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SECOND AND THIRD ORDER PERTURBATION SOLUTIONS OF A GENERALIZED BURGERS' EQUATION

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Abstract

The differential equation $u_\tau - uu_x = k(u_{xx} + cu_{x\tau})$ with initial values on $\tau = 0$ is considered. When $c \neq 0$ this represents a hyperbolic generalization of Burgers' equation. For $k << 1$ perturbation solutions are obtained, the outer solution being given completely up to third order, the inner solution (i.e. close to the shock) being given to second. The determination of the unknown functions in the second order inner solution is completed using an integral conservation technique. While the third order inner solution is not explicitly determined, it is shown that matching of the inner and outer solutions at third order is satisfied.

1. Introduction

The method of matched asymptotic expansions is widely used to investigate the solution of various generalizations of Burgers' equation in cases when the dissipation coefficient is small. At various stages in the historical development of this method, doubts have been expressed as to whether the method could be continued to higher orders. Thus Murray [1] in his original paper on a class of generalized Burgers' equations successfully matched the inner and outer solutions at first order, but expressed the opinion that matching at higher order could not determine the remaining unknown function in the first-order inner solution (which is essentially the quantity that Lighthill terms the "shock displacement due to diffusion" [2]). Subsequently, it has been shown for Burgers equation itself [3] as well as for certain generalized equations [4-6] that matching at second order is feasible and that it does determine Lighthill's displacement. After the second order matching, there remains an undetermined function in the second-order inner solution, and again doubts have been expressed as to whether this function can be determined by matching at higher order (see for example [3], p.360).

In a recent paper [7] we have looked into this question for Burgers' equation itself. It turned out that for this equation the inner and outer solutions can be matched at third order and that the third-order matching determines the remaining function in the second-order inner solution apart from a constant. This constant was determined using the integral conservation property introduced by Murray [1] and developed extensively by Crighton and Scott [4].

The purpose of the present paper is to investigate third-order matching for the generalized equation

$$u_\tau - uu_x = k(u_{xx} + cu_{x\tau}), \qquad k << 1 \tag{1}$$

with initial values $u(x,0) = \phi(x)$, given. This generalization is interesting in that when $c \neq 0$ equation (1) is hyperbolic (with initial values on a characteristic) as opposed to Burgers' equation $(c = 0)$ which is parabolic. It is one of the generalized equations considered by Murray [1] (see also [6]).

In Sections 2 and 3 of the paper, the outer and inner expansions of the solution are calculated up to third and second orders respectively. The outer solution is determined completely at each order in terms of the initial value $\phi(x)$ whereas the inner solution involves two unknown functions at each order, as well as the shock position. In Section 4 the two solutions are matched up to second order, and this determines the second order inner solution apart from one function. It is shown in Section 5 that this solution can be completed, without knowledge of the third order inner solution, by use of Crighton and Scott's technique of the conserved integral. Finally in Section 6 a matching of the inner and outer solutions to third order is made and it is shown that results are obtained consistent with those from the conserved integral.

2. The Outer Solution to Third Order

Expanding $u(x,\tau)$ in powers of k,

$$u(x,\tau) = u_0(x,\tau) + ku_1(x,\tau) + k^2u_2(x,\tau) + \dots , \tag{2}$$

substituting into (1) and comparing the coefficients of k^n we obtain the following set of equations:

$$u_{0\tau} - u_0u_{0x} = 0 \tag{3}$$

$$u_{1\tau} - u_0u_{1x} - u_{0x}u_1 = u_{0xx} + cu_{0x\tau} \tag{4}$$

$$u_{2\tau} - u_0u_{2x} - u_{0x}u_2 = u_{1xx} + cu_{1x\tau} + u_1u_{1x} \tag{5}$$

etc. The initial condition similarly leads to:

$$u_0(x,0) = \phi(x), \quad u_n(x,0) = 0 \quad (n \geq 1). \tag{6}$$

From equations (3) and (6_1) we have the usual first order solution [2,8]

$$u_0(x,\tau) = \phi(\beta) \tag{7}$$

where β is determined implicitly in terms of (x,τ) by

$$\beta = x + \tau\phi(\beta). \tag{8}$$

It is well-known that β ceases to be uniquely determined for all values of x when $\tau \geq \tau_f$ where $\tau_f = [\phi'_m]^{-1}$ and ϕ'_m is the maximum value of $\phi'(\beta)$. For $\tau \geq \tau_f$ the solution contains a shock and for values of x close to the shock position the solution is no longer given by (7) and (8).

