ is the longitudinal stress. We assume that this is given by a stress-strain relation of the form

$$\sigma(x,t) = E\{e(x,t) + M[e(x,t)]^2 - \int_0^\infty G(s)[e(x,t-s) - e(x,t)]ds\} . \tag{2}$$

The first two terms on the right represent the linear and quadratic elastic contributions to the stress while the integral term accounts for stress relaxation. E is the linear modulus of elasticity, M is the ratio of the quadratic modulus to the linear and $EG(s)$ is the viscoelastic relaxation modulus. In the case of a gas undergoing isentropic deformation according to the pressure-density relation $p\rho^{-\gamma} = \text{const.}$, the first two of these are given by $E = \gamma p_0$, $M = -\frac{1}{2}(\gamma+1)$ where p_0 is the pressure in the reference state and γ the ratio of specific heats.

The function $G(s)$ can be related to stress relaxation under the discontinuous strain history

$$e(x,t) = \begin{cases} 0 & (t < 0) \\ e_0 & (t > 0) . \end{cases}$$

The linear part of the stress in (2) then becomes

$$\sigma(x,t) = Ee_0[1 + \int_t^\infty G(s)ds]$$

so that the effective elastic modulus decays from an instantaneous value of $E[1 + \int_0^\infty G(s)ds]$ to the asymptotic value E. Some important special cases of $G(s)$ will be mentioned at the end of this section.

We consider motions in which the strain remains sufficiently small so that the nonlinear term in (2) is small compared to the linear elastic term. We also suppose that the relaxation term is of no larger order of magnitude than the nonlinear term (otherwise nonlinear effects would be small). In order to keep track of the various orders of magnitude we introduce re-scaled variables by writing

$$e = \varepsilon e^*, \; v = \varepsilon v^*, \; u = \varepsilon u^*, \; \sigma = \varepsilon\sigma^*, \; G(s) = \varepsilon G^*(s)$$

where $\varepsilon << 1$ and the starred variables are of order 1. Then substituting (2) into (1) and taking units of time such that $E/\rho_0 = 1$, we obtain the equation

$$u^*_{tt} - u^*_{xx} = \varepsilon\{2Mu^*_x u^*_{xx} - \int_0^\infty G^*(s)[u^*_{xx}(x,t-s) - u^*_{xx}(x,t)]ds\} . \quad (3)$$

From now on we shall drop the $*$'s, noting that the relations $e = u_x$, $v = u_t$ continue to hold for the re-scaled variables.

We seek the solution of equation (3) in the region $x > 0$ that satisfies the boundary condition $u(0,t) = F(t)$, given. This solution can be found in the form

$$u(x,t) = U(\theta,\xi) + O(\varepsilon) \qquad \theta = t - x, \quad \xi = \varepsilon x, \tag{4}$$

which represents a wave traveling with unit velocity in the positive x-direction with amplitude modulation over distances of order ε^{-1}. Substituting (4) into (3) and satisfying this latter equation to order ε, we obtain

$$h_\xi = Mhh_\theta - \frac{1}{2}\frac{\partial}{\partial\theta}\int_0^\infty G(s)[h(\theta-s,\xi) - h(\theta,\xi)]ds \tag{5}$$

where $h(\theta,\xi) = \partial U(\theta,\xi)/\partial\theta$. The boundary condition takes the form

$$h(\theta,0) = H(\theta) \tag{6}$$

where $H(t) = -F'(t)$ is given. An alternative form of equation (5) can be obtained by integrating the last term by parts:

$$h_\xi = Mhh_\theta + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\int_0^\infty G_1(s)h(\theta-s,\xi)ds \tag{7}$$

where

$$G_1(s) = \int_s^\infty G(s)ds.$$

An important special case is $G_1(s) = k\delta(s)$ where $\delta(s)$ is the Dirac delta. This corresponds physically to a relaxation time which is much shorter than the time scale for changes in the physical variables. Equation (2), in terms of the re-scaled variables, then has the form of a Kelvin-Voigt model with purely viscous dissipation:

$$\sigma(x,t) = E\{e(x,t) + M[e(x,t)]^2 + ke_t(x,t)\};$$

while equation (7) reduces to Burgers' equation:

$$h_\xi = Mhh_\theta + \frac{1}{2} kh_{\theta\theta}.$$

The simplest model allowing non-trivial relaxation is the exponential model

$$G(s) = de^{-s/\delta} \tag{8}$$

where δ is the relaxation time. In this case the stress-strain relation can be written in the equivalent differential form

$$\left(\frac{\partial}{\partial t} + \frac{1}{\delta}\right)\left[\frac{\sigma}{E} - (e + Me^2)\right] = d\delta e_t .$$

This model is often used for relaxing gases. The Kelvin-Voigt model may be obtained from it by taking the limit $\delta \to 0$, $d\delta^2 = k$, finite. The generalized Burgers' equation (5) or (7) has in this exponential case previously been investigated by Ockendon and Spence [5] and Crighton and Scott [7].

In the case of viscoelastic solids it is generally more appropriate to include a number of relaxation times by taking the model

$$G(s) = \sum_i d_i\, e^{-s/\delta_i}. \tag{9}$$

The equivalent differential form for the stress-strain relation is in the case of two relaxation times

$$\left(\frac{\partial}{\partial t} + \frac{1}{\delta_1}\right)\left(\frac{\partial}{\partial t} + \frac{1}{\delta_2}\right)\left[\frac{\sigma}{E} - (e + Me^2)\right] = (d_1\delta_1 + d_2\delta_2)\, e_{tt} + \left(\frac{d_1\delta_1}{\delta_2} + \frac{d_2\delta_2}{\delta_1}\right) e_t$$

A model involving relaxation and internal resonance which has been used to model the Bordoni internal friction peaks of micro-plastic relaxation

[8] is obtained by taking

$$G(s) = de^{-s/\delta}\sin \omega s. \tag{10}$$

In the case of viscoelastic polymers relaxation functions such as (8) or (9) decay too fast to be applicable. Suzimoto and Kakutani [9] have considered $G(s) = ds^{-\nu}$ $(0 < \nu < 1)$ in such cases. This relaxation function is too slowly decaying for most of the results in this paper to be valid. However an example with algebraic decay for which the results do apply is $G(s) = d(1+s/\delta)^{-K}$ $(K > 2)$.

