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COHOMOLOGY OF PROJECTIVE VARIETIES WITH REGULAR SL_2 ACTIONS

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Let G be a complex semisimple linear algebraic group, B a fixed Borel subgroup of G, H a maximal torus of G in B, g and h the Lie algebras of G and H, respectively. Kostant has expresed the cohomology ring of G/B as the coordinate ring $A(N \cap h)$ of the scheme theoretic intersection $N \cap h$ of the variety of nilpotent elements N of g with h. The purpose of this note is to give a similar description of the cohomology ring of a nonsingular complex projective variety X with a "regular" SL_2 action. We will show that there is an intrinsically defined subscheme Z of X whose coordinate ring $A(Z)$ is isomorphic to the cohomology ring of X. When $X = G/B$, we will identify $A(Z)$ with Kostant's description $A(N \cap h)$.

0. Introduction.

One of the most useful aspects of a flag manifold G/B is that its cohomology ring $H^*(G/B, \mathbb{C})$ admits several different descriptions. The classical semi-simple or Borel-Chevalley description says that $H^*(G/B,\mathbb{C})$ is the coinvariant algebra $A(h)/I^W$ associated to the Cartan subalgebra h of g. On the other hand, the nilpotent or Kostant description ([12]) says $H^*(G/B, \mathbb{C})$ is the coordinate ring $A(N \cap h)$ of the scheme theoretic intersection of the nilpotent variety $N \subset g$ and the Cartan h.

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In this paper we will study this semi-simple/nilpotent phenomenon as a special case of what happens when one has an action of $SL_2(\mathbb{C})$ on a smooth complex projective variety X.

Let us say that a holomorphic action of SL_2 on X is regular if any maximal unipotent subgroup has isolated fixed points. In this case, one knows that any maximal torus also has isolated fixed points and any maximal unipotent has a unique fixed point. We will now describe the general semi-simple/nilpotent situation. Let $\mathbb{B}$ denote a Borel subgroup of SL_2 and suppose V and V_s are respectively the holomorphic vector fields generated by the maximal unipotent and maximal torus in $\mathbb{B}$. The nilpotent description of $H^*(X, \mathbb{C})$ is given in Proposition 1.1 where it is shown that the coordinate ring $A(Z)$ of the zero scheme Z of V has a canonical grading making it isomorphic in the sense of graded rings with $H^*(X, \mathbb{C})$. In the semi-simple case, however, the coordinate ring $A(Z_s)$ of the zero variety Z_s of V_s (viewed as a finite unreduced subvariety X) is not graded. Rather $A(Z_s)$ admits a filtration $F_0 \subset F_1 \subset \ldots$ such that $F_p F_q \subseteq F_{p+q}$ and $\mathrm{Gr}\, A(Z_s) = \oplus F_p/F_{p-1} \cong \oplus H^{2p}(X, \mathbb{C}) = H^*(X, \mathbb{C})$.

For G/B, the filtration on $A(Z_s)$ is very well understood. Explicitly, let $h \in g$ be a regular semi-simple element that generates V_s. We may assume $h \in h$, so let $W \cdot h$ denote the orbit of h under W. The coordinate ring $A(W \cdot h)$ has a natural filtration and a fundamental result is that $A(W \cdot h) \cong A(Z_s)$ as filtered rings ([3, 7]). Thus $\mathrm{Gr}\, A(W \cdot h) \cong H^*(G/B, \mathbb{C})$. It is not hard to see that $\mathrm{Gr}\, A(W \cdot h)$ is the coinvariant algebra (Theorem 2.1), so this amounts to the semi-simple description.

In the nilpotent case we may assume that the unique zero of V is given by $eB \in G/B$. A natural coordinate system near eB is given by b_u^- and we may consider the grading on $A(b_u^-)$ induced by V_s explained in Proposition 1.1. With respect to this grading, $I(Z)$ is a homogeneous ideal, and we are able to find a graded homomorphism $\psi: A(h) \longrightarrow A(b_u^-)$. In Theorem 2.2 we show that ψ induces an isomorphism of graded rings $\bar{\psi}: A(h)/I^W \xrightarrow{\sim} A(b_u^-)/I(Z)$.

