King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 076

November 1985

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0. INTRODUCTION

In this article we define a tensor product for partially ordered sets acted on by a partially ordered monoid and study the related property of absolute flatness. As a by-product we show that a partially ordered commutative group is a strong amalgamation base in the category of partially ordered commutative monoids. This result originally due to Schreier in the case of groups has been generalized to include the case of monoids by Hall [6] and Howie [7].

1. PRELIMINARIES

Let S be a partially ordered monoid and E a partially ordered set. E is called a LEFT S-POSET if S acts on E in such a way that (i) the action is monotonic in each of the variables and (ii) for $s, t \in S$ and $x \in E$ we have $s(tx) = (st)x$ where sx stands for the result of the action of s on x. Then the order on E is called an S-ORDER. An S-MORPHISM or simply a morphism f from a left S-poset E to another F is a monotonic map which preserves S-action. These are the ordered variants of the notion of S-sets and S-morphisms, where S is a monoid ([1], [2], [8], [9] and [10]). The study of S-posets from the categorical point of view can be found in [4].

A right S-poset is defined similarity. An ordinary poset can be viewed as a (left or right) S-poset with trivial S-action.

Let E be a left S-poset and θ an equivalence on E compatible with S-action. It is called a CONGRUENCE if the quotient set E/θ is endowed with a natural S-order so that the canonical epimorphism is an S-morphism. Given one such equivalence θ, A θ-CHAIN is a finite sequence of elements $\{a, a_1, a_1', \ldots a_i, a_i', \ldots a_n, a_n', b\}$ of E such that $a \leq a_1$, $a_i \theta a_i'$, for $i = 1, \ldots, n$, $a_i' \leq a_{i+1}$ for $i = 1, \ldots, n-1$ and $a_n' \leq b$. It is CLOSED if $a = b$, otherwise OPEN.

The following result characterising a congruence can be established in a similar way to ([3] theorem 6.1, page 42 and theorem 6.4, page 46).

THEOREM 1.1. (a) *Let* E *be a* S-*poset*, θ *an equivalence on* E *compatible with the* S-*action*. *Then* θ *is a congruence iff every closed* θ-*chain is contained in a single equivalence class of* θ. *In this case the induced* S-*order on* E/θ *is given by* $[a] \leq [b]$ *in* E/θ *iff there is a* θ-*chain from* a *to* b *in* E.

(b) *In general, given an equivalence* ρ *on* E *compatible with the* S-*action, define* $a \leq_{\rho} b$ *in* E *iff there is a* ρ-*chain from* a *to* b. *Then* $\leq_{\rho}$ *is a quasi-order on* E *compatible with the* S-*action and the associated equivalence* R_{ρ} *defined by* $a R_{\rho} b$ *iff* $a \leq_{\rho} b \leq_{\rho} a$ *is a congruence, and in fact the congruence generated by* ρ.

2. TENSOR PRODUCT

Let S be a pomonoid, E a right S-poset, F a left S-poset and H a poset. A BALANCED MAP f fromm $E \times F$ to H is a monotonic map from the poset $E \times F$ with product order to H such that $f((xs, y) = f((x, sy))$ for $x \in E$, $y \in F$ and $s \in S$.

A TENSOR PRODUCT OF E AND F OVER S is a pair $(E \otimes_S F, \phi)$ where $E \otimes_S F$ is a poset and $\phi : E \times F \to E \otimes_S F$ given $\phi((x,y)) = x \otimes_S y$ is a balanced map such that for every other pair (G, ψ) with similar properties there exists a unique monotonic map $\bar{\psi}:E \otimes_S F \to G$ such that $\bar{\psi} \circ \phi = \psi$.

THEOREM 2.1. *With the data as above, a tensor product exists and is unique up to isomorphism.*

Proof. Let ρ be the equivalence relation on $E \times F$ generated by the binary relation obtained by identifying the pairs (xs, y) and (x, sy) for $x \in E$, $y \in F$ and $s \in F$. Let θ be the congruence generated by ρ, and consider the natural surjection $\Pi:E \times F \to E \times F/\theta$. Then Π is a balanced map and the pair $(E \times F/\theta, \Pi)$ satisfies all the conditions for a tensor product.

