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1. Introduction

In this paper R denotes an associative ring with identity, all modules are assumed to be left unital, and the category of left R-modules is denoted by R-Mod. For an R-module M, $A \leq' M$ will mean that A is an essential submodule of M, and Z(M) its singular submodule (which consists of those elements whose annihilators are essential left ideals of R). M is singular if Z(M) = M, and non-singular if Z(M) = 0. Furthermore, E(M) (resp. J(M)) will denote the injective hull (resp. Jacobson radical) of M and SOC(M) will represent the socle of M. For basic facts concerning torsion theories, we refer to [21]. Recall that a pair (G,F) of classes of left R-modules is called Goldie torsion theory if G is the smallest torsion class containing all modules B/A, where $A \leq' B$ and the torsion free class F is precisely the class of nonsingular modules. If (T,F) is a hereditary torsion theory then the associated filter of left ideals of R is denoted by F(T) and its members are called F-ideals. (T,F) is called stable if T is closed under injective hulls. As is well-known, R is semisimple artinian if and only if each left R-module is injective. Osofsky [17] proved that R is semisimple artinian if each cyclic R-module is injective. R is a left V-ring if each simple left R-module is injective. It is well known that a commutative ring is a V-ring if and only if it is regular (in the sense of von Neumann). R is called weakly regular if each left ideal of R is idempotent. V-rings are weakly regular [16]. A module M is quasi-injective if every homomorphism from a submodule of M into M can be lifted to an endomorphism of M. R is a left QC-ring if each cyclic left R-module is quasi-injective [1]. If R is left QC then R is semiperfect, and it is semisimple artinian if and only if $J(R) = 0$. Koehler [13] proved that QC-rings are direct sums of semisimple artinian rings and certain valuation duo rings. R is a left(right) duo ring if each left(right) ideal of R is two sided, and a left valuation ring if the set of its left ideals are linearly ordered. R is a left PCQI-ring in case each proper cyclic left R-module is quasi-injective. Note that a cyclic R-module C is called proper cyclic if $C \not\cong {}_R R$. PCQI-rings are either prime or semiperfect with nil radical [12]. R is called a splitting ring if the torsion submodule of every module is a direct summand. In the context of the Goldie theory, Cateforis and Sandomierski [5] proved that a commutative ring is splitting if and only if each singular module is injective. R is called a left SI-ring if each singular left R-module is injective. Left SI-rings have been thoroughly investigated in [9]. One object of this note is to examine the validity of the various characterizations of left SI-rings in the more general setting of quasi-injective modules. This has led us to initiate the study of rings all of whose torsion cyclic modules are quasi-injective. In general, these rings are not left SI.

2. Results

Theorem 1. Let R be a ring, and (G,F) be the Goldie theory for R-Mod. Then the following are equivalent:

1) R is left SI;

2) Each torsion finitely generated left R-module is injective and $R/Soc(R)$ is left noetherian;

3) Each torsion cyclic left R-module is injective and $R/Soc(R)$ is left noetherian;

4) Each torsion left R-module is quasi-injective;

5) Each finitely generated torsion left R-module is quasi-injective, $R/Soc(R)$ is left noetherian, and $Soc(R) = \bigcap_{I \in F(G)} I$

Proof. For the equivalence of statements of 1), 2) and 3) we refer to Smith [18, Theorem 3.1, p.100] and [19, p.257]. Now we prove 1) $\Longleftrightarrow$ 4). Suppose each torsion left R-module is quasi-injective. Let M be a torsion left R-module. Then $M \oplus E(M)$ is also a torsion module, and so it is quasi-injective. Hence $M \cong E(M)$, so M is injective. The converse is immediate from the properties of left SI-rings. Next, we prove 1) $\Longleftrightarrow$ 5). Suppose each torsion finitely generated left R-module is quasi-injective and $R/Soc(R)$ is left noetherian. Let M be any torsion left R-module. We claim that M is injective. Let I be an essential left ideal of R, and let $f: I \to M$ be a left R-homomorphism. Let $K = \operatorname{Ker} f$. Then I/K is torsion, and the exactness of the sequence $0 \to I/K \to R/K \to R/I \to 0$ implies that R/K is torsion, as both I/K and R/I are torsion. Now since R/K is a finitely generated torsion left $R/Soc(R)$ module and $R/Soc(R)$ is a left noetherian ring, it follows that R/K is a noetherian left $R/Soc(R)$-module. Hence R/K may be regarded as a noetherian left R-module, and so I/K is a finitely generated left R-module. Hence $I/K \oplus R/K$ is a torsion finitely generated left R-module. Therefore, $I/K \oplus R/K$ is (R-) quasi-injective. Since $0 \to I/K \to R/K$ is exact, quasi-injectivity of $(I/K \oplus R/K)$ implies that the sequence $0 \to I/K \to R/K$ splits. Hence I/K is a direct summand of R/K. This implies that f extends to a map $g:R \to M$. Hence M is injective. The converse is immediate since 1) $\iff$ 2).

