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CHORD FUNCTIONS OF CONVEX BODIES†

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Abstract. Various theorems are proved to show that chord functions, or the generalized k-chord functions, at certain sets of points in the plane determine the shape of any convex body uniquely. Consideration is given to special values of k which relate to problems of the equichordal type.

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1. Introduction.

If K is a convex body, and s is a direction, the chord function of K at s gives the length of each chord of K which is parallel to s. The information it gives is the same as the Steiner symmetral of K, or (up to translation) the (parallel) X-ray of K, in that direction. If p is a point, the chord function of K at p gives the length of each chord of K on a line through p, and corresponds to the point-source X-ray of K from p.

This paper continues the attempt to answer Hammer's X-ray problems ([H]), which are here interpreted as asking for a classification of those sets of directions, or points, or both, which have the property that the corresponding chord functions determine uniquely the shape of any convex body. For directions in the plane, this was done in [GM]. For points in the plane, the project was begun in [F_1] and [G], and continued in [V], which also contains the first results which apply to both points and directions, the natural setting here being the projective plane.

We begin by taking a wider viewpoint, and treating 'generalized' or 'k-chord' functions at points, the case $k = 1$ reducing to the usual chord functions at points described above. Though first referred to by name in [F_2], (constant) k-chord functions have appeared intermittently in the literature for at least 70 years. Some uniqueness results were obtained, for $k \geq 1$, in [F_2] by use of a version of the Stable Manifold theorem of differentiable dynamics. We avoid the use of this difficult theorem, and obtain these (see Theorem 1) and other results by instead generalizing the method introduced by A. Volčič in [V] for the case $k = 1$, thereby producing for each k a measure μ_k which has appropriate invariance properties for the corresponding k-chord functions. The measures are then applied to give a version (Theorem 2) of the main theorem of [GM], which deals with k-chord functions at points.

One motivation for considering k-chord functions for different k comes from problems of the equichordal type. A point p is an equichordal point of a convex body K if p is interior to K, and the chord function of p at K is constant and equal to α, say. The notorious equichordal problem asks if K exist with two such points. If we replace 'chord function' by 'k-chord function' in the definition of equichordal point, and specialize to $k = 0$ and $k = -1$, we obtain the definitions of equiproduct and equireciprocal points, respectively. These have also been investigated, as early as 1916. In fact, for $k = 1, 0,$ and -1, there are results which state that at most one convex body K exists with two given points of the appropriate type and given α, providing ∂K satisfies certain differentiability properties. In Theorem 4 we are able to obtain results of the same type which apply to all k-chord functions rather than just to those which are constant.

A second reason for working with values of $k \neq 1$ is that we are able to exhibit examples, particularly for $k = 0$ and -1, where the corresponding ones for $k = 1$ seem difficult to find (see Example 1); this puts our theorems in better perspective.

In Section 7, we return to the case $k = 1$ to state (Theorem 6) a projective generalization of the main theorem in [GM] which applies to directions, points, or both. This answers a question of A. Volčič. Finally, we indicate that for $k = 1$ there is a quite specific class of possible examples, whose existence only remains to be decided in order to completely settle our version of Hammer's problem.

2. Preliminaries and definitions.

If A is a set, the boundary, interior and closure of A are denoted by ∂A, $\text{int } A$ and $\overline{A}$, respectively.

Our setting, except in Section 7, will be $\mathbb{R}^2$. By a convex body we mean a compact convex subset of $\mathbb{R}^2$ with non-empty interior. A set E is starshaped at a point p if ∂E is a simple closed curve and every line through p meets E in a line segment. Note that such sets are compact.

Suppose k is an integer and p is a point in $\mathbb{R}^2$. The k-chord function f_p^k of a set E starshaped at p is defined as follows. Suppose $t \in [0, \pi]$ and ℓ is a line making an angle t with the x-axis. If $\ell \cap E = \emptyset$, we define $f_p^k(t) = 0$. Otherwise ℓ meets ∂E at two (possibly equal) points with distances $r(t)$ and $s(t)$ from p, and we define

$$f_p^k(t) = \begin{cases} |[s(t)]^k - [r(t)]^k|, & \text{if } p \notin E \\ [s(t)]^k + [r(t)]^k, & \text{if } p \in E \end{cases}$$

for $k \neq 0$, and

$$f_p^0(t) = \begin{cases} s(t)/r(t), & \text{if } p \notin E \\ s(t)\, r(t), & \text{if } p \in E \end{cases}$$

where $s(t) \geq r(t)$.

When $k = 1$, $f_p^1(t)$ is the usual chord function, which gives the length of the chord of E on a line ℓ through p making an angle t with the x-axis. For k a positive integer, k-chord functions were considered by

K. Falconer ([F_2]) under the name 'generalized chord functions'. Constant k-chord functions were defined much earlier, however, for $k \geq 1$ and $p \in E$, by W. Süss ([S]). We find it convenient to work with integral k, though most of what we do holds for arbitrary real k.

