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Mixed Boundary Value Problems for some Iterated P.D.E's for a Rectangle *

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ABSTRACT: It is well known that the Dirichlet problem for hyperbolic equations is a classical "not well posed" problem. Here we obtain uniqueness of solution for some mixed boundary value problems for some iterated partial differential equations for a rectangle that reduce to the Dirichlet problem for hyperbolic equations.

INTRODUCTION: Hadamard [1, 2] rejected the Dirichlet problem as unsuitable for hyperbolic equations. Bourgin and Duffin [3] and Fox and Pucci [4] treated the Dirichlet and Neumann problems for the wave equation $u_{xx} - u_{yy} = 0$ for a rectangle in standard position. John [5] treated the Dirichlet problem for the equation $u_{xy} = 0$ for more general shapes. Abdul-Latif and Diaz [6] used the equation $u_{xx} - u_{yy} = 0$ and treated the Dirichlet, Neumann and many mixed Dirichlet-Neumann boundary value problems including a "general mixed problem" for a rectangle in standard position. Abdul-Latif [7] treated the Dirichlet, Neumann and mixed Dirichlet-Neumann boundary value problems for the equation $u_{xy} = 0$ for moving rectangles.

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NECESSARY PRELIMINARIES: It is known classically that if $u(x, y) = 0$ on two adjacent sides of a rectangle in standard position (i.e. sides parallel to x-axis and y-axis) and $u_{xy} = 0$ everywhere in the rectangle, then $u(x, y)$ is identically zero. It is also known that if $u(x, y) = 0$ on the boundary of the rectangle and $u_{xx} + u_{yy} = 0$ everywhere in the rectangle then $u(x, y)$ is identically zero. These two classical results are needed in this paper.

In this paper, we mean by a rectangle in standard position to be a rectangle OIRS with O being the origin and OS on the positive x-axis Ox and OI on the positive y-axis Oy. It should be clear that all results mentioned in this paper for this standard positioned rectangle will also be true for any rectangle with sides parallel to x-axis and y-axis and of the same restricted dimensions.

We also have the inclined rectangle OIRS with O as the origin, the adjacent sides OS and OI make respective angles θ and $\pi/2 + \theta$ with the positive x-axis Ox, where $0 < \theta < \pi/4$ and the sides OS and OI have respective lengths of a and b. This is really a class of rectangles as θ moves from zero to $\pi/4$. Repeated reflection of this class of rectangles with respect to the lines $y = x$, x-axis, $y = -x$, y-axis, $y = x$, x-axis, and $y = -x$ in the given order give us seven other positions of the class of rectangles. Any result mentioned here for the inclined rectangle will also be true for the seven other positions and for any other rectangle with sides parallel to any of these eight positions and having the same restricted dimensions.

A theorem from Abdul-Latif and Diaz [6] which is of basic need here is:

Theorem A: Let $u(x, y) \in C^2$ in the standard positioned rectangle OIRS whose sides are of lengths a and b, $u(x, y) = 0$ on the boundary, and $u_{xx} - u_{yy} = 0$ everywhere. If a/b is irrational, then $u(x, y)$ is identically zero; if a/b is rational, then $u(x, y)$ need not be identically zero.

A theorem from Abdul-Latif [7] which is also of basic need here is:

Theorem B: Let the rectangle OIRS have adjacent sides OS and OI of respective lengths a and b with $a \geq b$, let $0 \leq \theta \leq \pi/4$ with $\tan\theta \leq b/a$, and let OS and OI make respective angles θ and $\pi/2 + \theta$ with the positive x-axis Ox. In this rectangle, let $u(x, y) \in C^2$, $u(x, y) = 0$ on the boundary and $u_{xy} = 0$ everywhere. If $\theta \neq \pi/4$ then $u(x, y)$ is identically zero; if $\theta = \pi/4$, then $u(x, y)$ need not be identically zero.

Now, let $\theta = (\frac{1}{2}) \cot^{-1}(-k/2)$ where k is any real number and let

$$x = \cos\theta\ x' + \sin\theta\ y' , \quad y = \sin\theta\ x' - \cos\theta\ y'$$

then the inclined rectangle OIRS which is inclined at an angle $\theta = (\frac{1}{2}) \cot^{-1}(-k/2)$ rotates into a standard positioned rectangle and $u_{xy} = 0$ transforms into the equations $u_{x'x'} + ku_{x'y'} - u_{y'y'} = 0$ for all real numbers k. Also, the standard positioned rectangle OIRS rotates into an inclined rectangle which is inclined at an angle $\theta = (\frac{1}{2}) \cot^{-1}(-k/2)$ and the equations $u_{xx} + ku_{xy} - u_{yy} = 0$, for all real numbers k, will transform into the single equation $u_{x'y'} = 0$.