We shall be concerned with the case in which the relaxation effects are small except within the shock layers that develop in the solution. We shall use δ in the general case to represent a typical decay time for $G(s)$ and shall make the basic assumption that δ is small compared to the time scale for variation of $H(t)$. The approach will be to seek solutions of (7) essentially as perturbation expansions in δ.

3. Perturbation solutions

A. Outer solution

Integrating the last term in eqn (7) by parts repeatedly, we can write this equation in the form

$$h_\xi - Mhh_\theta = \frac{1}{2}\{ k_1 \frac{\partial^2 h}{\partial\theta^2} - k_2 \frac{\partial^3 h}{\partial\theta^3} + \ldots + (-1)^n k_{n-1} \frac{\partial^n h}{\partial\theta^n} + (-1)^{n+1} \frac{\partial^{n+1}}{\partial\theta^{n+1}} \int_0^\infty G_n(s) h(\theta-s,\xi) ds \} \qquad (11)$$

where

$$G_n(s) = \int_s^\infty G_{n-1}(s')ds' = \frac{1}{(n-1)!} \int_s^\infty (s'-s)^{n-1} G(s')\, ds'$$
$$k_n = \int_0^\infty G_n(s')ds' = G_{n+1}(0). \qquad (12)$$

Let δ denote the time-scale for variations in $G(s)$, i.e., the decay-time, and write $s = \delta\beta$ in these expressions

$$G_n(\delta\beta) = \frac{\delta^n}{(n-1)!} \int_\beta^\infty (\beta'-\beta)^{n-1} G(\delta\beta')d\beta', \quad k_n = \frac{\delta^{n+1}}{n!} \int_0^\infty \beta^n G(\delta\beta)d\beta.$$

The integrands here are of order 1, showing that successive $G_n(s)$ and k_n's are of decreasing orders of magnitude in terms of δ.

For example, the exponential relaxation function $G(s) = de^{-s/\delta}$ leads to $G_n(s) = \delta^n G(s)$ and $k_n = d\delta^{n+1}$. The algebraic function $G(s) = d(1+s/\delta)^{-K}$ leads to $G_n(s) = [d\delta^n/(K-1)(K-2) \ldots (K-n)](1+s/\delta)^{-(K-n)}$ and $k_{n-1} = d\delta^n/(K-1)(K-2) \ldots (K-n)$. (In this latter case we must restrict $n < K$ which means that the following results are valid for $K > 2$ only.)

In the outer region (i.e. away from the shock), h and its derivatives are of order 1, so to first order in δ we need retain only the first term on the right in (11). Then we can take over directly the result for the outer solution of Burgers' equation to order k_1:

$$h(\theta,\xi) = H(\theta_1) + \frac{1}{2} k_1 \xi H''(\theta_1) [1 - M\xi H'(\theta_1)]^{-2} \qquad (13)$$

where $H(\theta_1)$ is the initial value function (6) and θ_1 is defined implicitly in terms of (θ,ξ) by

$$\theta_1 = \theta + M\xi H(\theta_1). \qquad (14)$$

Eqn. (14) defines θ_1 uniquely only for $\xi < \xi_f$ where

$$\xi_f = (MH'_m)^{-1}$$

and H'_m is the maximum value of $H'(\theta_1)$ when $M > 0$ and the minimum value when $M < 0$. For $\xi > \xi_f$ the solution must include a shock and in the vicinity of the shock the outer expansion (13) ceases to be valid.

B. Inner solution

Let the shock position be $\theta = \theta_s(\xi)$ and introduce the inner variable

$$\alpha = \frac{\theta - \theta_s(\xi)}{\delta} \qquad (15)$$

where δ is the decay time of the relaxation function $G(s)$. Using (α,ξ) as the independent variables and writing

$$h(\theta,\xi) = h(\theta_s + \delta\alpha, \xi) \equiv h^*(\alpha,\xi),$$

we can re-write eqn. (7) in the form

$$\delta \frac{\partial h^*}{\partial \xi} = (Mh^* + \theta_s') \frac{\partial h^*}{\partial \alpha} + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \int_0^\infty G_1(\delta\beta)\ h^*(\alpha-\beta,\xi)d\beta. \tag{16}$$

We now seek the solution as an expansion of the form

$$h^*(\alpha,\xi) = h_0^*(\alpha,\xi) + \delta h^*(\alpha,\xi) + \dots .$$

Substituting into (16) and comparing powers of δ we obtain for the first two terms the equations

$$(Mh_0^* + \theta_s') \frac{\partial h_0^*}{\partial \alpha} + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \int_0^\infty G_1(\delta\beta) h_0^*(\alpha-\beta,\xi)d\beta = 0$$

$$(Mh_0^*+\theta_s') \frac{\partial h_1^*}{\partial \alpha} + Mh_1^* \frac{\partial h_0^*}{\partial \alpha} + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \int_0^\infty G_1(\delta\beta) h_1^*(\alpha-\beta,\xi)d\beta = \frac{\partial h_0^*}{\partial \xi} .$$

These may be integrated with respect to α to give respectively

$$Mh_0^* + 2\theta_s' h_0^* + \frac{\partial}{\partial \alpha} \int_0^\infty G_1(\delta\beta) h_0^*(\alpha-\beta,\xi)d\beta = a(\xi) \tag{17}$$

$$(Mh_0^*+\theta_s')h_1^* + \frac{1}{2} \frac{\partial}{\partial \alpha} \int_0^\infty G_1(\delta\beta) h_1^*(\alpha-\beta,\xi)d\beta = \int_0^\alpha \frac{\partial h_0^*}{\partial \xi} d\alpha + c(\xi) \tag{18}$$

where $a(\xi)$ and $c(\xi)$ are the "constants" of integration.

