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Convex Bodies Equidecomposable by Locally Discrete Groups of Isometries

R.J. Gardner

CONVEX BODIES EQUIDECOMPOSABLE BY
LOCALLY DISCRETE GROUPS OF ISOMETRIES†

R.J. GARDNER

Abstract. We show that if a polytope K_1 in $\mathbb{R}^d$ can be partitioned into a finite number of sets, and these sets can be moved by isometries in a locally discrete group to form a convex body K_2, then K_2 is a polytope and a similar partition can be made where the sets involved are simplices with disjoint interiors. This gives partial answers to questions of Tarski, Sallee and Wagon.

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1. Introduction.

In 1925, Tarski ([T]) asked whether a circle (with interior) of unit area and the unit square are equidecomposable; that is, whether the one can be partitioned into a finite number of pieces which can then be rearranged to form the other. The problem seems difficult because the pieces may be arbitrary sets. The only directly relevant result seems to be that of Dubins, Hirsch and Karush ([DHK]), who show that this is not possible if the pieces are topological discs, and we ignore overlapping of the boundaries of the pieces.

This paper gives other negative results bearing on Tarski's problem. We only allow the pieces to be moved by isometries in a locally discrete group; for example, by rational translations. However, the novel feature is that in this case the 'circle-squaring' is shown to be impossible without any restriction whatsoever on the pieces themselves.

The main results are actually more general. From Theorem 1 it follows that if a polytope in $\mathbb{R}^d$ can be partitioned into a finite number of sets, which can be moved by isometries in a locally discrete group to form a convex body, then this convex body is also a polytope and a similar partition can be made using simplices (providing we ignore overlapping of boundaries). This gives a partial answer to a question of S. Wagon ([W]) concerning the unit cube and regular tetrahedron of volume one in $\mathbb{R}^3$ (see Corollary 4). We also prove an analogous result for any two convex bodies in the plane, in the case where the isometries are from a locally discrete group of translations. This relates to a question raised by Sallee ([S], Problem 1).

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2. Preliminaries and definitions.

If A is any set, we denote by $\mathrm{cl}\,A$, $\mathrm{int}\,A$ and $\mathrm{bd}\,A$, the closure, interior and boundary of A, respectively. When A is convex, ∂A will also denote the boundary of A. We write $\mathcal{P}(A)$ for the set of all subsets of A, and as usual $A \Delta B = (A\backslash B) \cup (B\backslash A)$.

If $1 \leq k \leq d$, λ_k will denote Hausdorff k-dimensional measure in $\mathbb{R}^d$. (Note that λ_d is just Lebesgue measure in $\mathbb{R}^d$.)

A polytope in $\mathbb{R}^d$ is a finite union of d-dimensional simplices. A convex body is a compact convex set with nonempty interior.

Let G be a group of isometries of $\mathbb{R}^d$, and suppose A and B are sets in $\mathbb{R}^d$. We say that A and B are G-equidecomposable if (i) $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$, (ii) the pieces A_i and B_i are sets with $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $1 \leq i \neq j \leq k$, and (iii) $B_i = \tau_i(A_i)$ for suitable $\tau_i \in G$, $i = 1, \ldots, k$. If G is the group of all isometries, we simply say that A and B are equidecomposable (this was the definition used by Banach and Tarski in [BT]).

We shall call A and B G-equidecomposable mod. null sets (respectively, G-equidecomposable mod. first category sets) if (ii) above is replaced by (ii)' A_i and B_i are sets with $\lambda_d(A_i \cap A_j) = \lambda_d(B_i \cap B_j) = 0$ (respectively, $A_i \cap A_j$ and $B_i \cap B_j$ are of first category) for $1 \leq i \neq j \leq k$. Finally, if (ii) above is replaced by (ii)" A_i and B_i are convex bodies with $\text{int } A_i \cap \text{int } A_j = \text{int } B_i \cap \text{int } B_j = \emptyset$ for $1 \leq i \neq j \leq k$, we say that A and B are G-convex equidecomposable. Again, if G is the group of all isometries, we just say A and B are convex equidecomposable.