<u>Theorem 2</u>. Let R be a commutative ring, and (G,F) be the Goldie theory for R-Mod. Then the following are equivalent:

1) R is a splitting ring;

2) Each torsion R-module is injective;

3) Each finitely generated torsion R-module is quasi-injective;

4) Each torsion cyclic R-module is quasi-injective and R is regular;

5) $R/Soc(R)$ is semisimple artinian.

<u>Proof</u>. 1) $\iff$ 2) is due to Cateforis and Sandomierski ([5], Th.2.1, p.156). In order to prove 1) $\iff$ 3), let us assume that each finitely

generated torsion left R-module is quasi-injective. Let $\overline{R} = R/I$, where I is an essential ideal of R. Then $\overline{R}$ is a QC-ring. Let M be a simple torsion ($\overline{R}$-)module. Then $M_{\overline{R}} \oplus \overline{R}_{\overline{R}}$ is quasi-injective ($\overline{R}$-)module. Hence $M_{\overline{R}}$ is ($\overline{R}$-)injective. Now by using the arguments found in the proof of Lemma 7 in [2], it can be shown that $\overline{R}$ is regular. Hence $\overline{R}$ is semisimple artinian as an R-module. Therefore R is an SI-ring by ([9], Prop. 3.1, p.46), and as R is commutative, it is a splitting ring. Conversely, if R is a splitting ring then R is nonsingular ([5], Prop. 2.2), so the class of torsion modules coincides with that of the singular modules which are injective by the results in [5]. The proof of 1) $\iff$ 4) is similar and hence omitted. Now we prove 1) $\iff$ 5). Assume that R is a splitting ring. Then R is an SI-ring. Hence R has only a finite no. of essential ideals ([9], Th.3.9, p.53). Thus $Soc(R) \leq' R$. Consequently, $R/Soc(R)$ is a ring all of whose cyclic modules are injective. Hence $R/Soc(R)$ is semisimple artinian [17]. Conversely, suppose $R/Soc(R)$ is semisimple artinian. Let M be a torsion R-module. Then M, considered as an $R/Soc(R)$-module, is quasi-injective. Hence M is quasi-injective as an R-module. Therefore, by Theorem 1(4), each torsion R-module is injective. Hence R is a splitting ring.

Note that rings in which only torsion cyclic modules are assumed to be quasi-injective need not be regular even in the commutative case, and so they need not be splitting rings. The ring, $\mathbb{Z}$, of integers (or more generally, any Dedekind domain which is not a field) is an example of a non-regular, non-splitting ring all of whose torsion cyclic modules are quasi-injective. In the remainder of this paper, we shall present connections of these rings with other related rings. First we give a definition for brevity.

<u>Definition</u>. A ring R will be called a left TCQI-ring if each torsion cyclic left R-module is quasi-injective (in the context of a torsion theory).

Now we give some examples of TCQI-rings.

Example 1. [9] Let K be a field, $K_n = K$ for $n = 1,2,\ldots$; $Q = K_n$, $S = \oplus K_n$, R the subalgebra of Q generated by 1 and S. Then R is a nonsingular ring whose socle is S, which is the only essential ideal of R, other than R itself. Since S is also a maximal ideal, each torsion cyclic module (considered in the context of the Goldie theory for R) is either zero or simple. Thus R is a commutative TCQI-ring. However, since R is neither prime nor semiperfect, R is not a PCQI-ring.