Let us suppose that

$$h_0^*(\alpha,\xi) \to h_\pm(\xi) \quad \text{as} \quad \alpha \to \pm \infty.$$

Then from (17) it follows that $Mh_\pm^2 + 2\theta_s' h_\pm = a$ and therefore

$$\theta_s' = -\frac{1}{2} M(h_+ + h_-), \quad a = -Mh_+h_-. \tag{19}$$

We now normalize h_0^* by writing

$$h_0^* = \frac{1}{2}(h_+ + h_-) + \frac{1}{2}(h_+ - h_-)f(\alpha,\xi) \tag{20}$$

so that $f(\alpha,\xi) \to \pm 1$ as $\alpha \to \pm\infty$. Then using (19) and (20) we can rewrite (17) in the form

$$\Delta[1 - f^2(\alpha)] = \frac{\partial}{\partial\alpha}\int_0^\infty G_1(\delta\beta)\ f(\alpha-\beta)d\beta \tag{21}$$

where the ξ-dependence of f has been suppressed and we define

$$\Delta = \frac{1}{2}M(h_+ - h_-). \tag{22}$$

Similarly in terms of this notation, eqn. (18) for h_1^* may be re-written

$$\Delta f(\alpha)h_1^*(\alpha) + \frac{1}{2}\frac{\partial}{\partial\alpha}\int_0^\infty G_1(\delta\beta)h_1^*(\alpha-\beta)d\beta = \frac{1}{M}\{-\theta_s''\alpha + \frac{\partial}{\partial\xi}(\Delta\int_0^\alpha f d\alpha)\} + c(\xi) \tag{23}$$

For the exponential relaxation function (8), eqn. (21) becomes

$$1 - f^2(\alpha) = \frac{d\delta}{\Delta}\ \frac{\partial}{\partial\alpha}\int_0^\infty e^{-\beta}f(\alpha-\beta)d\beta.$$

This is equivalent to the differential equation

$$(2f + \frac{d\delta}{\Delta})f' = 1 - f^2 \tag{24}$$

whose solution is given by

$$(\frac{d\delta}{2\Delta} - 1)\ln(1+f) - (\frac{d\delta}{2\Delta}+1)\ln(1-f) = \alpha - \alpha_0. \tag{25}$$

Here α_0 is chosen such that $f(\alpha_0) = 0$. For this model it is also possible to obtain an explicit solution of (23) for the second order inner solution h_1^* [7].

For a general relaxation function, eqns. (21) and (23) cannot be solved in closed form. Some properties of the solution of (21) will be derived in later sections of the paper. However, it is interesting to observe that matching of the inner and outer solutions can be performed without exact knowledge of the inner solution.

Equation (21) for $f(\alpha)$ is autonomous. We shall denote by $f_0(\alpha,\Delta)$ the solution that satisfies the condition $f_0(0,\Delta) = 0$, i.e., is centred at $\alpha = 0$. The general solution is then $f(\alpha) = f_0(\alpha-\alpha_0,\Delta)$ where $\alpha_0 = \alpha_0(\xi)$ is arbitrary. The corresponding inner solution h_0^* given by (2) then represents a shock centred at $\theta = \theta_s + \delta\alpha_0$, so that $\delta\alpha_0$ is the quantity that Lighthill terms the "shock displacement due to diffusion". The most significant outcome of the matching will be an expression for this displacement.

4. Matching

At any point $\theta = \theta_s(\xi)$ on the shock, there are two solutions θ_1 of eqn. (14) and we denote them by $\theta_1^{\pm}$.

The situation is illustrated in Figure 1 which shows the two characteristic lines $\theta_1 = \theta_1^{\pm}$ in the $\theta\xi$-plane [1]. We use the notation $H_{\pm}$, $H_{\pm}'$ etc. for $H(\theta_1^{\pm})$, $H'(\theta_1^{\pm})$ etc. Then since the two characteristics meet at $\theta = \theta_s(\xi)$,

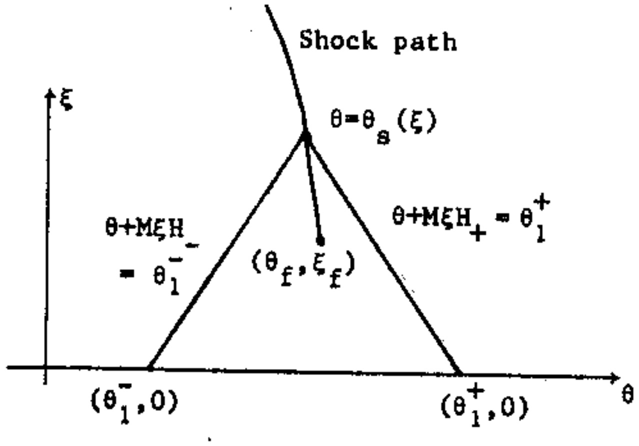

Figure 1

$$\theta_1^{\pm} = \theta_s + M\xi H_{\pm}. \tag{26}$$

We wish to expand the outer solution (13) in the inner variable given by (15). Writing $\theta = \theta_s + \delta\alpha$ in (14) we obtain

$$\theta_1 - \theta_1^{\pm} = \frac{\delta\alpha}{1 - M\xi H_{\pm}'} + O(\delta^2)$$

and, therefore, to order δ the outer solution (13) is given by

$$h(\theta,\xi) \sim H_{\pm} + \frac{\delta\alpha H_{\pm}'}{1-M\xi H_{\pm}'} + \frac{1}{2} k_1\xi \frac{H_{\pm}''}{(1-M\xi H_{\pm}')^2} \tag{27}$$

where the $\pm$ signs apply according as $\theta > \theta_s$ or $\theta < \theta_s$ respectively.