Notice that if A and B are polytopes, then they are G-convex equidecomposable if and only if they are G-geometrically equidecomposable; that is, G-equidecomposable in the classical sense (see [B] for example), where A_i and B_i above are simplices.

A group G of isometries of $\mathbb{R}^d$ is discrete if for every compact subset C of $\mathbb{R}^d$, $C \cap g(C) = \emptyset$ except for a finite number of g in G. Such a group is countable and has a convex fundamental region; that is, an open convex set R in $\mathbb{R}^d$ such that (i) $R \cap g(R) = \emptyset$ if $g \in G$ is not the identity and (ii) $\mathbb{R}^d = \cup\{\text{cl}(g(R)): g \in G\}$. (An extension of the arguments of [BG], Ch. 3, shows this.) The set R is not necessarily bounded, but ∂R is contained in a finite union of hyperplanes.

By a rational translation in $\mathbb{R}^d$ we mean an isometry g such that $g(\underline{x}) = \underline{x} + \underline{r}$, for all $\underline{x} \in \mathbb{R}^d$, and each coordinate of $\underline{r}$ is rational. The group of rational translations in $\mathbb{R}^d$ is denoted by Q_d.

In this paper we use the term <u>locally discrete</u> for a group with the property that each finite set of its elements generates a discrete group. Note that Q_d is locally discrete but not discrete.

3. Results.

The bulk of the work is done by the following theorem.

THEOREM 1. Let G be a discrete group of isometries of $\mathbb{R}^d$. Suppose that K_1 is a convex polytope in $\mathbb{R}^d$ and K_2 is a convex body in $\mathbb{R}^d$. If $\mu(K_1) = \mu(K_2)$ for every finitely additive, G-invariant measure on $P(\mathbb{R}^d)$, then K_1 and K_2 are G-convex equidecomposable.

Proof. Since G is discrete, we may write $G = \{g_n : n \in \mathbb{N}\}$. Let R be a fundamental region of G. For $i = 1, 2$, let $D_{in} = \mathrm{cl}(R \cap g_n^{-1}(\partial K_i))$ and $E_{in} = \mathrm{cl}(R \cap g_n^{-1}(K_i))$. Each set E_{in} is a convex body contained in $\mathrm{cl}\, R$ and D_{in} is a finite union of convex surfaces contained in ∂E_{in}. Since each K_i is bounded, the set N of n such that $E_{in} \neq \emptyset$ for $i = 1$ or 2 is finite. Let $D_i = \{D_{in} : n \in N\}$ and $D = D_1 \cup D_2 \cup \partial R$. Then D is a closed set.

Choose a point $\underline{x} \in R\backslash D$. For each n, $g_n(\underline{x}) \notin \partial K_i$, for $i = 1, 2$.

For any set $E \subset \mathbb{R}^d$, let $\mu(E) = \mathrm{card}\, \{n : g_n(\underline{x}) \in E\}$. Then μ, which depends on the point $\underline{x}$ chosen above, is a finitely additive, G-invariant measure on $P(\mathbb{R}^d)$.

It will be convenient to define $\psi_i(\underline{x}) = \mathrm{card}\, \{n : g_n(\underline{x}) \in K_i\}$, for $i = 1, 2$, so that $\psi_i(\underline{x}) = \mu(K_i)$ for $i = 1, 2$. By hypothesis, $\mu(K_1) = \mu(K_2)$, so $\psi_1(\underline{x}) = \psi_2(\underline{x}) = \psi(\underline{x})$ say. Now for any $\underline{x} \in R\backslash D$, $g_n(\underline{x}) \in K_i$ if and only if $\underline{x} \in \mathrm{int}\, E_{in}$, so we have shown that $\psi(\underline{x}) = \psi_i(\underline{x}) = \mathrm{card}\, \{n : \underline{x} \in \mathrm{int}\, E_{in}\}$, for $i = 1, 2$. This leads to the following simple relationship between the sets D_1 and D_2.