In order to calculate u_n for $n \geq 1$ it is best to use the variables (β,τ) in place of (x,τ). After substituting for u_0 from (7), equation (4) can be written in terms of these variables as

$$\frac{1}{D}\frac{\partial}{\partial\tau}(Du_1) = \frac{\phi''}{D^3} + c\left(\frac{\phi'^2}{D^2} + \frac{\phi\phi''}{D^3}\right) \tag{9}$$

where

$$D = 1 - \tau\phi'(\beta) \tag{10}$$

and the argument β is omitted through (9) for brevity. After integration and use of the initial condition (6_2) we thus obtain

$$u_1 = (1 + c\phi)\frac{\tau\phi''}{D^2} - c\,\frac{\phi'}{D}\ln D\,. \tag{11}$$

In a similar way, after substituting for u_0 and u_1, equation (5) takes the form

$$\frac{1}{D}\frac{\partial}{\partial\tau}(Du_1) = (1 + c\phi)^2\left(\frac{\tau\phi^{iv}}{D^4} + \frac{8\tau^2\phi''\phi'''}{D^5} + \frac{10\tau^3\phi''^3}{D^6}\right)$$

$$+ c(1 + c\phi)\left\{(6 - 5D)\frac{\phi'''}{D^4} + (19 - 12D)\,\frac{\tau\phi''^2}{D^5}\right.$$

$$\left. - \left[(2 - D)\,\frac{\phi'''}{D^4} + (3 - D)\,\frac{2\tau\phi''^2}{D^5}\right]\ln D\right\}$$

$$+ c^2\frac{\phi'\phi''}{D^4}\left\{7 - 4D - (5 - 2D)\ln D + (\ln D)^2\right\}$$

and integration together with condition (6_2) lead to the solution

$$u_2 = (1 + c\phi)^2\left(\frac{\tau^2\phi^{iv}}{2D^3} + \frac{8\tau^3\phi''\phi'''}{3D^4} + \frac{5\tau^4\phi''^3}{2D^5}\right)$$

$$+ c(1 + c\phi)\left\{(2 + 3\tau\phi')\frac{\tau\phi'''}{2D^3} + (21 + 13\tau\phi')\frac{\tau^2\phi''^2}{6D^4} - \left[\frac{\tau\phi'''}{D^3} + \frac{2\tau^2\phi''^2}{D^4}\right]\ln D\right\}$$

$$+ \frac{c^2\phi''}{2D^3}\left\{\tau\phi'(6 - \tau\phi' - 4\ln D) + (\ln D)^2\right\}. \qquad (12)$$

3. The Inner Solution to Second Order

For $\tau \geq \tau_f$ the solution involves a shock. Let the shock position be $x = x_s(\tau)$ and introduce the inner variable η, defined by

$$x - x_s(\tau) = k\eta. \qquad (13)$$

In terms of the variables (η, τ), equation (1) becomes

$$(1 - cx_s')u_{\eta\eta} + (u + x_s')u_\eta = k(u_\tau - cu_{\eta\tau}) \qquad (14)$$

where x_s' denotes $x_s'(\tau)$, the shock speed. We now seek an inner expansion of the solution in the form

$$u = u_0^*(\eta, \tau) + ku_1^*(\eta, \tau) + k^2u_2^*(\eta, \tau) + \ldots \qquad (15)$$

Substituting (15) into (14) and comparing the coefficients of the different powers of k, we obtain the set of equations

$$(1 - cx_s')u^*_{0_{\eta\eta}} + (u^*_0 + x_s')u_{0_\eta} = 0 \tag{16}$$

$$(1 - cx_s')u^*_{1_{\eta\eta}} + (u^*_0 + x_s')u^*_{1_\eta} + u^*_{0_\eta}u^*_1 = u^*_{0_\tau} - cu^*_{0_{\eta\tau}} \tag{17}$$

$$(1 - cx_s')u^*_{2_{\eta\eta}} + (u^*_0 + x_s')u^*_{2_\eta} + u^*_{0_\eta}u^*_2 = u^*_{1_\tau} - cu^*_{1_{\eta\tau}} - u^*_1u^*_{1_\eta} . \tag{18}$$

The solution of eqn. (16) is readily seen to be

$$u_0 + x_s' = a \tanh \xi \tag{19}$$

where

$$\xi = \frac{a}{2(1 - cx_s')}(\eta + b) . \tag{20}$$

Here $a(\tau)$ and $b(\tau)$ are the "constants" of integration; they must be determined by matching the inner and outer solutions. In fact it is clear from (19) that $a(\tau)$ is half the amplitude of the shock and from (13) and (20) that $-kb(\tau)$ is Lighthill's displacement of the centre of the shock.