This must be matched with the limit of the inner solution $h_0^* + \delta h_1^*$ as $\alpha \to \pm \infty$. We assume that as $\alpha \to \pm \infty$,

$$f(\alpha) \sim \pm 1 + \text{SST}$$
$$h_1^* \sim Y_\pm \alpha + Z_\pm + \text{SST} \tag{28}$$

where SST means sufficiently small terms not to enter the matching conditions. For this it is sufficient that these terms should be $o(\alpha^{-1})$ in $f(\alpha)$ and $o(1)$ in h_1^*. These remainder terms in $f(\alpha)$ will be investigated in Sections 6 and 7 where it will be shown that they are always exponentially small as $\alpha \to -\infty$, and as $\alpha \to \infty$ they are either exponentially small or of order $G_1(\delta\alpha)$ in the case of a relaxation function that does not decay exponentially. In either case they are $o(\alpha^{-1})$ since we require in (11) that k_1 exists, that is $G_1(s)$ is integrable over $[0,\infty)$.

The matching conditions are then as follows; from (20), (28) and (27):

$$h_\pm = H_\pm \tag{29}$$

$$Y_\pm = \frac{H'_\pm}{1-M\xi H'_\pm}, \quad \delta Z_\pm = \frac{1}{2} k_1 \xi \frac{H''_\pm}{(1-M\xi H'_\pm)^2} . \tag{30}$$

The lowest order matching condition (29) when combined with (19) and (26) gives the usual expression for the shock speed and leads to the familiar equal areas construction [1, 10]. In order to make use of the higher order matching conditions (30) we must obtain expressions for $Y_\pm$ and $Z_\pm$

from the solution h_1^*.

Substituting the asymptotic behaviors (28) into equation (23) we obtain, ignoring transcendentally small terms,

$$\pm\Delta(Y_\pm\alpha + Z_\pm) + \frac{1}{2} Y_\pm \int_0^\infty G_1(\delta\beta)d\beta \sim \frac{1}{M}\{-\theta_s''\alpha \pm \Delta'\alpha \mp \frac{\partial}{\partial\xi}[\Delta\int_0^{\pm\infty}(1 \mp f)d\alpha] + c.$$

Therefore

$$\pm \Delta Y_\pm = \frac{1}{M}(-\theta_s'' \pm \Delta') \tag{31}$$

$$\pm \Delta Z_\pm + \frac{1}{2} Y_\pm \int_0^\infty G_1(\delta\beta)d\beta = \mp \frac{1}{M}\frac{\partial}{\partial\xi}[\Delta\int_0^{\pm\infty}(1 \mp f)d\alpha] + c. \tag{32}$$

Differentiating the equation $-\theta_s' \pm \Delta = MH_\pm$, which follows from the matching conditions (29) together with (19) and (22), with respect to ξ and using (31) we readily obtain the first of the matching conditions (30) which therefore contains no new information. The second of these conditions does however provide two new equations which are sufficient to determine $c(\xi)$ and the displacement $\alpha_0(\xi)$ which is contained in f.

To determine α_0 we form the difference between the two equations (32):

$$\Delta(Z_+ + Z_-) + \frac{1}{2}(Y_+ - Y_-)\int_0^\infty G_1(\delta\beta)d\beta = -\frac{1}{M}\frac{\partial}{\partial\xi}\{\Delta[\int_0^\infty(1-f)d\alpha - \int_{-\infty}^0(1+f)d\alpha]$$

Substituting for $Y_\pm$ and $Z_\pm$ from (30) and noting the definition (12) of k_1 we can write this equation in the form

$$\frac{\partial}{\partial\xi}\{\frac{k_1}{2\delta}\ln(\frac{1-M\xi H_+'}{1-M\xi H_-'})\} = \frac{\partial}{\partial\xi}\{\Delta[\int_0^\infty(1-f)d\alpha - \int_{-\infty}^0(1+f)d\alpha]\}.$$

This equation may be integrated immediately. Since at the shock formation point $(\xi = \xi_f)$, $\theta_1^+ = \theta_1^-$ and $\Delta = 0$, the constant of integration is zero, and the result can be written

$$\int_{-\infty}^{\infty} [\sigma(\alpha) - f(\alpha,\xi)]d\alpha = \frac{k_1}{2\delta\Delta} \ln \left(\frac{1 - M\xi H_+'}{1 - M\xi H_-'}\right) \qquad (33)$$

where $\sigma(\alpha) = \mathrm{sgn}(\alpha) \equiv \pm 1$ according as $\alpha \gtrless 0$.

Now, replacing f by $f_0(\alpha-\alpha_0,\Delta)$ and setting $\beta = \alpha-\alpha_0$, we obtain the final result

$$\alpha_0 = \frac{k_1}{4\delta\Delta} \ln \left(\frac{1-M\xi H_+'}{1-M\xi H_-'}\right) + \frac{1}{2}\int_{-\infty}^{\infty}[f_0(\beta,\Delta) - \sigma(\beta)]d\beta. \qquad (34)$$

This first term in this expression for α_0 corresponds to Lighthill's formula for the shock displacement. The second term is zero whenever the shock is antisymmetrical about its centre, thus can be regarded as a shock displacement due to asymmetry of the shock structure.