Claim. $D_1 \Delta D_2$ is contained in a finite union of hyperplanes.

Proof. To stress that we do not use the assumption that K_1 is a polytope, we shall work with K_2.

For each pair m, n in N with $m \neq n$, there is at most one hyperplane H_{mn} which separates int E_{2m} and int E_{2n} and which meets both of the convex sets E_{2m} and E_{2n} in a set of positive λ_{d-1}-measure. Therefore $H = \cup\{H_{mn}: m, n \in N\}$ is a finite union of hyperplanes, and we show that $D_1 \Delta D_2 \subset H \cup \partial R$.

For each pair $m, n,$ in N with $m \neq n$, let ∇_{mn} be the set of points $\underline{d}$ in $D_{2m} \cap D_{2n} \cap R$ such that for each neighbourhood $U \subset R$ of $\underline{d}$, $D_{2m} \cap D_{2n} \cap U$ is strictly contained in $D_{2m} \cap U$ (or equivalently in $D_{2n} \cap U$). It is easy to see that ∇_{mn} is nowhere dense in D_{2m} and D_{2n}. Therefore $\nabla = \mathrm{cl}(\cup\{\nabla_{mn}: m, n \in N\})$ is also nowhere dense in D.

Suppose there is a $\underline{d} \in (D_2\backslash\nabla)\backslash(D_1 \cup H \cup \partial R)$, and therefore an $m \in N$ with $\underline{d} \in D_{2m}$. Also, there is a neighbourhood U of $\underline{d}$ such that if $\underline{d} \notin D_{2n}$, then $D_{2n} \cap U = \phi$, while if $\underline{d} \in D_{2n}$, then $D_{2m} \cap D_{2n} \cap U = D_{2m} \cap U$. Since $\underline{d} \notin D_1$, we may assume in addition that $D_1 \cap U = \phi$.

For all $\underline{x}_1, \underline{x}_2$ in U, $\underline{x}_1 \in \mathrm{int}\, E_{1n}$ if and only if $\underline{x}_2 \in \mathrm{int}\, E_{1n}$, because $D_1 \cap U = \phi$, and so $\psi(\underline{x}_1) = \psi(\underline{x}_2)$. Choose $\underline{x}_1 \in U \cap \mathrm{int}\, E_{2m}$ and $\underline{x}_2 \in U\backslash E_{2m}$. Then there is an $n_0 \neq m$ such that $\underline{d} \in D_{2n_0}$, $\underline{x}_2 \in \mathrm{int}\, E_{2n_0}$ and $\underline{x}_1 \notin \mathrm{int}\, E_{2n_0}$. For, if $\underline{d} \notin D_{2n}$, then $\underline{x}_1 \in \mathrm{int}\, E_{2n}$

if and only if $\underline{x}_2 \in \text{int } E_{2n}$ (as $D_{2n} \cap U = \phi$ in this case), while $\underline{x}_1 \in \text{int } E_{2m}$ but $\underline{x}_2 \notin \text{int } E_{2m}$.

Since $\underline{d} \notin \nabla$, $D_{2n_0} \cap D_{2m} \cap U = D_{2m} \cap U$, so these sets must be contained in a hyperplane which separates $\text{int } E_{2m}$ and $\text{int } E_{2n_0}$. This hyperplane meets both E_{2m} and E_{2n_0} in a set containing $D_{2m} \cap U$, which has positive λ_{d-1}-measure. Consequently the hyperplane is H_{mn_0}, and $\underline{d} \in H$, a contradiction.

We conclude that $(D_2 \backslash \nabla) \subset (D_1 \cup H \cup \partial R)$, and because ∇ is nowhere dense in D_2 and $(D_1 \cup H \cup \partial R)$ is closed, that $D_2 \subset (D_1 \cup H \cup \partial R)$. Reversing the roles of D_1 and D_2 gives $D_1 \subset (D_2 \cup H \cup \partial R)$ and completes the proof of the Claim.

