King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 083

April 1986

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DISTRIBUTIONAL (n+1) — DIMENSIONAL HEAT EQUATION

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ABSTRACT

Let $f(x) \in L_\infty(\mathbb{R})$. It is a classical result that the solution to the following initial value problem associated with the heat equation:

$$\frac{\partial U(x,t)}{\partial t} - c^2 \frac{\partial^2 U(x,t)}{\partial x^2} = 0 \qquad (1)$$
$$U(x,0^+) = f(x)$$

in the domain $\mathbb{R}^{2,+} = \{(x,t): x \in \mathbb{R},\ t > 0\}$ is:

$$U(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{|x-\xi|^2}{4c^2 t}\right) d\xi, \qquad (2)$$

$U(x,t)$ being the temperature function. We extend the notion of the temperature function to the space $\mathbb{R}^{n+1,+} = \{(x,t): x \in \mathbb{R}^n,\ t > 0\}$. We call $U(x,t)$ a temperature function in an open region of $\mathbb{R}^{n+1,+}$, if it is infinitely differentiable at each point of the region and satisfies:

$$U_t(x,t) - c^2 \sum_{i=1}^{n} U_{x_i x_i}(x,t) = 0$$

In this paper we exploit the above definition of the temperature function to solve the Heat equation in $\mathbb{R}^{n+1,+}$, with a distributional initial condition.

As it turns out, our solution is quite constructive and its two dimensional case is an extension of the classical solution to the above-mentioned initial value problem.

1980 Mathematics Subject Classification:

Primary 46F12

Secondary 44A15

Keywords and Phrases:

Heat equation, Distributional boundary value problem, generalized functions and distributions.

SCHWARTZ TESTING FUNCTION SPACE $\mathcal{D}_{L_p}(\mathbb{R})^n$ $(1 \leq p < \infty)$.

An infinitely differentiable function ϕ defined over $\mathbb{R}^n$ is said to belong to the space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ if:

$$D^{\alpha}\phi(x) = \frac{\partial^{|\alpha|}(x)}{\partial x_1^{\alpha_1} \; \partial x_2^{\alpha_2} \; . \; . \; . \; \partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \alpha_2 + . . . + \alpha_n$, belongs to $L_p(\mathbb{R}^n)$ for all $|\alpha| \geq 0$.

We introduce a sequence of semi-norms $\{\gamma_m\}_{m=0}^{\infty}$ on $\mathcal{D}_{L_p}(\mathbb{R}^n)$ as follows: For $\phi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$ we define:

$$\gamma_{|\alpha|}(\phi) = \left(\int_{-\infty}^{\infty} |D^{\alpha}\phi(x)|^p \, dx\right)^{1/p} \tag{3}$$

where $|\alpha| = 0,1,2,3,....$.

Since γ_0 is a norm, the sequence of semi-norms $\{\gamma_m\}_{m=0}^{\infty}$ is separating. The topology of $\mathcal{D}_{L_p}(\mathbb{R})^n$ is generated by the semi-norms in the usual manner [14, pp. 8-14]. We say that a sequence $\{\phi_\mu\}_{\mu=1}^{\infty}$ converges to ϕ in $\mathcal{D}_{L_p}(\mathbb{R}^n)$, if for each $m = 0,1,2,3,....$

$$\gamma_m(\phi_\mu - \phi) \to 0 \quad \text{as} \quad \mu \to \infty$$

The space $\mathcal{D}_{L_p}(\mathbb{R}^n)$ is a complete countably multinormal space [2, p. 87].

We denote by $(\mathcal{D}_{L_p}(\mathbb{R}^n))'$ the space of all continuous linear functionals on $\mathcal{D}_{L_p}(\mathbb{R}^n)$. We state here the well known result regarding the structure formula for $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ [4, p.201]. See also [2, pp.109-116].

<u>THEOREM 1</u>: Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ $(1 < p < \infty)$ then f is equal to a finite linear combination of the derivatives of functions in $L_q(\mathbb{R}^n)$, that is for each $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$

$$\langle f,\phi\rangle = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha\phi \cdot f_{|\alpha|}(x)\, dx; \quad \forall\, \phi \in \mathcal{D}_{L_p}(\mathbb{R}^n) \tag{4}$$

where $f_{|\alpha|}$'s are functions in $L_q(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$

Let us consider the Weierstrass kernel for $\mathbb{R}^{n+1,+}$ given by:

$$w(x,t) = \left(\frac{1}{\sqrt{4\pi tc^2}}\right)^n \exp\left(-\frac{|x|^2}{4tc^2}\right) . \tag{5}$$

The following properties of $w(x,t)$ are well known [3]:

i) $$\int_{\mathbb{R}^n} w(x,t)dx = 1 \quad \text{for each } t > 0 \tag{6}$$

ii) Let $\phi \in L_q(\mathbb{R})^n$ then $U_\phi(k,t)$ defined by:

$$U_\phi(x,t) = \int_{\mathbb{R}^n} \phi(x - \xi)\, w(\xi,t)d\xi \tag{7}$$

is a temperature function in $\mathbb{R}^{n+1,+}$, and

a) $\|U_\phi(x,t)\|_{L_q} \leq A_t \|\phi(\lambda)\|_{L_q}$, $t > 0$

b) $U_\phi(x,t)$ converges to $\phi(x)$ in the norm of $L_q(\mathbb{R}^n)$.