Consequently, we have two equivalent alternatives:

(I) Use a fixed equation $u_{xy} = 0$ and a moving rectangle for all θ, $0 < \theta < \pi/2$.

(II) Use $u_{xx} + ku_{xy} - u_{yy} = 0$ for all real numbers k and a rectangle in standard position.

In Theorem A, we have $k = 0$ in alternative (II) which corresponds to $\theta = \pi/4$ in alternative (I).

In this paper, Theorem $\bar{\bar{B}}$ is copied above from Abdul-Latif [7] and it is written according to alternative (I); however, we will use it in this paper according to alternative (II) and hence we rewrite it here in its equivalent form of alternative (II):

Theorem B: Let the rectangle OIRS be a rectangle in standard position, where the sides along the x-axis and y-axis are of respective lengths a and b with $a \geq b$. In this rectangle, let $u(x, y) \in C^2$, $u(x, y) = 0$ on the boundary, $u_{xx} + ku_{xy} - u_{yy} = 0$ everywhere, and $\theta = (\frac{1}{2}) \cot^{-1} (-k/2)$ with $\tan\theta \leq b/a$. If $k \neq 0$, then $u(x, y)$ is identically zero; if $k = 0$, then $u(x, y)$ need not be identically zero.

UNIQUENESS OF SOLUTION FOR SOME ITERATED EQUATIONS:

Theorem (I): Let OIRS be a standard positioned rectangle, where the sides are of lengths a and b with a/b being irrational. And,

1) Let $u(x, y) \in C^4$, $u_{xxxx} - u_{yyyy} = 0$, $u(x, y) = u_{nn}(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

2) Let $u(x, y) \varepsilon C^4$, $u_{xxxx} - 2u_{xxyy} + u_{yyyy} = 0$ everywhere, $u(x, y) = u_{nn}(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

3) Let $u(x, y) \varepsilon C^4$, $u_{xxxy} - u_{xyyy} = 0$ everywhere, $u(x, y) = 0$ on two adjacent sides, $u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

4) Let $u(x, y) \varepsilon C^4$, $u_{xxxx} - u_{xxyy} = 0$ everywhere, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on one vertical side, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

5) Let $u(x, y) \varepsilon C^4$, $u_{xxyy} - u_{yyyy} = 0$ everywhere, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on one horizontal side, $u_{nn}(x, y) = 0$ on the two hirizontal sides. Then $u(x, y)$ is identically zero.

6) Let $u(x, y) \varepsilon C^4$, $u_{xxxx} + u_{xxxy} - u_{xxyy} - u_{xyyy} = 0$ everywhere, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

7) Let $u(x, y) \varepsilon C^4$, $u_{xxxx} - u_{xxxy} - u_{xxyy} + u_{xyyy} = 0$ everywhere, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

8) Let $u(x,y) \in C^4$, $u_{xxxy} + u_{xxyy} - u_{xyyy} - u_{yyyy} = 0$ everywhere, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

9) Let $u(x,y) \in C^4$, $u_{xxxy} - u_{xxyy} - u_{xyyy} + u_{yyyy} = 0$ everywhere, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

10) Let $u(x, y) \in C^3$, $u_{xxx} - u_{xyy} = 0$ everywhere, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

11) Let $u(x, y) \in C^3$, $u_{xxy} - u_{yyy} = 0$ everywhere, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

12) Let $u(x, y) \in C^3$, $u_{xxx} + u_{xxy} - u_{xyy} - u_{yyy} = 0$ everywhere, $u(x, y) = u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

13) Let $u(x, y) \in C^3$, $u_{xxx} - u_{xxy} - u_{xyy} + u_{yyy} = 0$ everywhere, $u(x, y) = u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