Let us calculate this displacement for the exponential relaxation function (8), in which case the solution f is given by (25). The last term in (34) can be transformed to

$$\frac{1}{2}\left\{ \int_0^1 \frac{f-1}{f'} df + \int_{-1}^0 \frac{f+1}{f'} df \right\}$$

which can readily be evaluated using (24). The result is

$$\alpha_0 = \frac{d\delta}{4\Delta} \ln \left(\frac{1-M\xi H_+'}{1-M\xi H_-'}\right) + 2\ln 2 - 2.$$

Lighthill's result for Burgers' equation is obtained by taking the limit $\delta \to 0$, $d\delta^2 = k_1$, finite, $\delta\alpha_0 \to \theta_L(\xi)$, finite.

5. Integral conservation property

An alternative approach to the determination of α_0 which avoids the use of h_1^* entirely is via the integral conservation property intro-uced by Murray [11] and exploited extensively by Crighton and Scott [7] (see also [12]). Integrating eqn. (7) from $\theta = -\infty$ to $\theta = +\infty$, assuming that h and its derivatives tend to zero as $\theta \to \pm\infty$, we obtain the conservation property

$$\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty} h(\theta,\xi)d\theta = 0$$

or, alternatively,

$$\int_{-\infty}^{\infty} h(\theta,\xi)d\theta = \int_{-\infty}^{\infty} H(\theta)d\theta. \tag{35}$$

Now let $h^{(0)}(\theta,\xi)$ denote the outer expansion given by (13) and let $h^{(0)*}(\alpha,\xi)$ denote the inner expansion of the outer solution given by (27). Define $g(\alpha,\xi) = h^*(\alpha,\xi) - h^{(0)*}(\alpha,\xi)$, then the expression

$$h^{(0)}(\theta,\xi) + g(\alpha,\xi)$$

gives a uniform approximation to h, valid in both inner and outer regions. Using this approximation in (35), we obtain

$$\int_{-\infty}^{\infty} h^{(0)}(\theta,\xi)d\theta + \delta\int_{-\infty}^{\infty} g(\alpha,\xi)d\alpha = \int_{-\infty}^{\infty} H(\theta)d\theta. \tag{36}$$

It is clear that in order to calculate the left side of this equation to order δ, it is sufficient to use the lowest order approximation to $g(\alpha,\xi)$, and it is unnecessary to include the term h_1^* in the inner solution. From (20) and (27) this lowest approximation is

$$g(\alpha,\xi) \sim \frac{1}{2}(H_+ - H_-)\{f(\alpha,\xi) - \sigma(\alpha)\}$$

where $\sigma(\alpha) = \text{sgn}(\alpha) = \text{sgn}(\theta - \theta_s)$.

Substituting $h^{(0)}$ from (13) and changing to θ_1 as integration variable in the first term, we can write the terms of order 1 in (36) in the form

$$\int_{\theta_1^-}^{\theta_1^+} H(\theta_1)d\theta_1 = \frac{1}{2} M\xi[H_+^2 - H_-^2].$$

This is the usual equal areas rule, which is equivalent to the lowest order matching conditions. The terms of order δ in (36) are then as follows:

$$\int_{-\infty}^{\theta_1^-} \int_{\theta_1^+}^{\infty} \frac{1}{2} k_1\xi \frac{H''(\theta_1)}{1-M\xi H'(\theta_1)} d\theta_1 + \frac{\delta\Delta}{M}\int_{-\infty}^{\infty}\{f(\alpha,\xi) - \sigma(\alpha)\}d\alpha = 0.$$

This equation is identical with (33). Thus the shock displacement can be found without use of second order matching.

6. Behaviour of the inner solution as $\alpha \to -\infty$

We have $f(\alpha) \to -1$ as $\alpha \to -\infty$. Writing $f(\alpha) = -1 + \eta(\alpha)$ in (21) and linearizing in η we obtain the approximate equation

$$2\Delta\eta(\alpha) \sim \int_0^\infty G_1(\delta\beta)\eta'(\alpha-\beta)d\beta.$$

Integrating by parts, we can re-write this as

$$(k_0-2\Delta)\eta(\alpha) \sim \int_0^\infty K(\beta)\eta(\alpha-\beta)d\beta \qquad (\alpha \to -\infty) \tag{37}$$

where $K(\beta) = \delta G(\delta\beta)$ and k_0 is defined in (12). Note that the integral in (37) involves values of $\alpha-\beta$ for which $\eta(\alpha-\beta)$ is small. For any relaxation function, the solution $\eta(\alpha)$, if it exists, is asymptotically exponential, $\eta(\alpha) \sim ce^{p\alpha}$, where p is the smallest positive root of the equation

$$\int_0^\infty K(\beta)e^{-p\beta}\, d\beta = k_0 - 2\Delta. \tag{38}$$

For complex roots we take the roots with smallest real part.

For the exponential relaxation function (8), eqn. (38) has one solution, $p = (\frac{d\delta}{2\Delta} - 1)^{-1}$. Of course, this result is also obvious from the explicit solution (25) in this case. For $\Delta < \frac{1}{2}\, d\delta$, this value of p is real and positive while for $\Delta > \frac{1}{2}\, d\delta$, eqn. (38) has no positive solution. In this case there is no solution of (21) that approaches -1 as $\alpha \to -\infty$. The shock layer must still include a discontinuity in h, and it is said to be not fully dispersed [5, 7].

For $G(s)$ given by (9), eqn. (38) becomes

$$\sum_i \frac{d_i \delta}{\lambda_i + p} = k_0 - 2\Delta \tag{39}$$

where $\lambda_i = \delta/\delta_i$ and $k_0 = \sum_i d_i \delta_i$. Consider the graph of the left side of eqn. (39) as a function of p (see Figure 2 where, for definiteness we have taken $\lambda_1 < \lambda_2 < \ldots < \lambda_n$). When $k_0 - 2\Delta > 0$, (39) clearly has one positive and $(n-1)$ negative roots. When $k_0 - 2\Delta < 0$, there are n negative roots. Since (39) can be rearranged as a polynomial of degree n, there can be no complex roots. Hence we again have the result that if $\Delta < \frac{1}{2} k_0$ there exists a solution of (21) that approaches -1 as $\alpha \to -\infty$ while if $\Delta > \frac{1}{2} k_0$ a fully dispersed shock solution does not exist.