Now, for the first time, we apply the assumption that K_1 is a polytope. The set D_1 is contained in a finite union of hyperplanes. By the Claim, D is also contained in a finite union of hyperplanes, which divide cl R into a finite number of convex sets $P_1, \ldots, P_\ell$, with disjoint interiors. Each set $\text{int } P_j$ is contained in an open component of $(R\backslash D)$, so $\psi_i = \psi$ has the same value p_j at each point in $\text{int } P_j$, for $i = 1, 2$. If $p_j \geq 1$ then P_j is also bounded and so is a convex polytope. For $i = 1, 2$, $P_j \subset E_{in_{ik}}$, where n_{ik} are different natural numbers in N, for $1 \leq k \leq p_j$. So $g_{n_{ik}}(P_j) \subset K_i$ for $1 \leq k \leq p_j$ and $i = 1, 2$, and if $(j, k) \neq (j', k')$ then $\text{int } g_{n_{ik}}(P_j) \cap \text{int } g_{n_{ik'}}(P_{j'}) = \phi$, for $i = 1$ or 2. Finally, $K_i = \cup\{g_{n_{ik}}(P_j): 1 \leq k \leq p_j, 1 \leq j \leq \ell\}$, for

$i = 1, 2$, so K_1 and K_2 are G-convex equidecomposable. |||

<u>COROLLARY 2</u>. Let G be a locally discrete group of isometries of $\mathbb{R}^d$. Suppose that K_1 is a convex polytope and K_2 is a convex body in $\mathbb{R}^d$. If K_1 and K_2 are G-equidecomposable, then K_2 is a convex polytope and K_1 and K_2 are G-convex equidecomposable.

<u>Proof</u>. The finite set of isometries which witness the G-equidecomposability of K_1 and K_2 generate a discrete group G^*, and K_1 and K_2 are also G^*-equidecomposable. Then Theorem 1 can be applied. |||

The next corollary gives the new result on Tarski's problem.

<u>COROLLARY 3</u>. Let G be a locally discrete group of isometries of $\mathbb{R}^d$. Then a sphere (with interior) and cube of equal volume in $\mathbb{R}^d$ are not G-equidecomposable. In particular, they are not Q_d-equidecomposable.

In [W], S. Wagon asks whether a regular tetrahedron and cube in $\mathbb{R}^3$ are equidecomposable with Lebesgue measurable pieces. Dehn's solution to Hilbert's Third Problem (see [B]) yields that these polyhedra are not convex equidecomposable.

<u>COROLLARY 4</u>. Let G be a locally discrete group of isometries of $\mathbb{R}^3$. A regular tetrahedron and a cube in $\mathbb{R}^3$ are not G-equidecomposable. In particular, they are not Q_3-equidecomposable.

Notice that the statement of Theorem 1 does not hold for all non-discrete groups. For example, let G be a dense group of translations. If μ is a finitely additive, G-invariant measure on $P(\mathbb{R}^2)$, then μ must agree with a constant multiple of Jordan measure on the Jordan measurable sets in $\mathbb{R}^2$ (the proof is essentially the same as [W], Proposition 9.6). Since K_1 and K_2 are Jordan measurable, the hypothesis of Theorem 1 then simply says that K_1 and K_2 have the same Lebesgue measure, which is not enough to ensure that they are G-convex equidecomposable.

Theorem 1 and its corollaries may be strengthened, however. For example, K_1 can be any polytope, not necessarily convex, with essentially the same proof. To make further improvements, we must first discuss a special property of the groups we consider.

A group G is called <u>amenable</u> if there is a finitely additive measure μ on $P(G)$ such that $\mu(G) = 1$ and μ is left-invariant (i.e., $\mu(gA) = \mu(A)$ for $g \in G$ and $A \subset G$). Amenable groups have been extensively studied, and it is known for example that solvable groups are amenable, but any group containing a free subgroup of rank two is not amenable. The group of isometries of $\mathbb{R}^n$, $n \geq 3$, contains a free subgroup of rank two, and this allows the Banach-Tarski paradox ([BT]) which implies the equidecomposability of any two bounded sets in $\mathbb{R}^n$, $n \geq 3$, with nonempty interiors.

