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REPRESENTATIONS OF THE RIEMANN ZETA FUNCTION

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This report deals with a way of representing the zeta function via divergent (summable) Fourier transforms which leads to some new open questions and some new points of view on old ones.

We begin with the following three definitions and proceed to a famous calculation due to Riemann (1859) which is the basis for almost all subsequent investigations.

Definition 1. The zeta function $\zeta(s)$ is defined for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$ as the sum of the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} .$$

We shall shortly see that ζ extends to a meromorphic function on $\mathbb{C}$, having a simple pole at $s = 1$ and being analytic elsewhere.

Definition 2. The gamma function $\Gamma(z)$ is defined for $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt.$$

This definition can be extended to all of $\mathbb{C}$ with poles at $0, -1, -2, \ldots$.

Definition 3. The theta function $\theta(x)$ is defined for $x \in \mathbb{C}$, $\operatorname{Re} x > 0$ as the sum of the absolutely convergent series

$$\theta(x) = \sum_{-\infty}^{+\infty} e^{-n^2\pi x}.$$

The psi function $\psi(x)$ is defined as $\psi(x) = \frac{1}{2}(\theta(x) - 1) = \sum_{1}^{\infty} e^{-n^2\pi x}$.

The θ-function has a natural boundary along $\operatorname{Re} x = 0$ and satisfies $\theta(x + 2i) = \theta(x)$ and $\frac{1}{\sqrt{x}}\theta(\frac{1}{x}) = \theta(x)$. All this makes $\theta(x)$ one of the class of automorphic functions.

The proof that $\theta(x) = \frac{1}{\sqrt{x}}\theta(\frac{1}{x})$ uses the Poisson summation formula: for $f \in L^1(\mathbb{R})$, let $\hat{f}(u) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t u}\, dt$ be the Fourier transform of f. Poisson's formula states that if both f and $\hat{f}$ are continuous and in $L^1(\mathbb{R})$ then

$$\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \hat{f}(m).$$

The special case when $f(t) = e^{-\pi x t^2}$, $\hat{f}(u) = \frac{1}{\sqrt{x}} e^{-\pi u^2/x}$ gives the result for $\theta(x)$ at once.

Riemann's calculation begins with Euler's integral in the form $\Gamma(\frac{1}{2}s) = \int_0^{\infty} t^{\frac{1}{2}s-1} e^{-t}\, dt$. Substituting $t = n^2\pi x$, $n = 1, 2, 3, \ldots$, we obtain

$$\pi^{-\frac{1}{2}s}\, \Gamma(\tfrac{1}{2}s)\, n^{-s} = \int_0^{\infty} x^{\frac{1}{2}s-1}\, e^{-n^2\pi x}\, dx.$$

Summing now over n defines the lambda function $\Lambda(s)$ as

$$\Lambda(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\,\zeta(s) = \int_0^{\infty} x^{\frac{1}{2}s-1}\, \psi(x)\, dx,$$

with all the series and integrals being absolutely convergent for Re $s > 1$.

In this integral we divide the range into $\int_0^1 + \int_1^\infty$ and in the first we substitute $\frac{1}{x}$ for x, making use of the formula for $\theta(x)$ in the form $\frac{1}{2}+\psi(\frac{1}{x}) = \sqrt{x}(\frac{1}{2} + \psi(x))$. This gives us:

$$\int_0^1 x^{\frac{1}{2}s-1}\,\psi(x)\,dx = \int_1^\infty x^{-\frac{1}{2}s-1}\,\psi(\tfrac{1}{x})\,dx = \int_1^\infty x^{-\frac{1}{2}s-1}\{-\tfrac{1}{2} + \sqrt{x}(\tfrac{1}{2}+\psi(x))\}dx =$$
$$= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty x^{\frac{1}{2}(1-s)-1}\,\psi(x)\,dx.$$

Here we have used the result (about which there is more to say later) that $\int_1^\infty t^{z-1}\,dz = -\frac{1}{z}$ for Re $z < 0$.

