Distributional (n+1) – Dimensional Heat Equation

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DISTRIBUTIONAL \((n+1)\) — DIMENSIONAL HEAT EQUATION

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ABSTRACT

Let \(f(x) \in L_\infty(\mathbb{R})\). It is a classical result that the solution to the following initial value problem associated with the heat equation:

\[
\frac{\partial U(x,t)}{\partial t} - c^2 \frac{\partial^2 U(x,t)}{\partial x^2} = 0
\]

\[U(x,0^+) = f(x)\]

in the domain \(\mathbb{R}^{2,+} = \{(x,t) : x \in \mathbb{R}, t > 0\}\) is:

\[
U(x,t) = \frac{1}{2\sqrt{\pi}c} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x - \xi)^2}{4c^2t}\right) d\xi,
\]

\(U(x,t)\) being the temperature function. We extend the notion of the temperature function to the space \(\mathbb{R}^{n+1,+} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}\). We call \(U(x,t)\) a temperature function in an open region of \(\mathbb{R}^{n+1,+}\), if it is infinitely differentiable at each point of the region and satisfies:

\[
U_t(x,t) - c^2 \sum_{j=1}^{n} U_{x_jx_j}(x,t) = 0
\]

In this paper we exploit the above definition of the temperature function to solve the Heat equation in \(\mathbb{R}^{n+1,+}\), with a distributional initial condition.
As it turns out, our solution is quite constructive and its two dimensional case is an extension of the classical solution to the above-mentioned initial value problem.

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SCHWARTZ TESTING FUNCTION SPACE $\mathcal{D}_p^r(\mathbb{R}^n)$ ($1 \leq p < \infty$).

An infinitely differentiable function $\phi$ defined over $\mathbb{R}^n$ is said to belong to the space $\mathcal{D}_p^r(\mathbb{R}^n)$ if:

$$D^a \phi(x) = \frac{\partial^{|a|} \phi(x)}{a_1! a_2! \cdots a_n!}$$

where $|a| = a_1 + a_2 + \ldots + a_n$, belongs to $L_p(\mathbb{R}^n)$ for all $|a| \geq 0$.

We introduce a sequence of semi-norms $\{\gamma_m\}_{m=0}^\infty$ on $\mathcal{D}_p^r(\mathbb{R}^n)$ as follows: For $\phi \in \mathcal{D}_p^r(\mathbb{R}^n)$ we define:

$$\gamma_0|\phi| = \left( \int_{-\infty}^{\infty} |D^a \phi(x)|^p dx \right)^{1/p}$$

where $|a| = 0, 1, 2, 3, \ldots$.

Since $\gamma_0$ is a norm, the sequence of semi-norms $\{\gamma_m\}_{m=0}^\infty$ is separating. The topology of $\mathcal{D}_p^r(\mathbb{R}^n)$ is generated by the semi-norms in the usual manner [1, p. 8-14]. We say that a sequence $\{\phi_\mu\}_{\mu=1}^\infty$ converges to $\phi$ in $\mathcal{D}_p^r(\mathbb{R}^n)$, if for each $m = 0, 1, 2, 3, \ldots$

$$\gamma_m(\phi_\mu - \phi) \to 0 \quad \text{as} \quad \mu \to \infty$$

The space $\mathcal{D}_p^r(\mathbb{R}^n)$ is a complete countably multinormal space [2, p. 87].
We denote by \( (\mathcal{D}_p(\mathbb{R}^n))^\prime \) the space of all continuous linear functionals on \( \mathcal{D}_p(\mathbb{R}^n) \). We state here the well known result regarding the structure formula for \( f \in (\mathcal{D}_p(\mathbb{R}^n))^\prime \) \([4, \text{p.}201]\). See also \([2, \text{pp.}109-116]\).

**Theorem 1:** Let \( f \in (\mathcal{D}_p(\mathbb{R}^n))^\prime \) \((1 < p < \infty)\) then \( f \) is equal to a finite linear combination of the derivatives of functions in \( L_q(\mathbb{R}^n) \), that is for each \( f \in (\mathcal{D}_p(\mathbb{R}^n))^\prime \)

\[
<f, \phi> = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha \phi \cdot f(x) \, dx; \quad \forall \phi \in \mathcal{D}_p(\mathbb{R}^n)
\]

(4)

where \( f_{|\alpha}'s \) are functions in \( L_q(\mathbb{R}^n), \frac{1}{p} + \frac{1}{q} = 1 \).

Let us consider the Weierstrass kernel for \( \mathbb{R}^{n+1,+} \) given by:

\[
w(x,t) = \left(\frac{1}{\sqrt{4\pi t c^2}}\right)^n \exp\left(-\frac{|x|^2}{4tc^2}\right).
\]

(5)

The following properties of \( w(x,t) \) are well known \([3]\):

i) \( \int_{\mathbb{R}^n} w(x,t) \, dx = 1 \) for each \( t > 0 \)

(6)

ii) Let \( \phi \in L_q(\mathbb{R})^n \) then \( U_\phi(k,t) \) defined by:

\[
U_\phi(x,t) = \int_{\mathbb{R}^n} \phi(x - \xi) \, w(\xi,t) \, d\xi
\]

(7)

is a temperature function in \( \mathbb{R}^{n+1,+} \), and
a) \[ ||U_{\phi}(x,t)||_{L_q} \leq A_\xi ||\phi(x)||_{L_q}, \quad t > 0 \]

b) \( U_{\phi}(x,t) \) converges to \( \phi(x) \) in the norm of \( L_q(\mathbb{R}^n) \).