<u>THEOREM 2</u>: Let $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ $(1 < p < \infty)$. Then, for all $\phi \in \mathcal{D}_{L_p}(\mathbb{R}^n)$,

$$<<f(\xi),\ w(x-\xi,t)>,\ \phi(x)> \to <f(x),\phi(x)> \quad \text{as} \quad t \to 0^+$$

i.e.

$$<f(\xi),\ w(x-\xi,t)> \to f(x) \quad \text{in} \quad (\mathcal{D}_{L_p}(\mathbb{R}^n))' \quad \text{as} \quad t \to 0^+$$

<u>PROOF</u>: Using the structure formula for $f \in (\mathcal{D}_{L_p}(\mathbb{R}^n))'$ we have

$$I(t) = <<f(\xi),\ w(x-\xi,t)>,\ \phi(x)>$$

$$= \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} <<f_{|\alpha|}(\xi),\ D_\xi^\alpha w(x-\xi,t)>,\ \phi(x)>$$

where $f_{|\alpha|}$'s are in $L_q(\mathbb{R}^n)$.

However,

$$\frac{\partial}{\partial \xi_i} W(x-\xi,t) = -\frac{\partial}{\partial x_i} W(x-\xi,t)$$

Or more generally,

$$D_{\xi}^{\alpha}w(x-\xi,t) = (-1)^{|\alpha|}D_{x}^{\alpha}w(x-\xi,t). \qquad (9)$$

Therefore, we can write (8) as:

$$I(t) = \sum_{|\alpha|=0}^{r} <<f_{|\alpha|}(\xi),\ D_{x}^{\alpha}w(x-\xi,t)>,\ \phi(x)>$$

Using Fubini's theorem and integration by parts, we get:

$$I(t) = \sum_{|\alpha|=0}^{r}(-1)^{|\alpha|}\cdot<f_{|\alpha|}(\xi),\ <w(x-\xi,t,\ \phi^{|\alpha|}(x)>>$$

$$= \sum_{|\alpha|=0}^{r}(-1)^{|\alpha|}\cdot<f_{|\alpha|}(\xi),\ \int_{\mathbb{R}^n} w(x-\xi,t)\cdot\phi^{|\alpha|}(x)\cdot dx> \qquad (10)$$

Let $U_{\phi,\alpha}(\xi,t) = \int_{\mathbb{R}^n} w(x-\xi,t)\phi^{(\alpha)}(x)dx$

Then, it follows from (7) that for each fixed $t > 0$, $U_{\phi,\alpha}(\xi,t)$ belongs to $c^{\infty}(\mathbb{R}^{n+1,+})$ and

$$||U_{\phi,\alpha}(\xi,t)||_{L_p} \leq C_{\phi}||\phi^{(\alpha)}(\xi)||_{L_p} < \infty \qquad t > 0 \qquad (11)$$

and

$$||U_{\phi,\alpha}(\xi,t) - \phi^{(\alpha)}(\xi)||_{L_p} \to 0 \quad \text{as} \quad t \to 0^{+}. \qquad (12)$$

However,

$$D^{\beta}[U_{\phi,\alpha}(\xi,t)] = \int_{\mathbb{R}^n} w(x-\xi,t)\cdot\phi^{(\alpha+\beta)}(x)dx$$

$$(\forall\ |\beta| = 0,1,2,.......)$$

Therefore,

$$\gamma_{|\beta|}(U_{\phi,\alpha}(\xi,t) - \phi^{(\alpha)}(\xi)) \to 0 \quad \text{as} \quad t \to 0 \tag{13}$$

$$(\forall\ |\beta| = 0,1,2,3,\ldots\ldots)$$

Thus,

$$U_{\phi,\alpha}(\xi,t) \xrightarrow{\mathcal{D}_{L_p}(\mathbb{R}^n)} \phi^{(\alpha)}(\xi) \quad \text{as} \quad t \to 0. \tag{14}$$

From (10 and (14) we get

$$\lim_{t\to 0^+} I(t) = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \langle f_{|\alpha|}(\xi),\ \phi^{(\alpha)}(\xi)\rangle$$

$$= \langle f,\phi\rangle \ \forall\ \phi \in \mathcal{D}_{L_p}(\mathbb{R}^n) \quad \text{as} \quad t \to 0^+ \quad [\text{by } 4]$$

<u>EXAMPLE 1</u>: Consider the distributional heat problem:

$$U_t(x,t) - c^2 \sum_{i=1}^{n} U_{x_i x_i}(x,t) = 0 \tag{15}$$

$$\lim_{t\to 0^+} U(x,t) = f(x) \quad \text{in} \quad (\mathcal{D}_{L_p}(\mathbb{R}^n))' \tag{16}$$

Let us define a function $U(x,t)$ by:

$$U(x,t) = \langle f(\xi),\ w(x-\xi,t)\rangle,\ t > 0$$

Using the structure formula (4) for f we get:

$$U(x,t) = \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} <f_{|\alpha|}(\xi),\ D_{\xi}^{\alpha} w(x-\xi,t)$$

$$= \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} \cdot U_{\alpha}(x,t) \qquad (17)$$

According to (7), each $U_\alpha(x,t)$ is a temperature function, therefore $U(x,t)$ is a temperature function. In view of the theorem (2), $U(x,t)$ satisfies the distributional-initial condition (15).

EXAMPLE 2: Consider the case $n = 1$. The distributional initial value problem in $\mathbb{R}^{2,+} = \{(x,t): x,t \in \mathbb{R},\ t > 0\}$ becomes

$$U(x,t) - c^2 U_{xx}(x,t) = 0$$

$$U(x,0^+) = f(x) \quad \text{in} \quad (\mathcal{D}_{L_p}(\mathbb{R}))'$$

The above problem is the heat conduction problem for a straight infinite bar of uniform cross section and homogeneous material with the initial temperature distribution in the bar is given by the generalized function $f(x)$. The solution of the problem follows from (17) in the particular case when $n = 1$:

$$U(x,t) = <f(\xi),\ \frac{1}{2c\sqrt{\pi t}} \exp(-\ \frac{|x - \xi|^2}{4c^2 t})>.$$

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