If a group G of isometries of $\mathbb{R}^d$ is amenable, then λ_d can be extended to a finitely additive, G-invariant measure defined on $P(\mathbb{R}^d)$ (see, for example, [M], Theorem 5.1). Since the group of isometries of $\mathbb{R}^2$ is solvable and so amenable, 'paradoxes', such as the equidecomposability of a circle of area one and a circle of area two, cannot occur in $\mathbb{R}^2$.

Although the fact was not used directly in Theorem 1, every discrete group G of isometries of $\mathbb{R}^d$ is amenable, as S. Wagon has pointed out to the author. For, if G is not amenable, then G contains a free subgroup of rank two, by a theorem of Tits; but this is impossible, by the remarks following Theorem 1 of [MW]. We shall use the amenability of G in the next theorem.

THEOREM 5. Let G be a locally discrete group of isometries of $\mathbb{R}^d$. Suppose that K_1 is a polytope and K_2 is a convex body in $\mathbb{R}^d$. The following conditions are equivalent:

(i) K_1 and K_2 are G-equidecomposable mod. null sets.

(ii) K_1 and K_2 are G-equidecomposable mod. first category sets.

(iii) K_1 and K_2 are G-convex equidecomposable.

Proof. As in the proof of Corollary 2, we may take G to be discrete. The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are obvious.

To prove (i) $\Rightarrow$ (iii), we first note that if (i) holds then $\mu(K_1) = \mu(K_2)$ for every finitely additive, G-invariant measure μ on $P(\mathbb{R}^d)$ which is absolutely continuous with respect to λ_d. As in Theorem 1, we define the sets R and D and choose $\underline{x} \in R\backslash D$, so that for each n, $g_n(\underline{x}) \notin \partial K_i$, for $i = 1, 2$.

Let $B(\underline{y}, \varepsilon)$ denote the open ball, centre $\underline{y}$ and radius ε. Note that $g(B(\underline{y},\varepsilon)) = B(g(\underline{y}), \varepsilon)$ for any isometry g.

There exists an $\varepsilon > 0$ such that for each n, $\partial K_i \cap g_n(B(\underline{x},\varepsilon)) = \phi$, $i = 1, 2$. Put $B = \cup_n g_n(B(\underline{x}, \varepsilon))$, so that $g(B) = B$ for all $g \in G$.

By the remarks above, G is amenable, so there is an extension $\bar{\mu}$ of λ_d which is finitely additive, G-invariant, defined on $P(\mathbb{R}^d)$, and which must of course be absolutely continuous with respect to λ_d. Define μ by $\mu(E) = \bar{\mu}(E \cap B)$ for $E \subset \mathbb{R}^d$. Then μ has all the properties of $\bar{\mu}$ mentioned above.

For $i = 1, 2$, let $\psi_i(\underline{x}) = \text{card}\,\{n: g_n(\underline{x}) \in K_i\}$, as before. Then $\mu(K_i) = \bar{\mu}(K_i \cap B) = \psi_i(\underline{x})\cdot\lambda_d(B(\underline{x},\varepsilon))$, since $\partial K_i \cap B = \phi$. Now (i) implies that $\mu(K_1) = \mu(K_2)$, so $\psi_1(\underline{x}) = \psi_2(\underline{x}) = \psi(\underline{x})$, and the rest of the proof that (i) $\Rightarrow$ (iii) now follows that of Theorem 1.