**THEOREM 2:** Let \( f \in (D_{L_p}(\mathbb{R}^n))^\prime \quad (1 < p < \infty) \). Then, for all \( \phi \in D_{L_p}(\mathbb{R}^n) \),

\[ \langle f(\xi), w(x-\xi,t), \phi(x) \rangle + \langle f(x), \phi(x) \rangle \text{ as } t \to 0^+ \]

i.e.

\[ \langle f(\xi), w(x-\xi,t) \rangle \to f(x) \text{ in } (D_{L_p}(\mathbb{R}^n))^\prime \text{ as } t \to 0^+ \]

**PROOF:** Using the structure formula for \( f \in (D_{L_p}(\mathbb{R}^n))^\prime \), we have

\[ I(t) = \langle f(\xi), w(x-\xi,t), \phi(x) \rangle \]

\[ = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \langle f|_{\alpha}|(\xi), \partial^\alpha_w(x-\xi,t), \phi(x) \rangle \]

where \( f|_{\alpha} \)'s are in \( L_q(\mathbb{R}^n) \).

However,

\[ \frac{\partial}{\partial \xi_1} W(x-\xi,t) = - \frac{\partial}{\partial x_1} W(x-\xi,t) \]
Or more generally,

\[ D_{\xi}^\alpha w(x-\xi,t) = (-1)^{|\alpha|} D_{\chi}^\alpha w(x-\xi,t). \]  

(9)

Therefore, we can write (8) as:

\[ I(t) = \sum_{|\alpha|=0}^{r} \langle f |\alpha| (\xi), D_{\chi}^\alpha w(x-\xi,t) \rangle, \phi(x) \rangle \]

Using Fubini's theorem and integration by parts, we get:

\[ I(t) = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \langle f |\alpha| (\xi), w(x-\xi,t), \phi|\alpha| (x) \rangle \]

\[ = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \langle f |\alpha| (\xi), \int_{\mathbb{R}^n} w(x-\xi,t) \cdot \phi|\alpha| (x) \cdot dx \rangle \]  

(10)

Let \( U_{\phi,\alpha}(\xi,t) = \int_{\mathbb{R}^n} w(x-\xi,t) \phi^{(\alpha)}(x) \cdot dx \)

Then, it follows from (7) that for each fixed \( t > 0 \), \( U_{\phi,\alpha}(\xi,t) \) belongs to \( \mathcal{C}^\infty(\mathbb{R}^{n+1},+) \) and

\[ \|U_{\phi,\alpha}(\xi,t)\|_{L^p} \leq C_\phi \|\phi^{(\alpha)}(\xi)\|_{L^p} < \infty \quad t > 0 \]  

(11)

and

\[ \|U_{\phi,\alpha}(\xi,t) - \phi^{(\alpha)}(\xi)\|_{L^p} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0^+. \]

However,

\[ D_\beta [U_{\phi,\alpha}(\xi,t)] = \int_{\mathbb{R}^n} w(x-\xi,t) \cdot \phi^{(\alpha+\beta)}(x) \cdot dx \]

\( (|\beta| = 0,1,2,\ldots) \)
Therefore,

\[ \gamma|\beta| (U_{\phi,\alpha}(\xi, t) - \phi^{(\alpha)}(\xi)) \rightarrow 0 \quad \text{as} \quad t \rightarrow 0 \quad (13) \]

Thus,

\[ U_{\phi,\alpha}(\xi, t) \xrightarrow{D_{L^p}(\mathbb{R}^n)} \phi^{(\alpha)}(\xi) \quad \text{as} \quad t \rightarrow 0. \quad (14) \]

From (10 and 14) we get

\[ \lim_{t \rightarrow 0^+} I(t) = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \langle f_{\alpha}(\xi), \phi^{(\alpha)}(\xi) \rangle \]

\[ = \langle f, \phi \rangle \quad \text{as} \quad t \rightarrow 0^+ \quad \text{[by (4)]} \]

**Example 1**: Consider the distributional heat problem:

\[ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = 0 \quad (15) \]

\[ \lim_{t \rightarrow 0^+} U(x, t) = f(x) \quad \text{in} \quad D_{L^p}(\mathbb{R}^n) \quad (16) \]

Let us define a function \( U(x, t) \) by:

\[ U(x, t) = \langle f(\xi), w(x-\xi, t) \rangle, \quad t > 0 \]

Using the structure formula (4) for \( f \) we get:
\[
U(x,t) = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} f_{\alpha}(\xi, D_{\xi}^{\alpha} w(x-\xi, t)) \\
= \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} u_{\alpha}(x,t)
\] (17)

According to (7), each \( u_{\alpha}(x,t) \) is a temperature function, therefore \( U(x,t) \) is a temperature function. In view of the theorem (2), \( U(x,t) \) satisfies the distributional initial condition (15).

EXAMPLE 2: Consider the case \( n = 1 \). The distributional initial value problem in \( \mathbb{R}^{2,+} = \{(x,t): x,t \in \mathbb{R}, t > 0\} \) becomes

\[
U(x,t) - c^2 U_{xx}(x,t) = 0
\]

\[
U(x,0^+) = f(x) \text{ in } (\mathcal{D}_{\text{p}}(\mathbb{R}))'
\]

The above problem is the heat conduction problem for a straight infinite bar of uniform cross section and homogeneous material with the initial temperature distribution in the bar is given by the generalized function \( f(x) \). The solution of the problem follows from (17): in the particular case when \( n = 1 \):

\[
U(x,t) = \langle f(\xi), \frac{1}{2c\sqrt{\pi t}} \exp(-\frac{|x-\xi|^2}{4c^2t}) \rangle.
\]
References