The paper is organized as follows. In the first section we prove in Theorem 1.1 the basic fact that $A(Z) \cong H^*(X, \mathbb{C})$ for certain V. We also describe explicitly in Proposition 1.1 the grading of $A(Z)$. In section two, we will compare the various isomorphisms in the homogeneous case and in particular prove the above description of $A(Z)$ (cf. Theorem 2.2). In the third section we discuss an open problem relating to Schubert varieties.

I. A(Z) in the graded case

In this section we will prove the crucial results about $A(Z)$ and its grading. We will begin by reviewing the basic facts about holomorphic vector fields needed below. The basic references are [8, 9].

Let V be a holomorphic vector field on a smooth complex projective variety X with isolated but nontrivial zero set Z, and let $i(V)\colon \Omega_X^p \to \Omega_X^{p-1}$ be the contraction operator associated to V. Here Ω_X^p (resp. $\mathcal{O}_X$) denotes the sheaf of germs of holomorphic p-forms (resp. functions) on X. The structure sheaf $\mathcal{O}_Z$ of Z is by definition $\mathcal{O}_X/i(V)\Omega_X^1$. That is Z is the scheme (possibly unreduced) defined by the sheaf of ideals $I(Z) = i(V)\Omega_X^1$ in $\mathcal{O}_X$. Since $i(V)^2 = 0$, one may consider the complex of sheaves (where $n = \dim X$)

$$0 \longrightarrow \Omega_X^n \longrightarrow \Omega_X^{n-1} \longrightarrow \cdots \longrightarrow \Omega_X^1 \longrightarrow \mathcal{O}_X \longrightarrow 0$$

having differentials $i(V)$, and giving rise to a spectral sequence with $E_1^{-p,q} = H^q(X, \Omega_X^p)$. The basic property of this spectral sequence is that all differentials $d_1, d_2, \ldots,$ vanish, because X is Kaehler and $Z \neq \emptyset$ ([8, 9]). From this one obtains the following:

(i) $H^q(X, \Omega_X^p) = 0$ if $p \neq q$ (consequently $H^{2p+1}(X, \mathbb{C}) = 0$, and $H^{2p}(X, \mathbb{C}) = H^p(X, \Omega_X^p)$),

(ii) since Z is finite and $\mathcal{O}_Z$ is a coherent sheaf on X, $E_1^{-p,p} \Rightarrow H^0(X, \mathcal{O}_Z)$. Thus the ring $A(Z) = H^0(X, \mathcal{O}_Z)$ has a filtration

(1.1) $A(Z) = F_n \supset F_{n-1} \supset \cdots \supset F_0$ so that

(1.2) $F_p/F_{p-1} \cong H^p(X, \Omega_X^p)$ for all p, and

(iii) since $i(V)$ is a derivation on $\Omega^{\bullet}_X$ with respect to the wedge product pairing, the filtration satisfies $F_p F_q \subseteq F_{p+q}$, and (1.2) induces an isomorphism of graded algebras

(1.3) m_V: $\mathrm{Gr}\, A(Z) = \oplus F_p/F_{p-1} \xrightarrow{\sim} H^*(X, \mathbb{C})$, where $H^*(X, \mathbb{C})$ $(= \oplus H^p(X, \Omega^p_X))$ denotes the complex cohomology ring of X.

These results are proved in [8, 9]. The main difficulty in realizing the cohomology ring of X on Z lies in computing the mysterious filtration F_p. It is very difficult, in general, to calculate this filtration. In some of the standard examples, however, (e.g., vector fields on Grassmannians with exactly one zero). The ring $A(Z)$ turns out to be graded and already isomorphic to $H^*(X, \mathbb{C})$. The following theorem, which was announced in [5][(1)], explains this phenomenon.

<u>THEOREM 1.1</u>. Let X admit, in addition to V, an algebraic $\mathbb{C}^*$ action $(\lambda, x) \longrightarrow \lambda \cdot x$ with the property that there exists an integer $k \neq 0$ such that for any $\lambda \in \mathbb{C}^*$, $d\lambda \cdot V = \lambda^k V$. Then $A(Z)$ is graded, and the filtration F_p of $A(Z)$ is the canonical filtration associated to this grading. Hence $A(Z)$ is isomorphic, via m_V, to $H^*(X, \mathbb{C})$.