REMARKS.(i) The tensor product defined above is similar to the tensor product of S-sets ([1], [2], [8], [9] and [10].) Denoting as usual $\Pi(x,y))$ by $x \otimes y$ and using 1.1(b) it is possible but cumbersome to express the partial order in $E \times F$ in terms of ρ and component orders of E and F.

(ii) If E is a bi-S-poset then $E \otimes F$ acquires a natural left S-poset structure if we define $s(x \otimes y) = (sx) \otimes y$. A similar remark applies to F, and in particular $E \otimes_S S$ is a right S-poset naturally isomorphic to E, and $S \otimes_S F$ is a left S-poset naturally isomorphic to F.

(iii) Let $f:E \to F$ be a morphism of right S-posets and H a left S-poset. Then by the property of $\otimes$ there exists a unique monotonic map from $E \otimes H$ to $F \otimes H$ which is given by mapping $x \otimes y$ to $f(x) \otimes y$. We shall denote this map by $f \otimes H$. Then $\otimes H$ is a functor from the category of right S-posets to the category of posets. If H is a bi-S-poset then $f \otimes H$ is a right S-morphism and if E and F are bi-S-posets then it is a left S-morphism. Similar remarks apply for $G \otimes$ where G is a right S-poset.

3. ABSOLUTE FLATNESS

A left S-poset H is called LEFT FLAT if $\otimes H$ preserves all right S-monomorphisms.

Let $f:E \to F$ be a right S-monomorphism and θ_F (respectively θ_E) be the congruence over $F \times H$ (respectively over $E \times H$) defining the tensor product. Then H is left flat iff $\theta_F|_{f(E)\times H} = \theta_E$ (after obvious identification) for all monomorphisms $f:E \to F$.

The pomonoid S is LEFT ABSOLUTELY FLAT if all the left S-posets are flat. Right absolute flatness is defined similarly. S is absolutely flat if it is both right and left absolutely flat.

Our goal is to show that a pogroup is absolutely flat. As a prelude we analyze the congruence θ associated with the tensor product.

Let G be a pogroup, F a left and E right G-poset. Let ρ be the binary relation on $E \times F$ defined by $(a,b)\,\rho(c,d)$ if there exists a g in G such that $c = ag$ and $gd = b$. Then it is easy to see that ρ is an equivalence. Hence it follows by 1.1(b) that the congruence θ on $E \times F$ defining $E \otimes F$ is given by $(a,b)\theta(c,d)$ iff $(a,b) \underset{\rho}{\leq} (c,d) \underset{\rho}{\leq} (a,b)$.

Now we can prove

<u>THEOREM</u> 3.1. <u>A pogroup</u> G <u>is absolutely flat</u>.

<u>Proof</u>. After obvious identification, let E be a right G-subposet of F and H a left G-poset. We need to show $\theta_F|_{E\times H} = \theta_E$.

Suppose ρ and σ represent the corresponding quasi-orders on $F \times H$ and $E \times H$ respectively. Then it is enough to show that for (a,b) and (c,d) in $E \times H$, $(a,b) \underset{\rho}{\leq} (c,d)$ implies $(a,b) \underset{\sigma}{\leq} (c,d)$.

Let $(a,b) \leq (a_1,b_1)\ \rho(a_1',b_1') \leq (a_2,b_2)\ \rho(a_2',b_2') \ldots$ $\rho(a'_{n-1}, b'_{n-1}) \leq (a_n,b_n)\ \rho(a_n',b_n') \leq (c,d)$ be a ρ-chain from (a,b) to (c,d) where the intermediate elements belong to $F \times H$. Let $g_1, \ldots, g_n$ be elements of G carrying out this linking for ρ.

Indicating the fact that $a_i g_i = a_i'$ schematically by $a_i \xrightarrow{g_i} a_i'$ the relations in the first components of the chain can be represented by the scheme: $a \leq a_1 \xrightarrow{g_1} a_1' \leq a_2 \ldots \leq a_i \xrightarrow{g_i} a_i' \leq \ldots a_n \xrightarrow{g_n} a_n \leq c$.