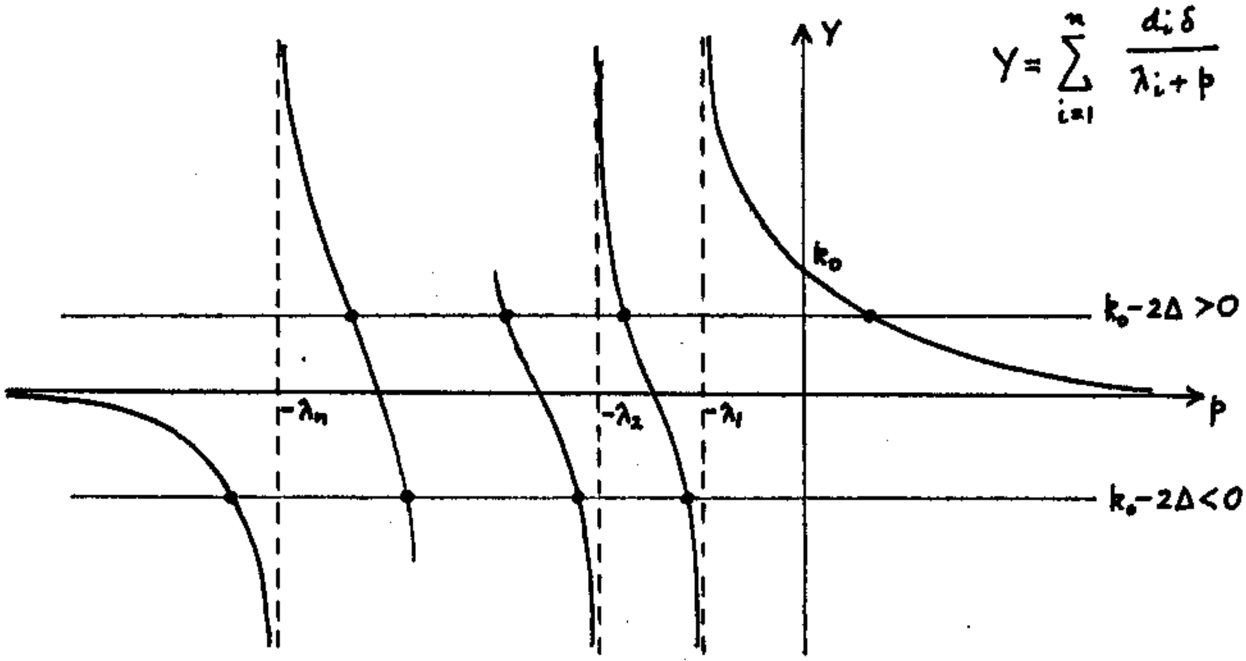

Figure 2

For general relaxation functions we can prove the following results regarding the roots of (38).

A. If $G(s)$ is bounded then (38) has at least one real positive root when $\Delta < \frac{1}{2} k_0$.

B. If $G(s)$ is bounded and positive then (38) has exactly one real positive root when $\Delta < \frac{1}{2} k_0$.

C. If $G(s) > 0$, then (38) has no real positive root when $\Delta > \frac{1}{2} k_0$.

D. If $G(s) > 0$ and $G'(s) \leq 0$ then (38) has no complex root with positive real part when $\Delta > \frac{1}{2} k_0$.

E. If $G(s) > 0$ then (38) has no complex root with positive real part when $\Delta > k_0$.

Thus for example for $G(s) = d(1+s/\delta)^{-K}$ we conclude that a fully dispersed shock exists if $\Delta < \frac{1}{2} k_0 = d\delta/2(K-1)$ and does not exist if $\Delta > \frac{1}{2} k_0$.

In order to prove the above results, define

$$g(p) = \int_0^\infty K(\beta) e^{-p\beta} \, d\beta + 2\Delta - k_0$$

so that (38) becomes $g(p) = 0$.

A. Under the assumed conditions, $g(0) = 2\Delta > 0$ and $g(p) \to 2\Delta - k_0 < 0$ as $p \to \infty$, hence $g(p)$ has at least one positive zero.

B. If $K(\beta) > 0$, $g(p)$ is a monotonically decreasing function of p, hence has exactly one real positive zero.

C. If $K(\beta) > 0$, $g(p)$ is monotonically decreasing from $g(0) = 2\Delta$ to $\lim_{p\to\infty} g(p) = 2\Delta - k_0 > 0$, hence has no positive real zero.

D. To prove D we apply the argument principle to the contour shown in Figure 3. From part C, $g(p)$ is real and positive along the segment OA. Along AB, $g(p) \sim 2\Delta - k_0$ as $R \to \infty$, so $g(p)$ is again real and positive. Along BO, set $p = is$ and consider $\text{Im } g(is)$:

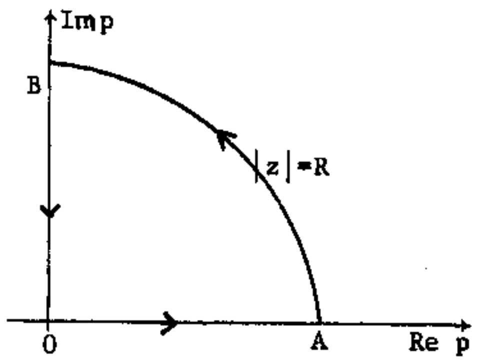

Figure 3

$$- \text{Im } g(is) = \int_0^\infty K(\beta)\sin s\beta \, d\beta$$

$$= \frac{1}{s} [K(0) + \int_0^\infty K'(\beta)\cos s\beta \, d\beta]$$

$$\geq \frac{1}{s} [K(0) + \int_0^\infty K'(\beta)d\beta] \geq 0$$

where we used $K'(\beta) \leq 0$. Since $\text{Im } g(is) \leq 0$, $\arg g(is)$ cannot change by a multiple of 2π along BO. It follows therefore that $g(p)$ has no zeros in the first quadrant, hence also none in the fourth quadrant.

