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Quasi Ordered Fields

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Abstract

We show that a quasi-ordered field (see below for definition) is either an ordered field or a Krull valued field.

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1. INTRODUCTION. The fact that order and valuation over a field are related goes back to Baer and Krull ([B] and [Kr]). Since then many structures have been introduced, essentially as an attempt to find a common generalization of valuations and orderings: Harrison primes [H], pseudo-order [M] and type V-topologies [Ka] to name a few. (See also [Be] and [D] for a different approach).

In this note we give a set of axioms on a field with a given binary relation closely related to the axioms for an ordered field; a field with such a relation is called "quasi-ordered". The main theorem says that quasi-ordered fields are either ordered fields or Krull valued fields.

It turns out that a quasi-order on a field defines a type V-topology on it.

A good treatment of both ordered fields and Krull valued fields may be found either in [J] or in [R].

2. AXIOMS. Let K be a field and $\leq$ be a total, reflexive and transitive (not necessarily antisymmetric) binary relation on K and $\sim$ be the associated equivalence relation: $a \sim b$ iff $a \leq b \leq a$. We shall write $a \not\sim b$ if $a \sim b$ is not true. $(K,\leq)$ is called QUASI-ORDERED if the following hold:

(Q1) If $x \sim 0$ for any $x \in K$, then $x = 0$.

(Q2) If $0 \le c$ and $a \le b$ then $ac \le bc$.

(Q3) if $a \le b$ and $b \not\sim c$ then $a + c \le b + c$.

Clearly an ordered field is a field with an antisymmetric quasi-order. Also, a Krull valued field K has a quasi-order induced by the valuation: with the multiplicative form of valuation, define $a \le b$ for a, b in K if the value of a is less than or equal to that of b.

These are the only possible types of quasi-orders on a field, as the following theorem shows.

THEOREM. A quasi-ordered field is either an ordered field or else a Krull valued field, such that the given quasi-order is identical to the one induced by the valuation.

The proof will be broken up into a series of lemmas.

3. THE EQUIVALENCE $\sim$.

Let K be a quasi-ordered field and $\sim$ be the associated equivalence relation: $a \sim b$ if $a \le b \le a$ for a, b in K. We define $E_x = \{y \in K: y \sim x\}$ for an element $x \in K$.

We remark that for a quasi-order induced by a valuation we have $x \sim -x$ for every x in K. Now we have

3.1. LEMMA. $x \sim -x$ iff $x \geq 0$ and $-x \geq 0$.

PROOF. We discard the trivial case $x = 0$. Suppose $x \sim -x$. Then E_x is a convex set containing x and $-x$ and not containing zero. Hence x and $-x$ cannot be on opposite sides of 0. But if $x \leq 0$ and $-x \leq 0$ then again by (Q1) and (Q2) we get $x = 0$. Thus both $x \geq 0$ and $-x \geq 0$.

Conversely, if $x \geq 0$ and $-x \geq 0$ and $x \not\sim -x$, (Q3) gives again $x = 0$. Thus $x \sim -x$ must hold.

Notice that if $E_x \neq \{x\}$ then $E_x = -E_x = E_{-x}$. Indeed if y is an element different from x in E_x then $-y$ also belongs to E_x: if not $y \leq x \not\sim -y$ implies that $0 \leq x - y$ and $x \leq y \not\sim -y$ gives $x - y \leq 0$ so that $(x - y) \sim 0$ then $x = y$ a contradiction. Consequently, if $E_x \neq \{x\}$ then $x \sim y$ implies $-x \sim -y \sim x$.

Moreover, $\sim$ is preserved by multiplication. Indeed if $x \sim y$ and $a \geq 0$ there is nothing to prove. But if $a \leq 0$ then $0 \leq -a$ and $-x \sim -y$ gives $ax \sim ay$.

We deduce

3.2. PROPOSITION. For an element $x \in K^*$, in order that $E_x \neq \{x\}$ it is necessary and sufficient that $E_1 \neq \{1\}$.

Indeed for any $x \in K^*$, $E_x = x \cdot E_1$.

Before proving the next theorem, we observe that the prime field of characteristic two has a unique quasi-order, $0 \leq 1$, induced by the trivial valuation and for this quasi-order $E_1 = \{1\}$. Aside from this all other quasi-ordered fields with $E_1 = \{1\}$ are ordered fields, as the following theorem shows.

3.3. THEOREM. If K is a quasi-ordered field such that $E_1 = \{1\}$, then it is either an ordered field or the prime field of characteristic two.

PROOF. By hypothesis, the equivalence relation reduces to equality, so that K is a totally ordered set in the usual sense. We consider two cases:

Case 1: Even characteristic. Then K is the prime field of characteristic two (or $K = \{0\}$). Indeed, first of all $b \geq 0$ for any $b \in K$ (if $b < 0$, then $0 = b + b < 0 + b = b$). And if b is different from 0 and 1, then $b \neq b + 1 \geq 0$, so that $1 = b + b + 1 \geq b + 0 = b$; similarly, $b \geq 1$, contradicting the assumption that E_1 has only one element.

Case 2: Characteristic not equal to two. Now we need only to check the monotonocity of addition: for any $a, b, c \in K$, $a \leq b$ implies $a + c \leq b + c$. If $c \neq b$ there is nothing to prove by (Q3). Suppose $c = b \neq 0$. Then $a - b \leq 0$ gives $a - b + 2b \leq 2b = b + c$. Thus $a + c \leq b + c$.

4. PROPER QUASI-ORDER. The result above shows that a quasi-ordered field is nothing but an ordered field in the extremal case when $\sim$ reduces to equality. Hereafter, we consider a proper quasi-ordered field K in which $\sim$ is not the equality relation, hence $x \sim -x$ for any $x \in K$ and nonzero elements are positive.

4.1. LEMMA. For any $a, b \in K^*$, $a + b \leq \max\{a, b\}$.

PROOF. Suppose that $a \leq b$ we want $a + b \leq b$. If not $-a \sim a \leq b < a+b$. Since $-b \not\sim a+b$, we get $a+b \sim -a-b \leq a$. Hence $a+b \leq b$ in any case.

To complete the proof of the main theorem, we note that if E_1 is not equal to $\{1\}$ then it is a convex subgroup of K^* and K^*/E_1 is an ordered group. The group homomorphism $U:K^* \to K^*/E_1$ is part of a Krull valuation whose induced quasi-order is the given one.

REMARK. We can define a QUASI-INTERVAL around a of length b by

$$]-b + a, a + b[= \{x \in K : \exists y \in K: 0 \leq y < b: -y \leq x - a \leq y\}.$$

Then these quasi-intervals form a subbase for a field topology, which is a V-topology in the sense of [Ka].

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