Proof. We will show that there is a one parameter group of automorphisms $\mathbb{C}^* \longrightarrow \mathrm{Aut}(A(Z))$ of $A(Z)$ preserving the filtration F_p, and giving $A(Z)$ the weight decomposition $A(Z) = \bigoplus_{0 \leq i \leq n} A_i$, where

$$A_i = \{f \in A(Z) : d\lambda \cdot f = \lambda^{-ki} f \text{ for all } \lambda \in \mathbb{C}^*\}.$$

This weight decomposition makes $A(Z)$ into a graded ring. The proof of theorem will be completed by showing that for each $i = 0, 1, \ldots,$ $n = \dim X$, $F_i = A_0 \oplus \ldots \oplus A_i$.

Suppose λ_p: $\Omega^p_X \to \Omega^p_X$ denotes the action on p-forms by $x \to \lambda \cdot x$.

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(1) It was further asserted that the condition $d\lambda \cdot V = \lambda^k V$ implies that V has a unique zero. Unfortunately this has not been proven, so we leave it as an open question.

Consider the folliwng commutative diagram of complexes:

$$\begin{array}{ccccc}
\cdots \longrightarrow & \Omega_X^p & \xrightarrow{\ i(V)\ } & \Omega_X^{p-1} & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle \lambda^{-kp}} & & \Big\downarrow {\scriptstyle \lambda^{-k(p-1)}} & \\
\cdots \longrightarrow & \Omega_X^p & \xrightarrow{\ i(\lambda\cdot V)\ } & \Omega_X^{p-1} & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle \lambda_p} & & \Big\downarrow {\scriptstyle \lambda_{p-1}} & \\
\cdots \longrightarrow & \Omega_X^p & \xrightarrow{\ i(V)\ } & \Omega_X^{p-1} & \longrightarrow \cdots
\end{array}$$

Obviously, the vertical maps are isomorphisms of complexes for all λ, so there exist isomorphisms

$$A(Z) \xrightarrow{\ \phi_\lambda\ } A(Z(\lambda)) \xrightarrow{\ \psi_\lambda\ } A(Z)$$

where $Z(\lambda) = \text{zero}\ (d\lambda\cdot V)$, so that if $\mu_\lambda = \psi_\lambda\cdot\phi_\lambda$, then $\mu_{\lambda_1\lambda_2} = \mu_{\lambda_1}\cdot\mu_{\lambda_2}$. Hence μ_λ is the desired one parameter group. Next, suppose the filtration of $A(Z(\lambda))$ is denoted $F_p(\lambda)$. By standard reasoning, $\phi_\lambda(F_i) = F_i(\lambda)$ and $\psi_\lambda(F_i(\lambda)) = F_i$ for all i and λ. It follows that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc}
H^i(X, \Omega_X^i) & \xrightarrow{\ \bar{\phi}_\lambda\ } & H^i(X, \Omega_X^i) \\
& {\scriptstyle \bar{\mu}_\lambda} \searrow \quad \swarrow {\scriptstyle \bar{\psi}_\lambda} & \\
& H^i(X, \Omega_X^i) &
\end{array}$$

induced by ϕ_λ, ψ_λ and μ_λ. However, $\bar{\psi}_\lambda = 1$ for all λ since the map $x \to \lambda\cdot x$ is homotopic to the identity. Thus $\bar{\mu}_\lambda(w) = \lambda^{-ki}w$ for all $w \in H^i(X, \Omega_X^i)$. Since this is true for all $i = 0, 1, \ldots, n$, it follows that $F_i = A_0 \oplus A_1 \oplus \ldots A_i$.

<u>REMARK</u>. The only step in the proof requiring isolated zeros is (1.1). In general, the proof shows that condition $d\lambda\cdot V = \lambda^k V$ renders the abuttment $\underline{H}_X^\bullet$ of $E_1^{-p,q}$ a graded ring whose filtration coincides with the canonical filtration. In other words, the isomorphism $\underline{H}_X^\bullet \cong \oplus H^q(X, \Omega_X^p)$ always obtains, even if Z is infinite.