If (ii) holds, then $\mu(K_1) = \mu(K_2)$ for every finitely additive, G-invariant measure on $P(\mathbb{R}^d)$ which vanishes on first category sets. To prove (ii) $\Rightarrow$ (iii) we follow the above argument but take $\bar{\mu}$ to be instead a finitely additive, G-invariant measure satisfying $\bar{\mu}(E) = \lambda_d(E)$ if $\lambda_d(\text{bd}\,E) = 0$ and $\bar{\mu}(F) = 0$ for all first category sets F (see [M], Theorem 6.1). |||

A few remarks are in order about the use of the amenability of G in the previous theorem. J. Mycielski and S. Wagon have recently proved (see [W], Theorem 11.20) that a group G of isometries of $\mathbb{R}^d$ is amenable _if and only if_ there is a finitely additive, G-invariant extension of λ_d which is defined on $P(\mathbb{R}^d)$. According to S. Wagon, the same methods which prove this also show that a group G of isometries of $\mathbb{R}^d$ is amenable _if and only if_ there is a finitely additive, G-invariant extension of Jordan measure, defined on $P(\mathbb{R}^d)$, which vanishes on first category sets. We note also that for certain non-amenable groups of area-preserving affine transformations of $\mathbb{R}^2$, any two bounded sets in $\mathbb{R}^2$ with nonempty interiors are G-equidecomposable (see [W], Theorem 7.3).

Theorem 5 gives the corresponding improvement in Corollaries 3 and 4. It is interesting that the assumption that K_1 and K_2 are G-equidecomposable is strictly stronger than (i) - (iii) of Theorem 5; see Example 9.

Some restriction on K_1 beyond convexity is needed in Theorem 1. In [G], the author exhibits two convex bodies K_1 and K_2 in $\mathbb{R}^2$, which are G-equidecomposable for a discrete group G of rotations, but which are not G-convex equidecomposable. This answered Problem 1 of [S] negatively. To obtain some positive results, we can weaken the conclusions as below.

THEOREM 6. Let G be a locally discrete group of isometries of $\mathbb{R}^d$. If K_1 and K_2 are convex bodies in $\mathbb{R}^d$ which are G-equidecomposable (mod. null sets or first category sets), then K_1 and K_2 are G-equidecomposable (respectively, mod. null sets or first category sets) with Borel pieces.

We omit the proof of Theorem 6, since it only differs from that of Theorem 1 in the final paragraph.

Alternatively, we may obtain a result in the spirit of Theorem 1 for arbitrary convex bodies K_1 and K_2 by restricting the isometries in G to be translations. To avoid complications we prove the next theorem only for the planar case, although it probably remains true in higher dimensions.

<u>THEOREM 7</u>. Suppose that G is a discrete group of translations in $\mathbb{R}^2$, and K_1 and K_2 are G-equidecomposable convex bodies in $\mathbb{R}^2$. Then K_1 and K_2 are G-convex equidecomposable.

<u>Proof</u>. We begin by repeating the proof of Theorem 1, up to and including the Claim in that proof, since the assumption that K_1 is a polytope is unnecessary for this. Next, another claim must be established.

<u>Claim</u>. For $i = 1, 2$, and n_1, n_2 in N with $n_1 \neq n_2$, the set $D_{in_1} \cap D_{in_2}$ is either finite or a line segment.

<u>Proof</u>. Suppose that for some i, n_1 and n_2 as above, $D' = D_{in_1} \cap D_{in_2}$ is infinite and not a line segment. Then we may choose two points $\underline{d}_1$, $\underline{d}_2$ in D' such that there are arcs C_{in_j} contained in D_{in_j} with endpoints $\underline{d}_1$ and $\underline{d}_2$, $j = 1, 2$, and with both C_{in_1} and C_{in_2} nonlinear and lying on the same side of the line segment $[\underline{d}_1, \underline{d}_2]$. Now choose points $\underline{c}_j \in C_{in_j}$, $j = 1, 2$, such that there are tangents to C_{in_j} at $\underline{c}_j$ which are parallel to $[\underline{d}_1, \underline{d}_2]$. By hypothesis g_{n_j} is a translation, say $g_{n_j}(\underline{x}) = \underline{x} + \underline{t}_j$ for $\underline{x} \in \mathbb{R}^2$. Then there are tangents to ∂K_i at $(\underline{c}_j + \underline{t}_j)$ which are parallel to $[\underline{d}_1, \underline{d}_2]$, with K_i on the same side of these tangents, $j = 1, 2$. The points $(\underline{c}_j + \underline{t}_j)$, $j = 1, 2$, must be the