E. To prove E we proceed as in D, but along BO consider $\text{Re } g(is)$:

$$\text{Re } g(is) = \int_0^\infty K(\beta)\cos s\beta \, d\beta + 2\Delta - k_0$$

$$\geq - \int_0^\infty K(\beta)d\beta + 2\Delta - k_0 = 2(\Delta - k_0).$$

Hence if $\Delta > k_0$, $\text{Re } g(is)$ remains positive along BO, so again $\arg g(is)$ does not change, and there are no zeros in the first quadrant.

These theorems say nothing about oscillatory relaxation functions such as (10). In this particular case, eqn. (38) becomes

$$[(1+p)^2 + \omega^2\delta^2]^{-1} = (1+\omega^2\delta^2)^{-1} - (2\Delta/d\delta^2\omega). \tag{40}$$

It is readily seen that if $\Delta < \frac{1}{2} k_0$, there are two real positive solutions for p, consistent with result A. (Here $k_0 = d\delta^2\omega/(1+\omega^2\delta^2)$.) If $\Delta > \frac{1}{2} k_0$, the two solutions are complex with negative real parts. Thus again in this case the condition $\Delta < \frac{1}{2} k_0$ appears as both necessary and sufficient for the existence of a fully dispersed shock, even though $G(s)$ does not satisfy the conditions of results C – E.

7. Behaviour of the inner solution as $\alpha \to \infty$

Since $f(\alpha) \to 1$ as $\alpha \to \infty$, we write $f(\alpha) = 1 - \eta(\alpha)$ in (21) and linearize in η, obtaining

$$2\Delta\eta(\alpha) \sim -\int_0^\infty K_1(\beta)\eta'(\alpha-\beta)d\beta \tag{41}$$

where $K_1(\beta) = G_1(\delta\beta)$. Unlike (37), the integral in (41) involves points where $\eta(\alpha-\beta)$ is not small. Following Sugimoto and Kakutani [9] we partition the integral into a near past where η is small and a distant past where, by virtue of fading memory, K_1 is small:

$$2\Delta\eta(\alpha) \sim -\int_{-\infty}^{M} K_1(\alpha-\beta)\eta'(\beta)d\beta - \int_M^\alpha K_1(\alpha-\beta)\eta'(\beta)d\beta.$$

Here the cut-off point M is held fixed as $\alpha \to \infty$, but is chosen in such a way that $\eta(M)$ is arbitrarily small. As $\alpha \to \infty$,

$$\int_{-\infty}^{M} K_1(\alpha-\beta)\eta'(\beta)d\beta \sim K_1(\alpha)[\eta(M) - 2] \simeq -2K_1(\alpha)$$

so we obtain

$$2\Delta\eta(\alpha) \sim 2K_1(\alpha) - \int_M^\alpha K_1(\alpha-\beta)\eta'(\beta)d\beta. \tag{42}$$

First consider the case when the dominant balance in (42) is between the first and last terms:

$$2\Delta\eta(\alpha) \sim -\int_M^\alpha K_1(\alpha-\beta)\eta'(\beta)d\beta.$$

The solution is exponential, $\eta(\alpha) \sim ce^{-p\alpha}$, where p is a root of the equation

$$2\Delta = p \int_0^\infty K_1(\beta)e^{p\beta}\, d\beta, \tag{43}$$

provided the integral converges. If this integral does converge then $K_1(\alpha)$ must be $o(e^{-p\alpha})$ so the assumed dominant balance in (42) is indeed satisfied. It is clear that this type of behaviour can only occur for relaxation functions that decay exponentially to zero at infinity.

For the exponential relaxation function (8), equation (43) gives $p = (1 + d\delta/2\Delta)^{-1}$, and the condition $K_1(\alpha) = o(e^{p\alpha})$ is satisfied. This result is also immediately obtained from the exact solution (25).

For the relaxation function (9), equation (43) becomes

$$\sum_i \frac{d_i\delta}{\lambda_i - p} = k_0 + 2\Delta$$

which is the same as (39) with $(p,\Delta) \rightarrow (-p,-\Delta)$. From the graph in Figure 2, we see that this equation has n real positive roots and no complex roots. The smallest root p lies between 0 and λ_1, hence again the condition $K_1(\alpha) = o(e^{p\alpha})$ is satisfied by this root. This result may be extended to similar relaxation functions that have continuous decay spectrum,

$$G(s) = \int_A^B d(\delta')e^{-s/\delta'}d\delta'$$

provided that $B < \infty$.

For the oscillatory function (10), equation (43) reduces to the same as (40) with $(p,\Delta) \to (-p,-\Delta)$. If $\Delta < d/2\omega(1+\omega^2\delta^2)$, there are two roots, both real and positive, and the smaller one satisfies $K_1(\alpha) = o(e^{p\alpha})$. If $\Delta > d/2\omega(1+\omega^2\delta^2)$; however, the roots become complex with real part 1, and $K_1(\alpha)$ and $\eta(\alpha)$ are in this case of the same order of magnitude. Thus the argument leading to (43) breaks down. Nevertheless the resulting asymptotic behaviour of $\eta(\alpha)$ is correct, as can be verified by replacing (21) by the equivalent second-order differential equation. The oscillatory asymptotic behaviour of the shock that this implies has been confirmed through numerical solutions by Lardner and Ramakesavan [8].