Proof: Similarities make it enough to sketch the proofs of (1), (3), (4), (6), (10) and (12). In (1), we have $u(x, y) \in C^4$ and hence $u_{xxxx} - u_{yyyy} = 0$ may be written as $u_{xxxx} + u_{xxyy} - u_{xxyy} - u_{yyyy} = 0$ which in turn may be written as

$$(u_{xx} + u_{yy})_{xx} - (u_{xx} + u_{yy})_{yy} = 0.$$

Let $v(x, y) = u_{xx} + u_{yy}$. Since $u(x, y) = 0$ on the horizontal sides, then applying the definition of the derivative we get $u_x(x, y) = u_{xx}(x, y) = 0$ on the horizontal sides. Also $u(x, y) = 0$ on the vertical sides gives $u_{yy}(x, y) = 0$ on the vertical sides. Now, $u_{nn}(x, y) = 0$ on the boundary means $u_{xx} = 0$ on the vertical sides and $u_{yy} = 0$ on the horizontal sides. Hence we have $v(x, y) = u_{xx} + u_{yy} = 0$ on the boundary. So that we have $v_{xx} - v_{yy} = 0$ everywhere, $v(x, y) = 0$ on the boundary and a/b is irrational, then by Theorem (A), we get $v(x, y)$ is identically zero. We now have $u_{xx} + u_{yy} = 0$ everywhere, $u(x, y) = 0$ on the boundary, then by the classical result, we get $u(x, y)$ is identically zero.

In (3): $u_{xxxy} - u_{xyyy} = 0$ may be written as $(u_{xy})_{xx} - (u_{xy})_{yy} = 0 \cdot u_n(x,y) = 0$ on the boundary and the definition of the derivative give $v(x, y) = u_{xy} = 0$ on the boundary. So that $v_{xx} - v_{yy} = 0$, $v = 0$ everywhere, a/b irrational give, by Theorem (A), $u_{xy} = 0$ everywhere. Now $u_{xy} = 0$ everywhere, $u = 0$ on two adjacent sides give, by a classical result, $u(x, y) = 0$ everywhere.

In (4): $u_{xxxx} - u_{xxyy} = (u_{xx})_{xx} - (u_{xx})_{yy} = 0$. Here, $u = 0$ on horizontal sides and $u_{nn} = 0$ on vertical sides give $u_{xx} = 0$ on the boundary and then Theorem (A) applies to give us $u_{xx} = 0$ everywhere. Now $u_{xx} = 0$ give us u_x = constant on every horizontal line, but $u_n = u_x = 0$ on one vertical side, and hence $u_x = 0$ on every horizontal line and this means $u_x = 0$ everywhere. Now, $u_x = 0$ everywhere give $u_{xy} = 0$ everywhere. We now have $u_{xy} = 0$ everywhere, $u = 0$ on two adjacent sides and this, by a classical result, give $u(x, y) = 0$ everywhere.

In (6): $u_{xxxx} + u_{xxxy} - u_{xxyy} - u_{xyyy} = (u_{xx} + u_{xy})_{xx} - (u_{xx} + u_{xy})_{yy} = 0$. Here, the boundary conditions and Theorem (A) give us $u_{xx} + u_{xy} = 0$ everywhere.

Now $u_{xx} + u_{xy} = \frac{\partial}{x}[(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})u] = 0$ give us u_x = constant on every line having direction numbers [1, 1], but the boundary conditions give us $u_x = 0$ on the boundary and hence $u_x = 0$ everywhere, which in turn give us $u_{xy} = 0$ everywhere, and since $u = 0$ on two adjacent sides, hence $u(x, y) = 0$ everywhere.

In (10): $u_{xxx} - u_{xyy} = (u_x)_{xx} - (u_x)_{yy} = 0$. The boundary conditions and Theorem (A) give $u_x = 0$ everywhere which in turn give $u_{xy} = 0$ everywhere, but $u = 0$ on two adjacent sides and hence $u(x, y) = 0$ everywhere.

In (12): $u_{xxx} + u_{xxy} - u_{xyy} - u_{yyy} = (u_x + u_y)_{xx} - (u_x + u_y)_{yy} = 0$. The boundary conditions and Theorem (A) give $u_x + u_y = 0$ everywhere and this means that u = constant on every line having direction numbers [1, 1], but $u = 0$ on the boundary and hence $u(x, y) = 0$ everywhere.