We observe, using (16), that the left side of (17) is equal to

$$(1 - cx_s')\frac{\partial}{\partial\eta}\left\{u^*_{0_\eta}\frac{\partial}{\partial\eta}\left(\frac{u^*_1}{u^*_{0_\eta}}\right)\right\}.$$

Thus after substituting for u^*_0 on the right, equation (17) may readily be integrated twice and the resulting solution for u_1^* takes the form

$$u_1^* = \left(b - \frac{ca'}{a}\right)\tanh^2\xi + \frac{x_s''}{a^2}(1 - cx_s')(\tanh^2\xi - 2\xi\tanh\xi - \xi^2\operatorname{sech}^2\xi)$$

$$- \frac{cx_s''}{a}\left[2(\tanh\xi + \xi\operatorname{sech}^2\xi)\ln\cosh\xi - \tanh\xi + \xi\operatorname{sech}^2\xi - 2\operatorname{sech}^2\xi\, G(\xi)\right]$$

$$+ \frac{2B}{a}(1 - cx_s')\xi - cB\xi^2\operatorname{sech}^2\xi - \frac{acb'}{1 - cx_s'}\xi\operatorname{sech}^2\xi$$

$$+ C(\tau)(\tanh\xi + \xi\operatorname{sech}^2\xi) + D(\tau)\operatorname{sech}^2\xi \qquad (21)$$

where

$$B = B(\tau) = \frac{a'}{a} + \frac{cx_s''}{1 - cx_s'}$$

$$G(\xi) = \int_0^\xi \xi\tanh\xi\, d\xi$$

and $C(\tau)$, $D(\tau)$ are two new "constants" of integration.

We note that when $c = 0$ (the Burger's equation case) equation (21) reduces to the result given in [7]. Also, in comparing with the result (30) of [6] it should be noted that the constants of integration are defined differently in the two cases.

4. Second Order Matching

It is known from the general results in [6] that for the present equation matching at second order may be carried through successfully. However, we shall outline the results here because we shall need many of them in the remainder of the paper.

First of all, the outer limit of the inner solution is simply obtained by allowing $\eta \to \pm\infty$ in (19) and (21). We get

$$u_0^* \sim -x_s' \pm a \tag{22}$$

$$u_1^* \sim [b' - \frac{ca'}{a} + \frac{x_s''}{a^2}(1 - cx_s')] \pm [\frac{cx_s''}{a}(2\ln 2 + 1) + C] + (\frac{\eta + b}{a})(a' \mp x_s''). \tag{23}$$

In each case the errors in these asymptotic formulas are exponentially small.

Next we want the inner limit of the outer solution. At any point $x = x_s(\tau)$ on the shock there are two values of β satisfying eqn.(18), and we denote them by $\beta_\pm$. Thus

$$\beta_\pm = x_s + \tau\phi(\beta_\pm). \tag{24}$$

The situation in the (x,τ) plane is illustrated in Figure 1 (see [2]).