- typing width (to be reduced to 4/5)

Consider the associated sequence of elements $\{a, a_1, a_1', \dots a_i, a_i', \dots a_n, a_n', c\}$ where a and c belong to E. Let i be the smallest index such that neither a_{i+1} nor a_{i+1}' belong to E. Then a_i, a_i' both belong to E, and $a_i' \leq a_{i+1} \xrightarrow{g_{i+1}} a_{i+1}' \leq a_{i+2}$. Let $\bar{a}_{i+1} \in E$ be such that $a_i' \leq \bar{a}_{i+1} \leq a_{i+1}$ and correspondingly put $\bar{a}_{i+1}' = \bar{a}_{i+1}\, g_{i+1}$. Replacing a_{i+1} by $\bar{a}_{i+1}$ and a_{i+1}' by $\bar{a}_{i+1}'$ we get a new ρ-chain where the smallest index i such that the pair of first components a_{i+1}, a_{i+1}' do not belong to E is increased. Thus, clearly after repeating the above process a finite number of steps, we obtain a ρ-chain where all intermediate first components belong to E. Thus, we have obtained a ρ-chain from (a,b) to (c,d) and so $(a,b) \underset{\sigma}{\leq} (c,d)$ and H is left flat, showing G is left absolutely flat. The proof for the other side is similar.

4. AMALGAMATION

Let S be a pomonoid. AN AMALGAM OVER S is a system $\mathcal{S} \equiv \{S_i : \psi_i : S \to S_i \quad i=1,2\}$ where S_i is a pomonoid and ψ_i is a monoid monomorphism $\mathcal{S}$ is WEAKLY EMBEDDABLE if there exists a pomonoid T and monoid monomorphisms $\phi_i : S_i \to T$ such that the following diagram is commutative. It is EMBEDDABLE if the diagram is cocartesian. If it is also cartesian, then $\mathcal{S}$ is STRONGLY EMBEDDABLE.

$$\begin{tikzcd} S \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & S_1 \arrow[d, "\phi_1"] \\ S_2 \arrow[r, "\phi_2"'] & T \end{tikzcd}$$

$\mathcal{S}$ is called an (respectively WEAK, STRONG) AMALGAMATION BASE if every amalgam over S is (respectively WEAKLY, STRONGLY) embeddable.

Hereafter we shall consider only commutative pomonoids. Let S be a commutative pomonoid with two over-pomonoids S_1 and S_2. Then $S_1 \otimes S_2$ is naturally a bi-S-poset. As in the case of commutative algebras over a commutative ring, one can equip $S_1 \otimes S_2$ with a natural multiplication by defining $(a \otimes b)\cdot(c \otimes d) = ac \otimes bd$ and

$S_1 \otimes S_2$ is indeed a pomonoid.

Now we can state and prove

THEOREM 4.1. *An absolutely flat commutative pomonoid* S *is an amalgamation base in the category of commutative pomonoids.*

Proof. Let $\{S:\psi_i:S \to S_i \quad i=1,2\}$ be an amalgam over S. Consider the diagram above with T replaced by $S_1 \otimes S_2 \cong S_2 \otimes S_1$ and we identify $S \otimes S_1$ with S_1 and $S_2 \otimes S$ with S_2 and let $\phi_1 = S_1 \otimes \psi_2$, $\phi_2 = \psi_1 \otimes S_2$. The absolute flatness of S implies that ϕ_1 and ϕ_2 are monoid monomorphisms.

Let $f_i:S_i \to M$ be monoid morphisms into M such that $f_i \circ \psi_i = f_j \circ \psi_j$. Then M is a S-poset (via $f_i \circ \psi_i = f_j \circ \psi_j$) and the map $f:S_1 \otimes S_2 \to M$ defined by $f(s_1,s_2) = f_1(s_1)f_2(s_2)$ is monotonic and balanced. Then there exists a monotonic map $g:S_1 \otimes S_2 \to M$ such that $g(x \otimes y) = f_1(x)f_2(y)$. But clearly g is a pomonoid morphism.

As for the strong amalgamation property, we have

THEOREM 4.2. *A commutative pogroup* G *is a strong amalgamation base in the category of commutative pomonoids.*

Proof. In view of 3.1 and the theorem above it is enough to show that the following diagram is cartesian:

$$\begin{tikzcd}
G \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & S_1 \arrow[d, "\phi_1"] \\
S_2 \arrow[r, "\phi_2"'] & S_1 \times S_2
\end{tikzcd}$$

namely $\phi_1 \circ \psi_1(G) = \phi_2 \circ \psi_2(G) = \phi_1(S_1) \cap \phi_2(S_2)$. One inclusion is obvious. Now suppose $(\alpha \otimes 1) = (1 \otimes \beta)$ for $\alpha \in S_1$ and $\beta \in S_2$. Let us suppose $G \subseteq S_i$ $i=1,2$. Now we have $(\alpha,1) \underset{\rho}{\leq} (1,\beta) \underset{\rho}{\leq} (\alpha,1)$ in $S_1 \times S_2$. We want to show $\alpha, \beta \in G$.