Recombining the integrals we obtain Riemann's formula

$$\text{(R)}\qquad \Lambda(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\,\zeta(s) = -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty (x^{\frac{1}{2}s} + x^{\frac{1}{2}(1-s)})\,\psi(x)\,\frac{dx}{x}.$$

This formula is of outstanding importance, since

(a) The integral is entire $(\psi(x) = \theta(e^{-\pi x}))$ and thus Λ and ζ are extended as meromorphic functions to $\mathbb{C}$.

(b) The functional equation $\Lambda(s) = \Lambda(1 - s)$ is evident, as is the fact that the only poles of Λ are at $s = 0, 1$. The strip $0 < \text{Re } s < 1$ is referred to as the critical strip and the line Re $s = \frac{1}{2}$ as the critical line. The famous Riemann hypothesis says that all the zeros of $\zeta(s)$ lie on the critical line. What is known is that (i) at least one third of all zeros are on the line and (ii) approximately the first eight million of them lie on the line.

(c) Via the Mellin transform the formula links an automorphic function with a Dirichlet series, so that clues to the behavior of $\zeta(s)$ might be sought among the properties of $\theta(x)$.

In Riemann's formula we can combine the rational terms with the integral in various ways, for instance:

(A) (Riemann) Multiply by $s(1-s)$ and integrate twice by parts. This gives $s(1-s)\pi^{-\frac{1}{2}s}\,\Gamma(\frac{1}{2}s)\,\zeta(s) = -4\int_1^\infty x^{3/4}\{x^{3/2}\,\psi'(x)\}'\,\cos(\frac{1}{2}t\log x)\frac{dx}{x}$, $(s = \frac{1}{2} + it)$. But this involves derivatives of ψ which are not easily related to ψ itself.

(B) (Ramanujan) Because $\int_1^\infty t^{z-1}dt = -\frac{1}{z}$ for $\mathrm{Re}\ z < 0$ we can rewrite $\Lambda(s)$ for $0 < \mathrm{Re}\ s < 1$ as $\Lambda(s) = \int_1^\infty (-\frac{1}{2}x^{-\frac{1}{2}} + \psi(x))\,(x^{\frac{1}{2}s} + x^{\frac{1}{2}(1-s)})\frac{dx}{x}$. This formula is unsatisfactory since we really need to involve $\theta(x)$ or $\frac{1}{2} + \psi(x)$.

(C) Replace $-\frac{1}{s} - \frac{1}{1-s}$ by $\frac{1}{2}\int_1^\infty x^{\frac{1}{2}s-1}dx + \frac{1}{2}\int_1^\infty x^{\frac{1}{2}(1-s)-1}dx$. For absolute convergence the first integral requires $\mathrm{Re}\ s < 0$, the second $\mathrm{Re}\ s > 1$. Since both cannot be true together, we must replace absolute convergence by summability. It is not hard to show that, if z is not on the positive real axis, that

$$\int_1^\infty t^{-\delta\log\log t+(z-1)}\,dt \to -\frac{1}{z} \quad \text{as } \delta \to 0_+$$

(or similarly, by putting $t = e^u$, that $\int_0^\infty e^{-\delta u^2} e^{zu}\, du \to -\frac{1}{z}$ as $\delta \to 0_+$).

We write $(G) \int_1^\infty t^{(z-1)} dt = -\frac{1}{z}$ (the Gauss sum of a divergent integral) and obtain $\Lambda(s) = (G)\, \frac{1}{2} \int_1^\infty (x^{\frac{1}{2}s} + x^{\frac{1}{2}(1-s)})\, \theta(x) \frac{dx}{x}$. If we now put $s = \frac{1}{2} + it$, $x = e^{2u}$, $\chi(u) = x^{\frac{1}{4}}\theta(x)$, then χ is an increasing even function of u, and we get

$$\Lambda(\tfrac{1}{2} + it) = (G) \int_0^\infty \chi(u) \cos tu\, du.$$

The Riemann hypothesis says that this integral has zeros only on the real axis.