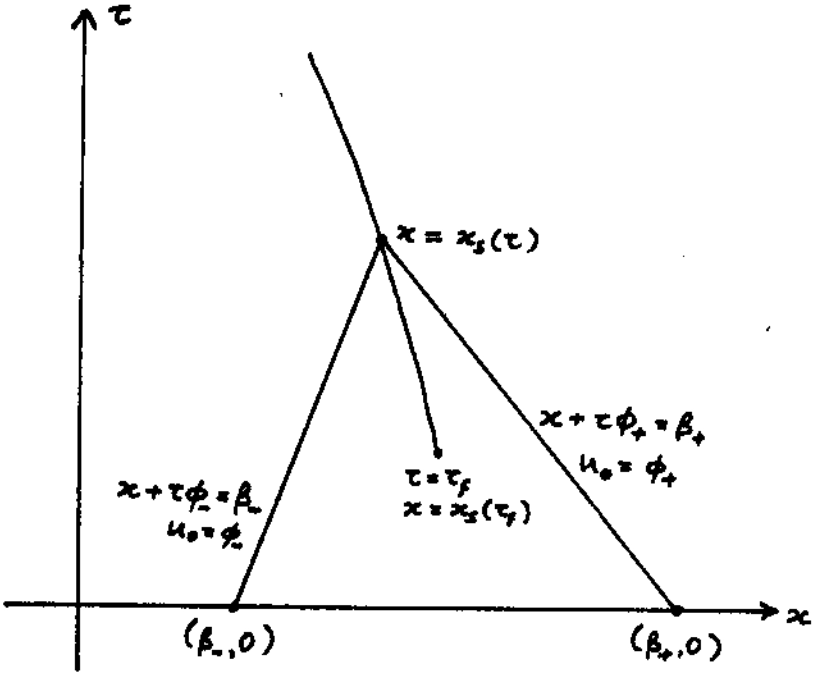

Figure 1

Through each point on the shock path there pass the characteristic lines of the family (8) corresponding to the two values $\beta = \beta_\pm$ of the characteristic variable. They carry signals $u_0 = \phi(\beta_\pm)$ respectively, which give the lowest order outer solution above and below the shock. In the figure we use the notation $\phi_\pm$ for $\phi(\beta_\pm)$, and below we also denote $\phi'(\beta_\pm)$, $\phi''(\beta_\pm)$, etc. by $\phi'_\pm$, $\phi''_\pm$, etc.

For points close to the shock, we write $x = x_s(\tau) + k\eta$ in terms of the inner variable and expand the outer solution in powers of k. From (8) we obtain

$$\beta = \beta_\pm + \frac{k\eta}{D_\pm} + \frac{k^2\tau\phi''_\pm\eta^2}{2D_\pm^3} + \ldots \tag{25}$$

where $D_\pm = 1 - \tau\phi'_\pm$, and the $\pm$ apply accordingly as $\eta > 0$ or $\eta < 0$ respectively. Substituting this into the outer solution given by eqns. (7), (11) and (12) and expanding in powers of k, we obtain the inner expansion of the outer solution to be

$$u^{(o)(i)} = \phi_\pm + k(Y_\pm\eta + Z_\pm) + k^2(U_\pm\eta^2 + V_\pm\eta + W_\pm) + \ldots \tag{26}$$

where

$$Y_\pm = \frac{\phi_\pm'}{D_\pm} \tag{27}$$

$$Z_\pm = \frac{\tau\phi_\pm'}{D_\pm^2}(1 + c\phi_\pm') - c\phi_\pm' \frac{\ln D_\pm}{D_\pm} \tag{28}$$

$$U_\pm = \frac{\phi_\pm''}{2D_\pm^3} \tag{29}$$

$$V_\pm = \left(\frac{\tau\phi_\pm'''}{D_\pm^3} + \frac{2\tau^2\phi_\pm''^2}{D_\pm^4}\right)(1 + c\phi_\pm) + \frac{2c\tau\phi_\pm'\phi_\pm''}{D_\pm^3} - \frac{c\phi_\pm''}{D_\pm^3}\ln D_\pm \tag{30}$$

and $W_\pm$ is given by the right side of (12) with β replaced by $\beta_\pm$. In this section we do not need the $O(k^2)$ terms in (26), but they will be needed later.

Matching the expansion (26) to order k with the inner solution $u_0^* + ku_1^*$ whose outer expansion is given by (22) and (23) we obtain the matching conditions

$$\phi_\pm = -x_s' \pm a \tag{31}$$

$$Y_\pm = (a' \mp x_s'')/a \tag{32}$$

$$Z_\pm = b' - \frac{ca'}{a} + \frac{x_s''}{a^2}(1 - cx_s') + \frac{ba'}{a} \pm \left\{ \frac{cx_s''}{a}(2\ln 2 + 1) + C - \frac{bx_s''}{a} \right\} \tag{33}$$

Equations (31), together with (24), are sufficient to determine the four quantities x_s, a, $\beta_\pm$. They can readily be shown to be equivalent to the equal areas construction [2,9] for the shock path. The conditions (32) can be obtained by differentiating (31) with respect to τ, hence do not provide anything new. The conditions (33) then provide two equations for $b(\tau)$ and $C(\tau)$ which after some manipulation take the forms

$$C = \frac{1}{2}(Z_+ - Z_-) - \frac{cx_s''}{a}(2\ln 2 + 1) + \frac{bx_s''}{a} \tag{34}$$

$$ab = -\frac{1}{2}\{(1 + c\phi_+)\ln D_+ - (1 + c\phi_-)\ln D_-\} + \text{const.} \tag{35}$$