Example 2. [9] Let $F_1 \subset F_2$ be distinct fields, Q the ring of 2×2 matrices over F_2, R the subring $\begin{pmatrix} F_2 & 0 \\ F_2 & F_1 \end{pmatrix}$ of Q. Let $J = \begin{pmatrix} F_2 & 0 \\ F_2 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ F_2 & F_1 \end{pmatrix}$. Then both J and K are two-sided ideals of R. Also, R is a right nonsingular ring, and J is the socle of R_R which is also a maximal essential right ideal of R. Similarly, R is left nonsingular, and K is the socle of ${}_RR$ which is the only essential left ideal of R, other than R itself. Thus each torsion cyclic left(right) R-module is either zero or simple. Hence R is both left and right TCQI-ring which is left and right hereditary and right artinian.

Example 3. Let F be a field and x an indeterminate and let W be the family of all well-ordered sets {i} of nonnegative real numbers, the order relation being the natural order of real numbers. Let R be the set of all formal power series $\sum_{i \in \{i\}} a_i x_i$, with a_i in F and {i} in W. Then R is a non-noetherian domain all of whose proper homomorphic images are self-injective ([15]). It is clear that R is a non-splitting TCQI-ring. Note that R is a PCQI-ring but not QC.

We also recall that Cozzens [6] has produced an important example of a nonartinian, two sided hereditary noetherian V-domain over which all cyclic modules are semisimple or free. Thus all torsion cyclic modules are obviously quasi-injective. Finally, we note that $\mathbb{Z} \oplus \mathbb{Z}$ is an example of a TCQI-ring which is neither a splitting ring nor a PCQI-ring.

Now we state the following propositions which can be proved by using basic facts concerning torsion classes, quasi-injective modules and QC-rings.

Proposition 1. Let R be a commutative ring and (T,F) be any hereditary torsion theory for R-Mod. Then R is a TCQI-ring $\Longleftrightarrow$ R/I is a QC-ring, for each ideal $I \in F(T)$.

Proposition 2. Let R be a ring with essential socle, and (G,F) be the Goldie theory for R-Mod. Then R is left TCQI $\iff$ R/Soc(R) is a left QC-ring.

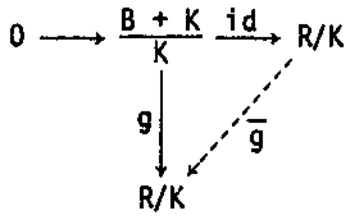
Proposition 3. Let R be a prime TCQI-ring which admits a non-trivial Goldie theory. Then R is a restricted QC-ring (i.e. each proper factor ring of R is a QC-ring). Moreover, if R is a prime ring with nonzero socle then R is left TCQI $\iff$ R/Soc(R) is left QC.

The aim of the next theorem is to obtain a characterization of left SI-rings in terms of left TCQI-rings. We, however, assume in Prop. 4 and Theorem 3 below, that each left ideal in the filter of the related theory is two-sided, as we particularly observed in Example 2.

Proposition 4. Let R be a weakly regular ring and (T,F) be a hereditary torsion theory for R-Mod. Then R is left TCQI $\iff$ each torsion cyclic left R-module is injective.

Proof. Suppose R is a left TCQI-ring. Let M be a torsion cyclic left R-module. Then $M \cong R/K$, where $K \in F(T)$. Hence R/K is R-quasi-injective. We claim that R/K is (R-)injective. Let B be a left ideal of R, and let $f : B \to R/K$ be an (R-) homomorphism. Then $f(B \cap K) = (0)$. For, if $f(B \cap K) \neq (0)$, then there is a nonzero element $a \in (B \cap K)$ such that $f(a) \neq 0$. Since R is weakly regular, $(B \cap K) = (B \cap K)^2$. Hence there exist x_i, $y_i \in (B \cap K)$ such that $a = \sum_{i=1}^{n} x_i y_i$. Thus $f(a) = f[\sum_{i=1}^{n} x_i y_i] =$