Theorem (II): Let OIRS be a rectangle in standard position, where the sides along the x-axis and y-axis are of respective lengths a and b with $a \geq b$. And,

(1) Let $u(x,y) \in C^4$, $u_{xxxx} + (k_1+k_2)u_{xxxy} + (k_1k_2-2)u_{xxyy} - (k_1+k_2)u_{xyyy} + u_{yyyy} = 0$ everywhere, $k_1 \neq 0$, $k_2 \neq 0$, $\theta_1 = \frac{1}{2}\cot^{-1}(-k_1/2)$ with $\tan\theta_1 \leq b/a$, $\theta_2 = \frac{1}{2}\cot^{-1}(-k_2/2)$ with $\tan\theta_2 \leq b/a$, $u(x, y) = u_n(x, y) = u_{nn}(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

(2) Let $u(x, y) \in C^4$, $u_{xxxx} + ku_{xxxy} + ku_{xyyy} - u_{yyyy} = 0$ everhwhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = u_{nn}(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

(3) Let $u(x, y) \in C^4$, $u_{xxxx} + ku_{xxxy} - 2u_{xxyy} - ku_{xyyy} + u_{yyyy} = 0$ everywhere $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, b/a is irrational $u(x, y) = u_{nn}(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

(4) Let $u(x, y) \in C^4$, $u_{xxxy} + ku_{xxyy} - u_{xyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on two adjacent sides, $u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

(5) Let $u(x, y) \in C^4$, $u_{xxxx} + ku_{xxxy} - u_{xxyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on one vertical side, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

(6) Let $u(x, y) \in C^4$, $u_{xxyy} + ku_{xyyy} - u_{yyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on one horizontal side, $u_{nn}(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

(7) Let $u(x, y) \in C^4$, $u_{xxxx} + (k + 1)\, u_{xxxy} + (k - 1)\, u_{xxyy} - u_{xyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

(8) Let $u(x, y) \varepsilon C^4$, $u_{xxxx} + (k - 1) u_{xxxy} - (k + 1) u_{xxyy} + u_{xyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2} \cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

(9) Let $u(x, y) \varepsilon C^4$, $u_{xxxy} + (k + 1) u_{xxyy} + (k - 1) u_{xyyy} - u_{yyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2} \cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

(10) Let $u(x, y) \varepsilon C^4$, $u_{xxxy} + (k - 1) u_{xxyy} - (k + 1) u_{xyyy} + u_{yyyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2} \cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the boundary, $u_{nn}(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

(11) Let $u(x, y) \varepsilon C^3$, $u_{xxx} + ku_{xxy} - u_{xyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2} \cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two horizontal sides and one vertical side, $u_n(x, y) = 0$ on the two vertical sides. Then $u(x, y)$ is identically zero.

(12) Let $u(x, y) \varepsilon C^3$, $u_{xxy} + ku_{xyy} - u_{yyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2} \cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = 0$ on the two vertical sides and one horizontal side, $u_n(x, y) = 0$ on the two horizontal sides. Then $u(x, y)$ is identically zero.

(13) Let $u(x, y) \in C^3$, $u_{xxx} + (k + 1)\, u_{xxy} + (k - 1)\, u_{xyy} - u_{yyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

(14) Let $u(x, y) \in C^3$, $u_{xxx} + (k - 1)\, u_{xxy} - (k + 1) + u_{yyy} = 0$ everywhere, $k \neq 0$, $\theta = \frac{1}{2}\cot^{-1}(-k/2)$ with $\tan\theta \leq b/a$, $u(x, y) = u_n(x, y) = 0$ on the boundary. Then $u(x, y)$ is identically zero.

<u>Proof:</u> The proofs here are similar to those of Theorem (I), however the use of Theorem B is needed in place of Theorem (A). We do the proof of part (1) only. In (1) the equation may be written as

$$(u_{xx} + k_1 u_{xy} - u_{yy})_{xx} + k_2 (u_{xx} + k_1 u_{xy} - u_{yy})_{xy} - (u_{xx} + k_1 u_{xy} - u_{yy})_{yy} = 0.$$

Let $v(x, y) = u_{xx} + k_1 u_{xy} - u_{yy}$, then our equation becomes

$$v_{xx} + k_2 v_{xy} - v_{yy} = 0.$$

Now, $u(x, y) = 0$ on the boundary $\Longrightarrow$ $u_{xx} = 0$ on horizontal sides and $u_{yy} = 0$ on vertical sides

$u_n(x, y) = 0$ on the boundary $\Longrightarrow$ $u_{xy} = 0$ on the boundary

$u_{nn}(x, y) = 0$ on the boundary $\Longrightarrow$ $u_{xx} = 0$ on vertical sides and $u_{yy} = 0$ on horizontal sides.

Hence, $v(x, y) = 0$ on the boundary. Now $v(x, y)$ satisfies the conditions of Theorem B and hence $v(x, y)$ is identically zero. We now have $u_{xx} + k_1 u_{xy} - u_{yy} = 0$ everywhere, $u(x, y) = 0$ on the boundary and the conditions of Theorem B are again satisfied to give us $u(x, y)$ identically zero.

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