One may argue that at the shock formation point $\tau = \tau_f$, $\beta_+ = \beta_-$ and $a(\tau_f) = 0$. Assuming that $b(\tau)$ remains bounded as $\tau \to \tau_f$ we can conclude that the constant in (35) is zero. (A perhaps more satisfactory basis for this conclusion will be given in the following section.) Expression (35) for $b(\tau)$ then reduces to Lighthill's classical expression [2] in the case $c = 0$.

5. The Integral Conservation Property

In order to complete the determination of the first order inner solution by matching, it is necessary to match at second order. Similarly after the second-order matching, the function $D(\tau)$ occuring in u_1^* remains unknown, and can only be determined by third-order matching, as we shall see in the next section. An alternative method of determining $D(\tau)$ which avoids calculation of the third-order inner solution is to use the integral conservation property first introduced by Murray [1] and extensively exploited by Crighton and Scott [4]. In this section we shall follow this approach.

Integrating equation (1) with respect to x from $-\infty$ to ∞, assuming that u, $u_x \to 0$ as $x \to \pm\infty$, we obtain that

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} u(x,\tau)dx = 0 .$$

Consequently,

$$\int_{-\infty}^{\infty} u(x,\tau)dx = \int_{-\infty}^{\infty} \phi(x)dx. \tag{36}$$

Let us denote the outer expansion given in (2) by $u^{(o)}(x,\tau)$, the inner expansion given in (15) by $u^{(i)}(\eta,\tau)$ and the inner expansion of the outer solution given in (26) by $u^{(o)(i)}(\eta,\tau)$. Let

$$f(\eta,\tau) = u^{(i)}(\eta,\tau) - u^{(o)(i)}(\eta,\tau). \tag{37}$$

Then the composite expansion

$$u = u^{(o)}(x,\tau) + f(\eta,\tau)$$

gives a uniform approximation valid in both regions [4]. Substituting this approximation into the conservation law (36), we obtain

$$\int_{-\infty}^{\infty} u^{(o)}(x,\tau)dx + k\int_{-\infty}^{\infty} f(\eta,\tau)d\eta = \int_{-\infty}^{\infty} \phi(x)dx. \tag{38}$$

Clearly in order to calculate the left side to any order k^n, we need to retain the terms of order k^n in $u^{(o)}$, but only of order k^{n-1} in f. In particular, if we let

$$f(\eta,\tau) = f_0(\eta,\tau) + kf_1(\eta,\tau) + \dots \tag{39}$$

and substitute the expansion (2) for $u^{(o)}$, we obtain at the different orders

$$\int_{-\infty}^{\infty} u_0(x,\tau)dx = \int_{-\infty}^{\infty} \phi(x)dx \tag{40}$$

$$\int_{-\infty}^{\infty} u_n(x,\tau)dx + \int_{-\infty}^{\infty} f_{n-1}(\eta,\tau)d\eta = 0 \qquad (n = 1,2,3,\dots). \tag{41}$$

Substituting from (7) and changing to β as variable of integration via (8), the condition (40) reduces to

$$\int_{\beta_-}^{\beta_+} \phi(\beta)d\beta = \frac{1}{2}\,\tau[\phi_+^2 - \phi_-^2] \tag{42}$$

which is exactly the equal areas property. As remarked before, it is equivalent to the first-order matching conditions (31).

Next, consider the condition (41) for $n = 1$. From (19) and the first term in (26), using (31), we obtain

$$f_0(\eta,\tau) = a[\tanh\xi - \sigma(\eta)] \tag{43}$$

where

$$\sigma(\eta) = \begin{cases} 1 & \text{if} \quad \eta > 0 \\ -1 & \text{if} \quad \eta < 0, \end{cases} \tag{44}$$

Substituting for u_1 from (11) and again changing to β as integration variable, we obtain from (41) that

$$\int_{-\infty}^{\beta_-} + \int_{\beta_+}^{\infty} \left\{\frac{\tau\phi''}{D}(1 + c\phi) - c\phi' \ln D\right]d\beta + a\int_{-\infty}^{\infty} [\tanh\xi - \sigma(\eta)]d\eta = 0. \tag{45}$$

After evaluation of the integrals, this leads to eqn. (35) with the constant on the right equal to zero.