$= \sum_{i=1}^{n} [x_i f(y_i)] \subseteq K(R/K) = (0)$. But this is a contradiction. Now f induces an R-homomorphism $g : \frac{B + K}{K} \to R/K$ given by $g(x + K) = f(x)$; for all $x \in B$. As R/K is (R-) quasi-injective, $\exists$ an (R-) homomorphism $\overline{g} : R/K \to R/K$ such that the diagram:



is commutative. Let $\Pi : R \to R/K$ be the natural projection. Then $\overline{g}\Pi$ is an extension of f. Hence R/K is (R-) injective. The converse is immediate.

Note that in the above proof it is enough to assume that B is an essential left ideal of R.

Theorem 3. Let (G,F) be the Goldie theory for R-Mod. Then the following are equivalent:

(1) R is left SI;

(2) R is left TCQI-ring all of whose essential left ideals are idempotent and R/Soc(R) is left noetherian.

<u>Proof</u>. First, suppose R is a left SI-ring. Then R is left hereditary ([9], Th.3.3) and so R is a nonsingular ring. Thus the class of torsion modules is precisely that of the singular modules. Hence R is certainly a left TCQI-ring. Moreover, each essential

left ideal of R is idempotent by ([2], Lemma 6) and R/Soc(R) is left noetherian by ([9], Prop. 3.6). Conversely, assume that R is left TCQI-ring in which each essential left ideal is idempotent and R/Soc(R) is noetherian. If $K \subseteq' R$ then R/K is (R-) quasi-injective. Now by using the arguments given in the proof of Prop.4, it can be shown that R/K is (R-) injective. Hence by ([18], Th.3.1, p.100), R is an SI-ring.

Next, we study the relationship of TCQI-rings with PCQI-rings. First, we state some lemmas.

Lemma 1 (Koehler [14]). If a duo ring R has only a finite number, n, of maximal ideals and each prime ideal is maximal, then R is the direct sum of n local rings.

Lemma 2. Let R be a left duo left PCQI-ring, which is not a domain. Then each prime ideal is maximal.

Proof. Since R is not a domain, (0) is not a prime ideal. Let P be a prime ideal of R. Then R/P is an integral domain, and considered as an R-module, R/P is proper cyclic. Hence R/P is (R-) quasi-injective, as R is a PCQI-ring. This implies that R/P is a self-injective domain. Hence R/P is a divisible R/P-module [4, Prop.12, Ch.7]. Then R/P is a field. Hence P is maximal.

Lemma 3. Let R be a local left TCQI-ring, and (T,F) any hereditary torsion theory. Then ideals in $F(T)$ are linearly ordered.

Proof. Let A, B be any two ideals in $F(T)$. Then $A \cap B \in F(T)$. Hence $R/A \cap B$ is quasi-injective. Since R is local, $R/A \cap B$ is indecomposable. Hence $E(R/A \cap B)$ is also indecomposable. This implies that $R/A \cap B$ is uniform. Since $(A/A \cap B) \cap (B/A \cap B) = (0)$, either $A/A \cap B = A \cap B$ or $B/A \cap B = A \cap B$. Thus either $A = A \cap B$ or $B = A \cap B$ i.e., either $A \leq B$ or $B \leq A$.

Lemma 4. Let (T,F) be a nontrivial hereditary stable torsion theory over a local ring R such that ${}_R R \in F$. If R is left PCQI then R is an integral domain, and (T,F) is Goldie torsion theory.

Proof. First we observe that if M is any cyclic left R-module then M is either torsion or torsion free. For, if $M \cong R$ then M is torsion free. In case, $M \not\cong R$ then M is a proper cyclic left R-module. Hence M is quasi-injective, as R is a left PCQI-ring. Then by ([3], Th.2.3), M splits into its torsion and torsion free parts. But cyclic modules over local rings are indecomposable, so M is either torsion or torsion free. If K is the unique maximal left ideal of R then either $K \leq' R$ or K is a direct summand of R. Since R is local and admits a nontrivial theory, K is neither zero nor a summand of R. Hence all simple modules are torsion. Now the arguments given in the proof of Th.3.3 on p.277 in [20], can be used to complete the proof of the present lemma.