Theorem 1.1 can be strengthened by describing the grading of $A(Z)$. We will assume V has exactly one zero x_0. The general case is also similar. Since x_0 is also a fixed point of the $\mathbb{C}^*$ action λ, $\mathbb{C}^*$ acts on the tangent space $T_{x_0}X$ of X at x_0, and consequently on the symmetric algebra $A = \mathrm{Sym}(T^*_{x_0}X)$ of the cotangent space $T^*_{x_0}X$ of X at x_0. The weight decomposition of this action makes A into a graded algebra. In the following proposition A will be regarded as a graded algebra with this gradation.

PROPOSITION 1.1. There exists a $\mathbb{C}^*$-invariant open affine neighbourhood U of x_0 such that

(i) U is $\mathbb{C}^*$-equivariantly isomorphic to Spec(A), and consequently the ring of regular functions $A(U)$ on U admits a graded algebra structure,

(ii) the ideal $I(Z)$ of the variety Z is homogeneous in the graded algebra $A(U)$,

(iii) the graded algebra $A(U)/I(Z)\,(= A(Z))$ is isomorphic to $H^*(X, \mathbb{C})$ via m_V.

Proof. We may assume, without loss of generality, that $k > 0$ in $d\lambda \cdot V = \lambda^k V$. Since x_0 is an isolated fixed point of the $\mathbb{C}^*$ action λ, and also the sink of X, we know by [10] that $x_0^- = \{x \in X : \lim_{\lambda\to\infty} \lambda\cdot x = x_0\}$ is open in X, and $\mathbb{C}^*$ equivariantly isomorphic to the affine variety $T_{x_0}X$. So, if we let $U = x_0^-$, we get (i) immediately. For (ii), let $\phi\colon \mathbb{C}\times X \to X$ be the one parameter group associated to the vector field V. Since the condition $d\lambda\cdot V = \lambda^k V$ translates to the identity $\lambda\cdot\phi(t)\cdot\lambda^{-1} = \phi(\lambda^k t)$, the associated comorphisms acting on the regular functions satisfy $\lambda^*\phi^*(t) = \phi^*(\lambda^{-k}t)\lambda^*$. Let $z_i \in A(U)$, $i \le i \le n$, be a holomorphic local coordinate system around x_0. Since $\lambda^*(\phi^*(t)(z_i) - z_i) = \phi^*(\lambda^{-k}t)(\lambda^*(z_i)) - \lambda^*(z_i)$, the ideal generated by $\phi^*(t)(z_i) - z_i$ in $A(U)$ is invariant under the $\mathbb{C}^*$-action λ. This gives (ii), because $I(Z)$ is generated by $\phi^*(t)(z_i) - z_i$, $t \in \mathbb{C}$ and $1 \le i \le n$. (iii) follows from (1.3) and the proof of Theorem 1.1.

In the case that SL_2 acts regularly on X we may take V to be generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and V_s to be generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In this case $d\lambda \cdot V = \lambda^2 V$ where $\lambda: \mathbb{C}^* \longrightarrow \mathrm{Aut}(X)$ is the one parameter subgroup associated to V_s.

II. SL_2 actions on G/B

Let G be a complex semisimple linear algebraic group, B a fixed Borel subgroup of G, H a maximal torus of G in B, g, b, and h the Lie algebras of G, B, and H, respectively. We denote by Δ the set of roots of H in G, Δ_+ the set of positive roots in Δ associated to B, $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of simple roots in Δ_+, $\{e_\beta \in g: \beta \in \Delta\}$ the set of root vectors such that $\{[e_\beta\ e_{-\beta}]: \beta \in \Delta\}$ is dual to Δ (i.e., $h_\beta = [e_\beta, e_{-\beta}]$ is the co-root associated to $\beta \in \Delta$). The integer obtained from the canonical perfect pairing between one parameter subgroup $\lambda: \mathbb{C}^* \to H$ and the character $\chi: H \to \mathbb{C}^*$ is denoted by $\langle\lambda, \chi\rangle$. The height $\sum_1^\ell m_i$ of any $\beta = \sum_1^\ell m_i \alpha_i$ in Δ is denoted by $h(\beta)$.