Finally, consider (41) for $n = 2$. For the first term we substitute from (12) for u_2, changing again to β, and obtain after evaluating the integrals that

$$\int_{-\infty}^{\infty} u_2 dx = -F(\beta_+) + F(\beta_-) \tag{46}$$

where

$$F(\beta) = (1 + c\phi)^2\left\{\frac{\tau^2\phi'''}{2D^2} + \frac{5\tau^3\phi''^2}{6D^3}\right\} + c(1 + c\phi)\frac{\tau\phi''}{2D^2}(2 + \tau\phi' - 2\ln D)$$

$$+ \frac{c^2\phi'}{D}\left\{\frac{1}{2}(\ln D)^2 + \tau\phi'\right\}. \tag{47}$$

For f_1, from (21) and (26) and making use of (32) and (33) we obtain

$$f_1(\eta,\tau) = (D + \frac{ca'}{a} - b')\operatorname{sech}^2\xi - \frac{x_s''}{a^2}(1 - cx_s')[\operatorname{sech}^2\xi + 2\xi(\tanh\xi - \sigma(\eta)) + \xi^2\operatorname{sech}^2\xi]$$

$$- \frac{cx_s''}{a}[2(\tanh\xi \ln\cosh\xi - \xi + \sigma(\eta)\ln 2) - (\tanh\xi - \sigma(\eta))]$$

$$- cB\xi^2\operatorname{sech}^2\xi + C[\tanh\xi - \sigma(\eta)] + R(\xi) \tag{48}$$

where the remainder $R(\xi)$ is an odd function of ξ that tends to zero exponentially as $\xi \to \pm\infty$, and hence makes no contribution to the integral in (41). Thus (41) with $n = 2$ leads to the following expression for $D(\tau)$:

$$D = b' + \frac{x_s''(1 - cx_s')}{a^2} - \frac{ca'}{a}(1 - K) +$$

$$+ \frac{1}{4(1 - cx_s')}\{x_s''b - 2bcx_s''(2\ln 2 + 1) + 4K\dot{c}^2x_s'' - 2abC + a(F_+ - F_-)\} \tag{49}$$

where [10]

$$K = \int_0^\infty \xi^2\operatorname{sech}^2\xi d\xi = \pi^2/12$$

and $F_\pm$ denotes $F(\beta_\pm)$.

6. Third Order Matching

To complete the third order matching we must solve equation (18) for u_2^*, then match the solution to the terms of order k^2 in (26). The third-order matching conditions are therefore that

$$u^* \sim U_\pm\eta^2 + V_\pm \dot{+} W_\pm \qquad (\eta \to \infty) \tag{50}$$

where $U_\pm$, $V_\pm$ and $W_\pm$ are given by (29), (30) and the sentence following (30).

Integrating (18) once with respect to η, we obtain

$$(1-cx_s')u^*_{2_\eta} + (u^*_0 + x_s')u^*_2 = \int u^*_{1_\tau}\,d\eta - cu^*_{1_\tau} - \frac{1}{2}u^{*2}_1 + E(\tau) \qquad (51)$$

where $E(\tau)$ is the "constant" of integration and the antiderivative will be made unambiguous below (see eqn. (A1)). To perform the matching, it is not necessary to solve (51) completely for u^*_2: we can simply substitute the asymptotic behaviours of all the terms.

From the second order matching conditions we have that

$$u^*_0 + x_s' \sim \pm a, \qquad u^*_1 \sim Y_\pm\eta + Z_\pm \qquad (52)$$

as $\eta \to \pm\infty$. It follows that

$$u^*_{1_\tau} \sim Y_\pm'\eta + Z_\pm' \qquad (53)$$

$$\int u^*_{1_\tau}\,d\eta \sim \frac{1}{2}Y_\pm'\eta^2 + Z_\pm'\eta + T_\pm \qquad (54)$$

where primes denote τ-derivatives and $T_\pm$ for the moment remains unknown. Substituting the asymptotic forms (52)-(54) and the matching condition (50) into (51) and comparing the coefficients of η^2, η and 1, we obtain

$$\pm aU_\pm = \frac{1}{2}(Y_\pm' - Y_\pm^2) \qquad (55)$$

$$2(1 - cx_s')U_\pm \pm aV_\pm = Z_\pm' - cY_\pm' - Y_\pm Z_\pm \tag{56}$$

$$(1 - cx_s')V_\pm \pm aW_\pm = T_\pm - cZ_\pm' - \frac{1}{2} Z_\pm^2 + E \ . \tag{57}$$