Suppose $(a,b), (c,d) \in S_1 \times S_2$ such that $c = ag$ and $b = gd$.

We shall indicate this by $(a,b) \xrightarrow{g} (c,d)$. Then a α-chain from $(\alpha,1)$ to $(1,\beta)$ can be represented by

$$(\alpha,1) \le (s_1,t_1) \xrightarrow{g_1} (s_1',t_1') \le (s_2,t_2) \xrightarrow{g_2} (s_2',t_2') \longrightarrow \;\ldots$$
$$(s_m,t_m) \xrightarrow{g_m} (s_m',t_m') \le (1,\beta) \;\ldots \tag{1}$$

and a α-chain from $(1,\beta)$ to $(\alpha,1)$ is represented by

$$(1,\beta) \le (u_1,v_1) \xrightarrow{h_1} (u_1',v_1') \le (u_2,v_2) \xrightarrow{h_2} (u_2,v_2) \;\ldots$$
$$\ldots\; (u_m\; v_m) \xrightarrow{h_n} (u_n',v_n') \le (\alpha,1) \tag{2}$$

Let $g_{m,k} = g_m\, g_{m-1} \cdots g_{m-k}$ for $m > k \ge 0$ and

$h_{n,\ell} = h_n\, h_{n-1} \cdots h_{n-\ell}$ for $n > \ell \ge 0$

Without loss of generality we may suppose that $(s_i,t_i) \xrightarrow{g_i} (s_i',t_i')$ means $s_i' = s_i\, g_i$ and $g_i^{-1} t_i = t_i'$ and a similar remark applies to a h-link.

Then we have the following chain of inequalities from (1) & (2)

$$g_{m,m-1}\alpha \le g_{m,m-2}s_1 \le \cdots g_{m,m-i}s_i \le \cdots g_{m,0}s_m \le 1 \tag{3}$$

$$g^{-1}_{m,m-1} \quad g^{-1}_{m,m-2}t_1 \le \;\cdots\; g^{-1}_{m,m-i}t_i \quad \cdots g^{-1}_{m,0}t_m \le \beta \tag{4}$$

$$h_{n,n-1} \le h_{n,n-2}u_1 \le \;\cdots\; h_{n,n-j}u_j \le \cdots h_{n,0}u_n \le \alpha \tag{5}$$

$$h^{-1}_{n,n-1}\beta \le h^{-1}_{n,n-2}v_1 \le \cdots h^{-1}_{n,n-j}v_j \le \cdots h^{-1}_{n,0}v_n \le 1 \tag{6}$$

The relations (3) and (5) belong to S_1 whereas the relations (4) and (6) belong to S_2.

By (3) and (5) we have $g_{m,m-1}h_{n,n-1} \le g_{m,m-1}\alpha \le 1$ and (4) and (6) give $g^{-1}_{m,m-1}\, h^{-1}_{n,n-1} \le h^{-1}_{n,n-1}\beta \le 1$. But then $g_{m,m-1}h_{n,n-1} = g_{m,m-1}h^{-1}_{n,n-1} = 1$ and hence $\alpha = g^{-1}_{m,m-1}$ and $\beta = h_{n,n-1}$ both belong to T.

5. CONCLUSION

It would be interesting to see if the above results also remain valid for partially ordered non-commutative groups. One possible tool could be the notion of extension properties introduced by Hall, suitably formulated in the ordered context.

It is possible to define amalgams for a family $\{S:\psi_i:S \to S_i\}_{i\in I}$ indexed by an arbitrary set I. The main results remain valid for this case, by exploiting the tensor product of many factors.

Following Kil'p [8] one can show that a pomonoid which is a union of pogroups is absolutely flat and hence a weak amalgamation base.

The ideas of this work developed during a Meditation Walk in Audubon Park with I. Fleischer and the author gratefully acknowledges the initial impulse given by him to this work.

The author also thanks Prof. J.M. Howie and the referee for their incisive comments which helped to improve the content of the paper.

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