Let n be a regular nilpotent element in g. By Jacobson-Morosov Lemma ([11]) there exists an $s\ell_2$-triple $\{n, f, s\}$ associated to n (i.e., $[n, f] = s$, $[s, f] = -2f$, and $[s,n] = 2n$). This gives a regular SL_2 action on G/B. Conversely, each regular SL_2 action on G/B is obtained in this way. We will take, without loss of generality, the principal regular nilpotent element $n = \sum_1^\ell e_{\alpha_i}$ throughout the rest of the paper. In this case $s \in h$ is a regular semisimple, while $f \in b^-$ is a regular nilpotent element of g. Let $\lambda: \mathbb{C}^* \to H$ be the one parameter subgroup of H associated to s (i.e., $d\lambda(1) = s$). Note that λ is uniquely determined by the condition $\langle\lambda, \alpha_i\rangle = 2$ for any $1 \le i \le \ell$. Therefore, for any $\beta \in \Delta$, $2h(\beta) = \langle\lambda, \beta\rangle$. As before, V (resp. V_s) denotes the vector field induced from the $\mathbb{C}$ (resp. $\mathbb{C}^*$) action $\phi(t) = \exp(tn)$ (resp. λ). Then V admits $x_0 = B \in G/B$ as its unique zero. Since the tangent action of λ on $T_{x_0}(G/B) \cong b_u^- = \sum_{\alpha\in\Delta_+} \mathbb{C} e_{-\alpha}$ is equivalent to the Adjoint action of λ on b_u^-, we have $\lambda \cdot e_{-\alpha} = \mathrm{Ad}\lambda(e_{-\alpha}) = \lambda^{-2h(\alpha)}\, e_{-\alpha}$ for any $\alpha \in \Delta_+$. Thus

$\mathrm{Sym}(T^*_{x_0}(G/B))$ is $\mathbb{C}$ equivariantly isomorphic to $A(b_u^-)$ $(=\mathrm{Sym}(b_u^{-*}))$. The following lemma gives the explicit gradation of $A(Z)$.

LEMMA 2.1. There exists an H invariant open affine neighborhood U of x_0 in G/B together with a natural holomorphic local coordinate system z_α, $\alpha \in \Delta_+$, at x_0 such that $A(U)$ is isomorphic, as a graded algebra to $\mathbb{C}[z_\alpha : \alpha \in \Delta_+]$, where the grading on $\mathbb{C}[z_\alpha : \alpha \in \Delta_+]$ is given by degree of $z_\alpha = h(\alpha)$ for $\alpha \in \Delta_+$.

Proof. Let $U = x_0^-$. By [1], $U = B^- x_0$ where B^- is the Borel subgroup of G opposite to B. Choose z_α, $\alpha \in \Delta_+$, the natural coordinate functions at x_0 defined by the isomorphism $b_u^- \xrightarrow{\sim} B^- x_0$ given by $x = \sum z_\alpha e_{-\alpha} \to \exp(x)\, x_0$. The rest of the claim follows from Proposition 1.1 and the observations above.

We will now give the geometric description of the isomorphism $m_V : A(Z) \xrightarrow{\sim} H^*(G/B, \mathbb{C})$. To do this we need to recall the theory of V-equivariant bundles ([2], [9]). Let V be a holomorphic vector field with isolated zeroes $Z \neq \emptyset$ on a Kaehler manifold X. A holomorphic vector bundle E on X is said to be V-equivariant if there exists a V-derivation $\tilde{V} : \mathcal{O}_X(E) \to \mathcal{O}_X(E)$, i.e., a $\mathbb{C}$-linear map satisfying $\tilde{V}(fs) = V(f)s + f\tilde{V}(s)$, for $f \in \mathcal{O}_X$, $s \in \mathcal{O}_X(E)$. Let $\tilde{V}(Z)$ be a matrix representation of $\tilde{V}$ around Z. Then we know that the function $c_k(\tilde{V})$, defined by the identity $\det(I + x\tilde{V}) = \sum c_k(\tilde{V})\, x^k$ is in $F_k \subset A(Z)$ for each k, and moreover $m_V(c_k(\tilde{V})) = c_k(E)$, k-th Chern class of E, ([2], [9]).