We now introduce a small diversion.

Pólya and Szegö [1] have the following result: Let f be positive, increasing, and integrable on $[0, A]$. Then $F(z) = \int_0^A f(u) \cos(zu)\, du$ has only real zeros.

So, if we let $A \to \infty$, we get a short (and incomplete) "proof" of this interesting property of $\Lambda(\frac{1}{2} + it)$. But the zeros of $\int_0^\infty e^{-\delta u^2} \chi(u) \cos tu\, du$ are not all real [2].

Returning to safer ground, we shall look at the natural boundary of $\theta(x)$ along the imaginary axis. As $x \to \frac{im}{n}$, $\theta(x)$ either tends to 0 (if m, n are odd) or behaves like $\frac{\gamma(m,n)}{\sqrt{nx - im}}$ if $2|mn$:

The coefficients (m,n) are eighth roots of unity and are related to the Jacobi symbol $(\frac{m}{n})$.

Hence the (formal) series $\sum_m \sum_n \frac{\gamma(m,n)}{\sqrt{nx - im}}$ is a kind of partial fraction expansion for θ: it is not absolutely convergent but is the analytic continuation of $\sum_m \sum_n \frac{\gamma(m,n)}{(nx - im)^{\frac{1}{2}}|nx - im|^z}$ to $z = 0$.

If we take the formal Mellin transform of this series we obtain a double series

$$\rho(s) \overset{\text{DEF}}{=\!=} \frac{2\pi^{\frac{1}{2}(1-s)}}{\Gamma(\frac{1}{2}(1 - s)}\,\zeta(s) = \sum_{u=\pm 1} e^{iu\frac{\pi}{4}(1-s)} \sum_{m,n\geq 1} \frac{\gamma(um,n)}{m^{\frac{1}{2}s}\, n^{\frac{1}{2}(1-s)}} \ .$$

Here $\rho(s)$ is an entire function satisfying $\rho(s) = \rho(1 - s))$. The details of this are in [3]. All this is kind of a Dirichlet series in two summands, and the functional equation $\rho(s) = \rho(1 - s)$ is embodied in a property of the coefficients $(\gamma(m,n) = e^{-i\frac{\pi}{4}}\gamma(-n,m)$ for $m,n > 0)$ which is equivalent to the law of quadratic reciprocity.

We know that an ordinary Dirichlet series $\sum_1^{\infty} \frac{a(n)}{n^s}$ whose coefficients satisfy $a(mn) = a(m)\,a(n)$ for $(m,n) = 1$ factorises into $\prod_{p\geq 2} S_p$, where $S_p = \sum_{n=1}^{\infty} \frac{a(p^n)}{p^{ns}}$. For $\zeta(s)$, $a(n) = 1$ and $S_p = \sum_1^{\infty} \frac{1}{p^{ns}} = (1 - p^{-s})^{-1}$.
But the zeros of S_p are not those of $\zeta(s)$ — this may be because the functional equation (i.e., the symmetry under the mapping $s \to 1 - s$) is absent from these factors.

An open problem is whether a double sum of the type $\sum_{m,n} \sum \frac{\gamma(m,n)}{m^s\, n^t}$ can be factorised — e.g., as $\prod_p \prod_q S_{p,q}$ where p, q are primes, and $S_{p,q}$ is some sort of double sum involving only the primes p and q. If so, the zeros of $S_{p,q}$ might be easier to investigate than those of ρ: since the functional equation is embodied in the sum (unlike the case of the single series), they will at least be symmetrically located.

References

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[2] P.L. Walker: On an integral summable to $2\xi(s)/s(s-1)$, Math. of Comp. 32(1978), 1311-1316.

[3] P.L. Walker: The Mellin transform of the partial fraction expansion for $\theta(x)$, Math. Ann. 253(1980), 103-110.