endpoints of a line segment in ∂K_i. Note that one of the points $\underline{d}_1 + \underline{t}_j$ or $\underline{d}_2 + \underline{t}_j$ must also be on this line segment. Thus $\underline{c}_j$, $j = 1, 2$, and either $\underline{d}_1$ or $\underline{d}_2$, all lie on the same line segment, parallel to $[\underline{d}_1, \underline{d}_2]$, in D'. So both arcs C_{in_j}, $j = 1, 2$, coincide with $[\underline{d}_1, \underline{d}_2]$, a contradiction.

It follows from the Claim that for $i = 1, 2$, D_i can be considered as a finite graph in $\mathrm{cl}\,R$. Further, by the Claim of Theorem 1, $D_1 \Delta D_2$ is contained in ∂R together with a finite set of lines H. Therefore, the set $D \cup H \cup \partial R$ may also be considered as a finite graph in $\mathrm{cl}\,R$, and its finite set of nodes denoted by V.

Consider two points $\underline{v}$, $\underline{w}$ in V at the ends of an edge in this graph; that is, endpoints of an open arc $C \subset D_{1m_1}$ say, for some $m_1 \in N$, such that $V \cap C = \phi$. If C is not a line segment, there is an $m_2 \in N$ such that $C \subset D_{2m_2}$, since $C \subset D_1$, $V \cap C = \phi$, and $D_1 \Delta D_2 \subset H \cup \partial R$.

If $n \neq m_1$, and $D_{1n} \cap C \neq \phi$, then $C \subset D_{1n}$ since $V \cap C = \phi$. However, an argument similar to that used in the Claim above shows that this is not possible. Therefore if $n \neq m_1$, $D_{1n} \cap C = \phi$, and similarly, if $n \neq m_2$, $D_{2n} \cap C = \phi$.

Let J denote the convex body bounded by C and the line segment $[\underline{v}, \underline{w}]$. For $i = 1, 2$, put $J_i = g_{m_i}(J)$. Then J_i is a compact convex subset of K_i, containing a nonlinear arc in ∂K_i, and $J_2 = g_{m_2} g_{m_1}^{-1}(J_1)$. Further, the elements g_{m_1}, g_{m_2} of G are the only ones with these properties.

Let $\{J_{ik} : 1 \le k \le k_0,\ i = 1, 2\}$ list all such distinct pairs of compact convex sets, obtained in the manner above from pairs of points in V at the ends of open nonlinear arcs in D which contain no point of V. For each k, and $i = 1, 2$, J_{ik} is a subset of K_i containing a nonlinear arc of ∂K_i, and $J_{2k} = g(J_{1k})$ for some $g \in G$ depending on k. Further, for $i = 1, 2$, $P_i = \mathrm{cl}(K_i \setminus \cup \{J_{ik} : 1 \le k \le k_0\})$ is a convex polygon. We have shown that $(K_1 \setminus \mathrm{int}\ P_1)$ and $(K_2 \setminus \mathrm{int}\ P_2)$ are G-convex equidecomposable. Finally, P_1 and P_2 are also G-convex equidecomposable, as in the last paragraph of the proof of Theorem 1. |||

We may now extend Theorem 7 in the same way we extended Theorem 1.

<u>THEOREM 8</u>. Let G be a locally discrete group of translations in $\mathbb{R}^2$. If K_1 and K_2 are convex bodies in $\mathbb{R}^2$, the following conditions are equivalent.

(i) K_1 and K_2 are G-equidecomposable mod. null sets.

(ii) K_1 and K_2 are G-equidecomposable mod. first category sets.