PROPOSITION 2.1. For each character χ of H, the homogeneous line bundle L_χ is V-equivariant, and moreover the function

$$\sum_{i=1}^{\ell} d\chi([e_{\alpha_i}, e_{-\alpha_i}])z_{\alpha_i} \quad \text{in} \quad \mathbb{C}[z_\alpha : \alpha \in \Delta_+]$$

represents the first Chern class $c_1(L_\chi)$ of L_χ via m_V. In particular, if $\{w_i : 1 \le i \le \ell\}$ is the set of fundamental dominant weights associated to the simple roots $\{\alpha_i : 1 \le i \le \ell\}$, then

$$m_V(z_{\alpha_i}) = c_1(L_{w_i}), \quad 1 \le i \le \ell.$$

Proof. The first assertion is well known ([2], [9]). For the second, one can use a technique introduced in [2] to compute $c_1(L_\chi)$. Namely, one lifts the $\mathbb{C}$-action $\phi(t)$ on L_χ naturally, and then takes the t-derivation $\tilde{V}$ of this lift on L_χ. In particular when this is carried out in the neighborhood $U = B^- x_0$ of x_0 in Lemma 2.1, one obtains (cf. [3, p. 4])

$$\tilde{V} = \frac{d}{dt} (d\chi(\mathrm{Ad}(\phi(t)) (\sum_{\alpha\in\Delta_+} z_\alpha e_{-\alpha}))\Big|_{t=0}$$

$$= d\chi(\mathrm{ad}(n) (\sum_{\alpha\in\Delta_+} z_\alpha e_{-\alpha})) = d\chi([n, \sum_{\alpha\in\Delta_+} z_\alpha e_{-\alpha}])$$

$$= \sum_{\alpha\in\Delta_+} z_\alpha d\chi([n, e_{-\alpha}]) = \sum_{i=1}^{\ell} z_{\alpha_i} d\chi([e_{\alpha_i}, e_{-\alpha_i}]),$$

because for $h(\alpha) > 0$, $[n, e_{-\alpha}]$ is in b_u^- and $d\chi(b_u^-) = 0$. In particular, if $\chi = w_i$ is a fundamental weight, then $dw_i([e_{\alpha_j}, e_{-\alpha_j}]) = \delta_{i,j}$. Thus $m_V(z_{\alpha_i}) = c_1(L_{w_i})$ for $1 \leq i \leq \ell$. This finishes the proof of the proposition.

The vector field V_s, which is associated to the regular semisimple element $d\lambda(1) = s \in h$, was studied in [3, 7]. We will now recall the cohomology description of G/B through V_s. These results are given in [3, 7]. The zero set Z_s of V_s coincides with $\{wB | w \in W\}$, where W is the Weyl group of (H, G). Let $A(W\cdot s)$ denote the coordinate ring of the orbit $W\cdot s \subset h$, i.e., $A(W\cdot s) = A(h)/I(W\cdot s)$, $I(W\cdot s)$ is the ideal of all f in $A(h)$ (the coordinate ring of h) such that $f|W\cdot s = 0$. It was shown in [3, 7] that the homomorphism $\theta: A(W\cdot s) \to A(Z_s)$ defined by $\theta(f)(wB) = f(w\cdot s)$ maps the natural filtration of $A(W\cdot s)$ coming from the degree in $A(h)$ onto the filtration F_p of $A(Z_s)$, and in fact one has the following:

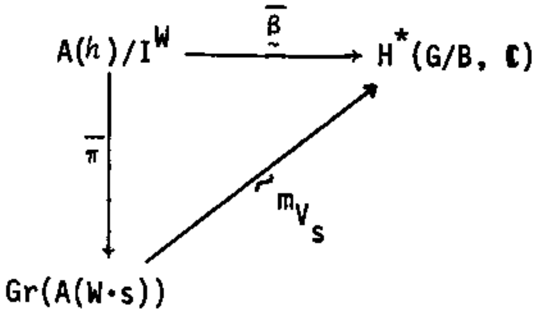
THEOREM 2.1. The cohomology ring of G/B is the graded ring associated to the filtration of $A(W\cdot s)$ induced by degree in $A(h)$, i.e.,

$$m_{V_s} : \mathrm{Gr}(A(W\cdot s)) \xrightarrow{\sim} H^*(G/B, \mathbb{C}).$$

Under this isomorphism the element $[\chi]$ of $A(W\cdot s)$ defined by $d\chi|W\cdot s$, where χ is any character of H, corresponds modulo F_0 (constants) to the first Chern class $c_1(L_\chi)$ of the line bundle L_χ on G/B associated to χ.