It is easily seen that (55) and (56) can be obtained by differentiating (27) and (28) with respect to τ. Thus the third order matching provides only the two new conditions (57). After substituting from (28) - (30) and making some manipulations, we can cast these conditions in the form

$$T_\pm + E = dF_\pm/d\tau \tag{58}$$

where $F_\pm$ is defined by (47) with $\beta = \beta_\pm$.

Finally, we must compute $T_\pm$ which are defined by (54), $u_1^*(\eta,\tau)$ being given by (21) and (20). The calculation is straightforward but very laborious (see Appendix), and the final answer can be written as

$$T_\pm = \frac{d}{d\tau}\Big\{bZ_\pm - \frac{1}{2} b^2 Y_\pm - 2cb' \ln 2 + \frac{2(1 - cx_s')}{a} [C + \frac{cx_s''}{a} (1 - K)$$

$$\pm (D + bY_\pm - Z_\pm - cKB)]\Big\}. \tag{59}$$

Thus eqns. (58) and (59) provide two equations for E and D. By adding the two equations we obtain an expression for E directly, while the difference between them leads to the result

$$\frac{d}{d\tau}\left\{\frac{2(1-cx_s')}{a}\,[2D + b(Y_+ + Y_-) - (Z_+ + Z_-) - 2cKB] + b(Z_+ - Z_-) - \frac{1}{2}\,b^2(Y_+ - Y_-) - (F_+ - F_-)\right\} = 0$$

which upon integration determines $D(\tau)$ apart from an unknown constant of integration. This constant could presumably be determined by matching with an appropriate solution at the shock formation point. By comparing with (49) we see that the constant of integration is in fact zero, and apart from this the value of D obtained by third order matching is identical to that obtained from the conserved integral approach.

Integrating (51) once more we can obtain the complete solution for u_2^*. The expression is extremely lengthy and we shall not write it down. However, we observe that u_2^* will include a further unknown function, say $E_1(\tau)$, multiplying $\text{sech}^2\xi$. It is perhaps not unreasonable to speculate that this function could be determined by a fourth order matching. The solution u_2^* in the special case $c = 0$ is given explicitly in [7].

Appendix

$u^*_{1_\tau}$ in eqn. (51) is obtained by differentiating (21) with η held constant. We note that $\partial\xi/\partial\tau = B(\tau)\xi + B_1(\tau)$ where

$$B_1(\tau) = ab'/2(1 - cx_s').$$

If a typical term in u^*_1 has the form

$$u_1^{(t)} = f(\tau)g(\xi)$$

then

$$\int u^{(t)}_{1_\tau} d\eta = \frac{2(1 - cx_s')}{a}(B\xi + B_1)u_1^{(t)} + \left[\frac{2(1 - cx_s')}{a} f(\tau)\right]' \int g(\xi)d\xi$$

Using this for each of the terms in (21) we arrive at the result that

$$\int u^*_{1_\tau} d\eta = \frac{2(1 - cx_s')}{a}(B\xi + B_1)u_1^* + \frac{\partial}{\partial\tau}\left\{\frac{2(1 - cx_s')}{a}\left[(b' - \frac{ca'}{a})(\xi - \tanh\xi)\right.\right.$$

$$+ \frac{x_s''(1 - cx_s')}{a^2}(\xi - \tanh\xi - \xi^2\tanh\xi) + C\ \xi\tanh\xi + D\ \tanh\xi$$

$$- \frac{cx_s''}{a}(2\xi\tanh\xi \ln\cosh\xi - 2\tanh\xi\ G(\xi) + \xi\tanh\xi - 2\ln\cosh\xi)$$

$$\left.\left. + \frac{B(1 - cx_s')}{a}\xi^2 - cB(\xi^2\tanh\xi - 2G(\xi)) - \frac{acb'}{1 - cx_s'}(\xi\tanh\xi - \ln\cosh\xi)\right]\right\} \qquad (A.1)$$

where here ξ is held fixed for the τ-differentiation. The asymptotic behaviour (54) is now readily extracted and expression (59) found for $T_\pm$.

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