Corollary ([12], Th.17). Let R be a local prime left PCQI-ring. Then R is a valuation domain.

Proof. Since R is prime and the associated torsion theory is nontrivial, $Z(R) = 0$. Hence R is an integral domain and (T,F) is the Goldie theory. It also follows from the proof of the above lemma that each nonzero left ideal of R is an element of F(T). Hence by Lemma 3, R is a valuation domain.

Theorem 4. Let R be a left duo ring, and (T,F) be a nontrivial stable hereditary torsion theory for R-Mod such that ${}_R R \in F$. If R is left TCQI then R is PCQI $\Leftrightarrow$ R is either semisimple artinian or an integral domain.

Proof. Suppose R is a left PCQI-ring. Then R is either prime, or it is semiperfect ([12], Prop.3). If R is prime then R is an integral domain, as R is a left duo ring. If R is a non-local semiperfect ring, then by Lemmas 1 and 2, we can write $R = R_1 \oplus \cdots \oplus R_n$, with each R_i local. Since each R_i is a proper cyclic R-module and R is a PCQI-ring, it follows that each R_i is, in fact, a QC-ring. Hence R is a QC-ring. This implies that $J(R) = Z(R) = T(R) = 0$. Hence R is semisimple artinian by the results in [1]. Now suppose that R is a local PCQI-ring. Then by Lemma 4, R is an integral domain. Conversely, assume that R is semisimple artinian or an integral domain. If R is semisimple artinian then R is obviously a PCQI-ring. If R is an integral domain then each proper cyclic R-module is in fact a torsion cyclic R-module, and so it is quasi-injective as R is left TCQI by the hypothesis. Hence R is a left PCQI-ring.

<u>Theorem 5</u>. Let R be a commutative ring with zero socle, and (G,F) be a (nontrivial) Goldie theory for R-Mod. Then the following are equivalent:

(1) R is a PCQI-ring;

(2) R is a TCQI-ring all of whose nonzero ideals are F-ideals.

<u>Proof</u>. Suppose R is a PCQI-ring. Let M be a torsion cyclic R-module. As (G,F) is nontrivial, $M \not\cong {}_RR$. Hence M is a proper cyclic R-module, and so it is quasi-injective. Moreover, R is either prime or a semiperfect ring [12]. In case R is prime then, being a commutative ring, R is an integral domain. Hence each nonzero ideal of R is an essential ideal, and so an F-ideal for the given theory. Suppose R is semiperfect. Then we can write $R = \bigoplus_{i=1}^{n} R_i$, where each R_i is a local PCQI-ring. If $n > 1$, then each R_i is a proper cyclic R-module. Hence R_i is (R-) quasi-injective. This implies that R_i is a QC-ring. Hence $J(R_i) = Z(R_i) \neq 0$. For otherwise, R_i will have to be semisimple artinian and then R will have nonzero socle. Now, since R_i is a local QC-ring, it follows that $Z(R_i) \subseteq' R_i$. This implies that R_i is torsion, for $i = 1,\cdots,n$. Hence R is torsion but this contradicts the nontriviality of (G,F). Hence $n = 1$, and so R is a local (semiperfect) PCQI-ring with zero socle. If K is the unique maximal ideal of R

then either $K \leq' R$ or K is a direct summand of R. Since R is local and $Soc(R) = 0$, K cannot be a direct summand of R, so $K \leq' R$. Hence each simple R-module is torsion. Now let A be a nonzero ideal of R. As $Soc(R) = 0$, it is possible to choose an element $x\ (\neq 0)$ in A and a nonzero submodule $B \subseteq A$, such that B is maximal with respect to $x \notin B$. Then $\frac{Rx + B}{B}$ is an essential simple submodule of A/B. Then as noted above, all simple modules are torsion and so $\frac{Rx + B}{B}$ is an essential torsion submodule of A/B. Since Goldie theory is stable, it follows that A/B is also torsion. Also, since $B \neq 0$, $R/B \not\cong_R R$. Hence R/B is a proper cyclic R-module and so it is quasi-injective. Therefore, by ([3], Th.2.3) R/B split into its torsion and torsion free parts. But cyclic modules over local rings are indecomposable, so R/B is either torsion or torsion free. However, since R/B contains a nonzero torsion submodule A/B, R/B is torsion. Hence the exactness of the sequence $R/B \to R/A \to 0$ implies that R/A is also torsion. Therefore A is an F-ideal. Conversely, suppose R is a TCQI-ring all of whose nonzero ideals are F-ideals. Let M be a proper cyclic R-module. Then $M \cong R/A$ and A is a nonzero ideal. Hence A is an F-ideal, and so M is a torsion cyclic R-module. Therefore M is quasi-injective. This completes the proof.