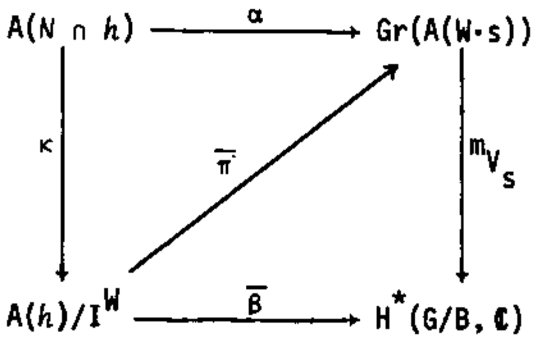
Let $\beta: A(h) \to H^*(G/B; \mathbb{C})$ be the graded algebra homomorphism determined by $\beta(d\chi) = c_1(L_\chi)$, where χ is any character of H, and let $\pi: A(h) \to A(W\cdot s)$ be the natural homomorphism $\pi(f) = f|W\cdot s$. One obtains from Theorem 2.1 the semi-simple description of $H^*(G/B, \mathbb{C})$.

<u>COROLLARY</u>. The algebra homomorphism $\beta: A(h) \to H^*(G/B; \mathbb{C})$ is surjective, and $\ker \beta = I^W$, the ideal of W-invariant functions in $A(h)$ vanishing at the origin. Moreover, the following diagram commutes



In [12], it was shown that the restriction map res: $A(g) \to A(h)$ induces an isomorphism $\kappa: A(N \cap h) \xrightarrow{\sim} A(h)/I^W$. The main point is that the ideal $I(N)$ of N is generated by the Ad-invariant polynomials $f \in A(g)$ such that $f(0) = 0$. In the next proposition, we use an alternate description of $I(N)$ due to Borho and Kraft ([6]) to directly connect $\mathrm{Gr}\, A(W\cdot s)$ and $A(N \cap h)$. This gives, for example, a very simple proof that $A(N \cap h)$ is the regular representation of W.

<u>PROPOSITION 2.2</u>. There exists a natural graded algebra isomorphism $\alpha: A(N \cap h) \to \mathrm{Gr}(A(W\cdot s))$ forming the following commutative diagram



where κ is the graded algebra isomorphism induced from the restriction map $\text{res}: A(g) \to A(h)$.

Proof. Let $I(O_s)$ be the ideal of the smooth variety $O_s = G\cdot s$ in g. Since N is a normal variety ([12]) and $N = \overline{G\cdot n}$, the ideal $I(N)$ of N is equal to $\text{gr}(I(O_s))$ ([6]). Thus

$$A(N \cap h) = A(g)/I(N) + I(h) = A(g)/\text{gr}(I(O_s)) + I(h).$$

Since $O_s \cap h = W\cdot s \subset h$ is smooth, $I(W\cdot s) = I(O_s) + I(h)$ in $A(g)$. Thus the restriction map $\text{res}: A(g) \to A(h)$ induces an isomorphism $A(g)/I(O_s) + I(h) \xrightarrow{\sim} A(W\cdot s)$. On the other hand for an arbitrary pair of ideals I_1 and I_2 in $A(g)$, we know that $\text{gr}(I_1) + \text{gr}(I_2) \subseteq \text{gr}(I_1 + I_2)$. Thus, since $\text{gr}(I(h)) = I(h)$, we get a natural surjective graded algebra homomorphism $\alpha: A(N \cap h) \to A(g)/\text{gr}(I(O_s) + I(h)) = \text{gr}(A(g)/I(O_s) + I(h)) \xrightarrow{\sim} \text{gr}(A(W\cdot s))$. Clearly α makes the above diagram commute. Hence α is an isomorphism since κ and $\overline{\pi}$ are.