(iii) K_1 and K_2 are G-convex equidecomposable.

Theorem 8, applied with $G = Q_2$ for example, gives some results which relate to Problem 1 of [S].

If two convex bodies K_1 and K_2 in $\mathbb{R}^2$ are convex equidecomposable, then K_1 and K_2 are equidecomposable. To see this, one uses a trick used in [BT], Lemma 17, whereby boundaries of convex pieces may be absorbed into the interiors of these pieces. The same trick cannot always be used when G is locally discrete, and the statement that K_1 and K_2 are G-equidecomposable cannot be added to the list of equivalences in Theorems 5 and 8. The next example shows this.

<u>EXAMPLE 9</u>. Let K_1 be the rectangle $\{(x, y): 0 \leq x \leq 1, -2 \leq y \leq 2\}$ and K_2 the rectangle $\{(x, y): 0 \leq x \leq 2, 0 \leq y \leq 2\}$. Then K_1 and K_2 are clearly Q_2-convex equidecomposable. We show below that they are not Q_2-equidecomposable.

Since Q_2 is locally discrete, it suffices to show that K_1 and K_2 are not G-equidecomposable when G is a discrete subgroup of Q_2. We need only consider $G = G_r$, where G_r is generated by the translations $(0, 1/r)$ and $(1/r, 0)$, for some natural number r. Our aim will be achieved if we can find a finitely additive, G_r-invariant measure μ on $P(\mathbb{R}^2)$ such that $\mu(K_1) \neq \mu(K_2)$.

To this end, let $L = \{(x, y): x$ or $y = m/r$ for some integer $m\}$ and $L_i = K_i \cap L$, $i = 1, 2$. According to [M], 5.2, the Hausdorff 1-dimensional measure λ_1 (equal to Lebesgue linear measure) in $\mathbb{R}^2$ can be extended to a finitely additive, isometry invariant measure $\bar{\mu}$ on $P(\mathbb{R}^2)$. Define μ by $\mu(E) = \bar{\mu}(E \cap L)$ for $E \subset \mathbb{R}^2$. Then μ is finitely additive and G_r-invariant, because if $g \in G_r$ then $g(L) = L$.

However, $\mu(K_1) = \bar{\mu}(L_1) = \lambda_1(L_1) = 8r + 5$, while $\mu(K_2) = \bar{\mu}(L_2) = \lambda_1(L_2) = 8r + 4$. |||

If a group G is locally discrete, then it is amenable, since any group is amenable if and only if all its finitely generated subgroups are. Therefore our results are evidence for a positive answer to the following question, which would settle Tarski's problem.

QUESTION. Does Theorem 5 remain true for any amenable group G of isometries of $\mathbb{R}^d$?

References

[BT] S. Banach and A. Tarski, Sur la décomposition des ensembles des points en parties respectivement congruentes, *Fund. Math.* 6(1924), 244-277.

[BG] C.T. Benson and L.C. Grove, *Finite reflection groups*, Bogden and Quigley, New York (1971).

[B] V. Boltianskii, *Hilbert's third problem*, Winston, Washington (1978).

[DHK] L. Dubins, M. Hirsch and J. Karush, Scissor congruence, *Israel J. Math.* 1(1963), 239-247.

[G] R.J. Gardner, A problem of Sallee on equidecomposable convex bodies, *Proc. Amer. Math. Soc.*, to appear.

[M] J. Mycielski, Finitely additive invariant measures, I, *Colloq. Math.* 42(1979), 309-318.

[MW] J. Mycielski and S. Wagon, Large free groups of isometries, *L'Enseignement* Math., to appear.

[S] G.T. Sallee, Are equidecomposable plane convex sets convex equidecomposable?, *Amer. Math. Monthly* 76(1969), 926-927.

[T] A. Tarski, Problème 38, *Fund. Math.* 7(1925), 381.

[W] S. Wagon, *The Banach-Tarski Paradox*, Cambridge University Press, New York (1985).

University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran, Saudi Arabia