<u>Corollary ([7], p.208)</u>. If R is a commutative noetherian PCQI-ring with zero socle then R is a Dedekind domain which is not a field.

<u>Proof</u>. Let A be a nonzero ideal of R. Then $R/A \not\cong {}_RR$, as A is an F-ideal by the above theorem. Hence R/A is a noetherian QC-ring and so it is uniserial by the results in [1]. Thus each proper factor ring of R is uniserial, i.e., R is a restricted uniserial ring. Since $Soc(R) = 0$, the desired implication follows from Th.1 on p.208 in [7].

The purpose of the next theorem is to obtain a representation of left PCQI-rings which may be of independent interest. Let us recall that if R_1 and R_2 are two left nonsingular rings then by an essential product of R_1 and R_2 we mean any subdirect product R of R_1 and R_2 which contains an essential left ideal of $R_1 \times R_2$. Note that if R is a subdirect product of R_1 and R_2 then R is a subring of $R_1 \times R_2$ such that the projections $R \to R_1$ and $R \to R_2$ are both surjective. We also require for the purposes of the next theorem that a subring of a ring has the same identity.

<u>Theorem 6</u>. If R is a semiprime, left PCQI-ring (with zero socle), then R is an essential product of finitely many prime rings (with zero socle).

<u>Proof</u>. As left PCQI-rings are either prime or semiperfect, it follows that R has no infinite direct sum of two-sided ideals. Hence by Handelman [11, Prop.9] R is an essential product of finitely many prime rings. Since any simple submodule of one of these prime rings would also be simple submodule of R, it follows

each of these prime rings must have zero socle if R has zero socle.

Theorem 7. Let R be a commutative nonsingular PCQI-ring. Then R is isomorphic to an essential product of two rings R_1 and R_2 such that R_1 is a finite direct product of fields, and R_2 is a valuation domain.

Proof. By (Goodearl [10], Theorem 10) R is isomorphic to an essential product of two nonsingular rings R_1 and R_2 such that R_1 has essential socle and R_2 has zero socle. Since both R_1 and R_2 are homomorphic images of R, R_1 and R_2 are also PCQI-rings. Now let us consider R_1 first. Since R_1 is a PCQI-ring, R_1 is either prime or semiperfect. If R_1 is prime then, since $S(R_1) \leq' R_1$, R_1 is primitive. Hence R_1 is a field. If R_1 is semiperfect then as argued in Theorem 4, R_1 is either semisimple artinian or an integral domain. Since R_1 is commutative and has essential socle, this implies that R_1 is a finite direct product of fields. Now consider R_2. Since $Soc(R_2) = 0$, it follows from Theorem 4 that R_2 is an integral domain. Hence by Lemma 3, R_2 is a valuation domain.

Finally, we establish a necessary and sufficient condition in order that a left TCQI-ring is left QC. First we state the following lemmas, in the context of the Goldie theory.

Lemma 5. Let R be a local self-injective left TCQI-ring. Then R is left QC.

<u>Proof</u>. Let I be a nonzero left ideal of R. Then $I \subseteq' R$, since R is local and self injective. Hence R/I is a torsion cyclic left R-module. Therefore, R/I is quasi-injective. Hence R is left QC.

<u>Lemma 6</u> (Faith and Utumi [8], Theorem 4.5, p.174). If R is a <u>left</u> self-injective ring and I is a <u>right</u> ideal of R then idempotents in R modulo I can be lifted to idempotents in R.