We now come to the principal result of this section which is the relation between V_s and V descriptions of $H^*(G/B, \mathbb{C})$. The one-parameter family $\lambda: \mathbb{C}^* \to H$ associated with $s \in h$ acts on the Lie algebra g by the adjoint representation, $(d\lambda(1) = s)$. Since $\{n, f, s\}$ forms a principal $s\ell_2$-triple, n lies in the eigenspace of this action with eigenvalue $+1$. This, in return, implies $[b_u^-, n] \subset b^-$. Thus, the algebra homomorphism $\psi: A(h) \to A(b_u^-)$ obtained from $\psi(dx)(y) = d\chi([n, y])$ is well defined and grading preserving. Here χ is a character of H which is extended naturally on B^-, and $A(b_u^-)$ is graded in the sense of section 1. We remark that the grading of $A(b_u^-)$ has been considered in various places, e.g. [13].

<u>THEOREM 2.2</u>. The graded algebra homomorphism $\psi: A(h) \to A(b_u^-)$ induces graded algebra isomorphism $\overline{\psi}: A(h)/I^W \xrightarrow{\sim} A(b_u^-)/I(Z)$ making the following diagram commutative

$$\begin{array}{ccccc}
A(N \cap h) & \xrightarrow{\kappa} & A(h)/I^W & \xrightarrow{\bar{\psi}} & A(b_u^-)/I(Z) \\
& \overset{\alpha}{\searrow}\!\sim & \Big\downarrow \bar{\pi} & & \Big\downarrow m_V \\
& & Gr(A(W\cdot s)) & \xrightarrow[\sim]{m_{V_s}} & H^*(G/B, \mathbb{C})
\end{array}$$

Proof. It follows from Theorem 2.1 and Propositions 2.1, 2.2, because $\beta(d\chi) = m_V(\psi(d\chi)) = c_1(L_\chi)$ for any character χ of H, and $d\chi$ generates $A(h)$.

III, A Complement

The principal result of [4] is that if Y is any subvariety of a smooth projective variety X with holomorphic vector field V having only isolated zeros such that V is tangent to the set of smooth points of Y, then the coordinate ring $A(Y \cap Z)$ of the scheme theoretic intersection of Y and Z, the zero scheme of V, has a filtration such that $Gr\ A(Y \cap Z)$ admits a homomorphism into $H^*(Y, \mathbb{C})$ making the following diagram commute:

$$\begin{array}{ccc}
Gr\ A(Z) & \xrightarrow{\sim} & H^*(X, \mathbb{C}) \\
\Big\downarrow & & \Big\downarrow \\
Gr\ A(Y \cap Z) & \xrightarrow{\psi} & H^*(Y, \mathbb{C})
\end{array}$$

In general, ψ is not injective or surjective, but if V is a semi-simple vector field and $H^*(X, \mathbb{C})$ surjects onto $H^*(Y, \mathbb{C})$, then ψ is an isomorphism, i.e. $H^*(Y, \mathbb{C})$ admits a semi-simple description. Moreover, if X admits a regular SL_2 action and V and V_s are the vector fields arising from $\mathbb{B}$ as above, then the coordinate ring $A(Y \cap Z)$ of the scheme $Y \cap Z$ is graded, and the natural map $A(Z) \to A(Y \cap Z)$ is a graded algebra homomorphism. The following seems to be a reasonable conjecture: For any $s\ell_2$ triplet $\{n, f, s\}$, where n is a regular nilpotent element in g, and for any Schubert variety $Y = \overline{BwP/P}$ in G/P, the graded ring $A(Y \cap Z)$, Z the zero scheme of n, is canonically isomorphic with $H^*(Y, \mathbb{C})$. Here P stands for a parabolic subgroup of G. The reason this conjecture is interesting is that it would say that in the cohomology ring of G/P, i.e. $A(Z)$, the ideal $I(Y \cap Z)$ defining $Y \cap Z$ determines $H^*(Y,\mathbb{C})$ in the obvious way, namely as $A(Z)/I(Y \cap Z)$. Recently this conjecture has been verified if G/P is the Grassmann manifold by the first author.

Finally we add that some of the above results have analogous for other parabolics in G.

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