Returning to (42) let us consider the possibility that the first term on the right dominates the second. Then

$$\eta(\alpha) \sim K_1(\alpha)/\Delta \qquad (\alpha \to \infty). \tag{44}$$

A sufficient condition for this is

$$\int_M^\alpha K_1(\alpha-\beta)K(\beta)d\beta = o(K_1(\alpha)) \qquad (\alpha \to \infty). \tag{45}$$

Consider the algebraic relaxation function $G(s) = d(1+s/\delta)^{-K}$ $(K > 2)$. Then $K(\beta) = d\delta(1+\beta)^{-K}$, $K_1(\beta) = [d\delta/(K-1)](1+\beta)^{-(K-1)}$ and one can show that

$$\int_M^\alpha K_1(\alpha-\beta)K(\beta)d\beta \simeq \frac{(d\delta)^2}{(K-1)(K-2)}\,\alpha^{-K} \qquad (\alpha \to \infty)$$

so that condition (45) is satisfied. Thus in this case the asymptotic behaviour of the shock solution is, from (44),

$$\eta(\alpha) \sim [d\delta/(K-1)\Delta]\,\alpha^{-(K-1)} \qquad (\alpha \to \infty).$$

As long as $K > 2$, this is still sufficient to allow the matching process to go through (see the remark following (28)).

We note also that this type of asymptotic behaviour was found also by Sugimoto and Kakutani [9] for the relaxation function $G(s) = As^{-\nu}$ $(0 < \nu < 1)$. This relaxation function however does not satisfy the requirements of Sections 3 and 4, so the matching does not go through as presented in Section 4.

8. Estimates of the shock width

It is sometimes useful to have an estimate of the width of the shock layer. The usual way of forming this [1] can be illustrated with the solution (25) for the exponential relaxation function. The values of α at which $f = 1 - 1/N$ and $f = -1 + 1/N$ are given respectively by

$$\alpha_+ \approx \alpha_0 + (\frac{d\delta}{2\Delta} + 1)\ \ell n\, N\,, \qquad \alpha_- \simeq \alpha_0 - (\frac{d\delta}{2\Delta} - 1)\ \ell n\, N$$

and the difference $\Delta\alpha = \alpha_+ - \alpha_-$ can be taken as a measure of the shock width:

$$\Delta\alpha = \frac{d\delta}{\Delta}\ \ell n\, N\,. \tag{46}$$

This procedure may be extended to more general cases. Suppose that $f(\alpha) \sim -1 + c_1 e^{p_1\alpha}$ as $\alpha \to -\infty$ and $f(\alpha) \sim 1 - c_2 e^{-p_2\alpha}$ as $\alpha \to \infty$, where p_1 and p_2 are the smallest positive roots of (38) and (43) respectively. Then proceeding as above, we estimate the width as

$$\Delta\alpha = (\frac{1}{p_1} + \frac{1}{p_2})\ \ell n\, N + (\frac{\ell n\, c_1}{p_1} + \frac{\ell n\, c_2}{p_2})\,. \tag{47}$$

For N sufficiently large, the second term may be neglected. A similar procedure may be used in principle when the behaviour as $\alpha \to \infty$ is given by (44) rather than by an exponential function.

In neglecting the second term in (47) we are taking the width to be dominated by the two tails of the inner solution. A more satisfactory approach which takes account of the centre of the shock layer may be to <u>define</u> the width as follows:

$$\Delta\alpha = \int_{-\infty}^{\infty} [1-f^2(\alpha)]\,d\alpha\,. \tag{48}$$

The motivation for this definition is made clear by Figure 4. From (24) this integral is readily evaluated for the exponential case, and the result is $\Delta\alpha = 2d\delta/\Delta$, similar to (46).

The advantage of the definition (48) is that a simple closed expression can be obtained for $\Delta\alpha$ in the general case. For, integrating eqn. (21) over $-\infty < \alpha < \infty$ we obtain the identity

$$\int_{-\infty}^{\infty} [1-f^2(\alpha)]d\alpha = \frac{2k_1}{\delta\Delta}$$

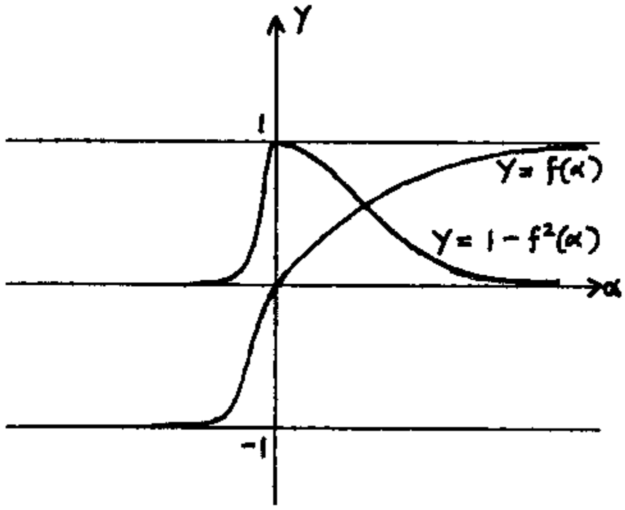

Figure 4

and therefore we have the estimated width $\Delta\alpha = 2k_1/\delta\Delta$. In terms of the physical variable θ, the width is $\Delta\theta = 2k_1/\Delta$.

Particular results for the various relaxation functions are as follows:

$$G(s) = \sum_i d_i \, e^{-s/\delta_i} \quad : \quad \Delta\theta = \frac{2}{\Delta}\sum d_i\delta_i^2$$

$$G(s) = de^{-s/\delta}\sin \omega s \quad : \quad \Delta\theta = 4\omega d\delta^3/\Delta(1+\omega^2\delta^2)^2$$

$$G(s) = d(1+s/\delta)^{-K} \quad : \quad \Delta\theta = 2d\delta^2/\Delta(K-1)(K-2)$$

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