<u>Lemma 7</u> (Smith [19], Theorem 2.10, p.252). A ring R is semiprime artinian $\Leftrightarrow$ R is a left selfinjective ring such that for each essential left ideal E of R, R/E is an injective left R-module.

The proof of the next lemma is an adaption from ([14], Lemma 2.5).

<u>Lemma 8</u>. Let R be a duo selfinjective left TCQI-ring, which has only a finite no., n, of maximal ideals then R is the direct sum of n local rings.

<u>Proof</u>. We use the principle of induction on the no. of maximal ideals in R. If R has one maximal ideal M, then R is a local ring. Now suppose that if R has k maximal ideals with $k < n$ then R is the direct sum of k local rings. Assume now that R has exactly n maximal ideals. Let $M_1, M_2, \cdots, M_n$ be the

distinct maximal ideals of R. Since $\prod_{i=2}^{n} M_i \subseteq \bigcap_{i=2}^{n} M_i$, and since two-sided maximal ideals are prime, $\bigcap_{i=2}^{n} M_i \nsubseteq M_1$. Hence $R = M_1 + \bigcap_{i=2}^{n} M_i$. Let $N = \bigcap_{i=1}^{n} M_i$. Then $R/N = M_1/N \oplus (\bigcap_{i=2}^{n} M_i/N)$ is a direct sum. Thus $\overline{1} = \overline{a} + \overline{b}$; where $\overline{a}$ and $\overline{b}$ are orthogonal idempotents in R/N. Since R is left selfinjective, we can lift the idempotents $\overline{a}$ and $\overline{b}$ in R/N to orthogonal idempotents e and f in R, by Lemma 6. Thus $1 = e + f$, where $\overline{a} = \overline{e}$ and $\overline{b} = \overline{f}$. Hence $R = Re \oplus Rf$ is a direct sum. We now apply induction to complete the proof.

<u>Theorem 8</u>. Let R be a left duo ring. Then the following are equivalent for the Goldie theory in R-Mod:

(1) R is left selfinjective left TCQI;

(2) R is left QC.

<u>Proof</u>. Let us assume that R is left selfinjective left TCQI ring. First we show that R has only a finite no. of maximal left ideals. We do this by showing that R/J is semisimple artinian. With no loss of generality, we assume that R is selfinjective, regular and left TCQI. Let M be any torsion cyclic left R-module. Then $M \cong R/K$, where K is a left ideal of R. Since R is left TCQI, R/K is (R-) quasi-injective. Since R is a left duo ring, K is a two-sided ideal of R. Hence R/K is (R/K)-quasi-injective, i.e. R/K is (R/K-) injective. We claim that R/K is

(R-) injective. Let B be a left ideal of R and let $f : B \to R/K$ be an R-homomorphism. Then $f(B \cap K) = (0)$. For if $f(B \cap K) \neq (0)$, then since R is regular, every principal ideal of R is generated by an idempotent. Hence $B \cap K$ contains an idempotent e (say) such that $f(e) \neq 0$. But then $0 \neq f(ee) = ef(e) \subseteq K(R/K) = 0$, and this is a contradiction. Now f induces an R-homomorphism $\overline{f} : \frac{B + K}{K} \to R/K$. Clearly, $\overline{f}$ is an R/K-homomorphism. Since R/K is (R/K-) injective, $\overline{f}$ extends to a map $\overline{g} : R/K \to R/K$. Let $\pi : R \to R/K$ be the natural map. Then $\overline{g}\pi$ is an extension of f. This implies that R/K is (R-) injective. Hence by Lemma 7, R is semisimple artinian. Thus R has a finite no., n, of maximal ideals. Hence by Lemma 8, we can write $R = R_1 \oplus \cdots \oplus R_n$; where each R_i is a local, selfinjective TCQI-ring. By Lemma 5, each R_i is a QC-ring. Hence R is a QC-ring. The converse follows immediately from the definitions.

Cor. R is either semisimple artinian, or else it does not admit nontrivial torsion theories.

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Department of Mathematical Sciences
University of Petroleum & Minerals
Dhahran, Saudi Arabia