If p is a point, and ℓ is a line through p meeting the interior of a convex body K, suppose that ℓ meets ∂K at x and y. Then we shall write $y = p(x)$ or $y = p^{-1}(x)$, according as x lies between p and y on ℓ, or y lies between p and x on ℓ.

If H and K are convex bodies, and $\text{int}(H \Delta K)$ is non-empty, let A be a component of $\text{int}(H \Delta K)$. (A <u>component</u> means a non-empty maximal connected subset.) If H and K have the same k-chord functions at p, the set

$$A' = \cup\{\ell \cap \text{int}(H \Delta K): p \in \ell, \quad \ell \cap A \neq \emptyset\} \setminus A$$

is also a component of $\text{int}(H \Delta K)$, disjoint from A. Then we shall write $A' = p(A)$ or $A' = p^{-1}(A)$, according as A lies between p and A' on ℓ, or A' lies between p and A on ℓ.

If H and K have the same k-chord functions at p, then the components A and A' (starshaped at p, but not necessarily convex) also have equal k-chord functions at p. Let $k = 1$, and suppose A is a component of $\text{int}(H \Delta K)$ with $p^{-1}(A)$ defined. Then $\lambda(p^{-1}(A)) < \lambda(A)$, where λ denotes Lebesgue measure in $\mathbb{R}^2$ (in contrast to the equality we have, by the well-known Cavalieri principle, when p is at infinity; see Section 7). In [V], A. Volčič introduced a new measure giving $p^{-1}(A)$ and A equal values. Our first task is to obtain a family of measures which do this for k-chord functions.

3. The measures μ_k.

Let L be the class of bounded Lebesgue measurable subsets of $\mathbb{R}^2$. If $A \in L$, define for each integer k,

$$\mu_k(A) = \iint_A |y|^{k-2}\, dx\, dy.$$

Then μ_k is a measure on L, and $\mu_2 = \lambda$. If $k > 1$, $\mu_k(A) < \infty$ for each $A \in L$, and $\mu_k(A) < \infty$ for all k if A has positive distance from the x-axis.

Suppose now that $A_1, A_2 \in L$ and $\lambda(A_i) > 0$, $i = 1, 2$. Let $p_0 = (x_0, y_0) \in \mathbb{R}^2$, and suppose A_1 and A_2 are starshaped at p_0 and have equal k-chord functions at p_0 for some integer k, and $p_0 \notin A_i$, $i = 1, 2$.

Then, putting $x = x_0 + r\cos\theta$, $y = y_0 + r\sin\theta$, we have

$$A_1 = \{(r, \theta)\colon r_1(\theta) \le r \le s_1(\theta),\ \alpha \le \theta \le \beta\} \quad \text{and either}$$

$$A_2 = \{(r, \theta)\colon r_2(\theta) \le r \le s_2(\theta),\ \alpha \le \theta \le \beta\} \quad \text{or}$$

$$A_2 = \{(r, \theta)\colon r_2(\theta) \le r \le s_2(\theta),\ \alpha + \pi \le \theta \le \beta + \pi\}.$$

We will assume former, since the latter can be treated similarly. The assumptions on the k-chord functions give $s_2^k(\theta) - r_2^k(\theta) = s_1^k(\theta) - r_1^k(\theta)$ $(k \ne 0)$ or $s_2(\theta)/r_2(\theta) = s_1(\theta)/r_1(\theta)$ $(k = 0)$, for $\alpha \le \theta \le \beta$.

With the change of coordinates above we obtain, for $i = 1, 2$,

$$\mu_k(A_i) = \int_\alpha^\beta \int_{r_i}^{s_i} |r \sin\theta + y_0|^{k-2}\, r\, dr\, d\theta.$$

Now if $k \neq 0$ we put $\rho = r^k$ and if $k = 0$ we set $\rho = \ln r$. Then

$$\mu_k(A_i) = \frac{1}{k}\int_\alpha^\beta \int_{r_i^k}^{s_i^k} |\rho^{1/k} \sin\theta + y_0|^{k-2}\, \rho^{(2-k)/k}\, d\rho\, d\theta \quad (k\neq 0), \qquad (1)$$

and

$$\mu_0(A_i) = \int_\alpha^\beta \int_{\ln r_i}^{\ln s_i} |e^\rho \sin\theta + y_0|^{-2\rho}\, e^{2\rho}\, d\rho\, d\theta. \qquad (2)$$

<u>LEMMA 1</u>. If $y_0 = 0$, $\mu_k(A_1) = \mu_k(A_2)$.

<u>Proof</u>. From (1) and (2) we obtain $\mu_k(A_i) = \frac{1}{k}\int_\alpha^\beta \int_{r_i^k}^{s_i^k} |\sin^{k-2}\theta|\, d\rho\, d\theta$ $(k \neq 0)$, and $\mu_0(A_i) = \int_\alpha^\beta \int_{\ln r_i}^{\ln s_i} \sin^{-2}\theta\, d\rho\, d\theta$. Because A_1 and A_2 have the same k-chord functions, the length of the interval of integration in these expressions is the same for $i = 1, 2$. Since the integrand is independent of ρ, the lemma follows.

For the next two lemmas we shall assume in addition that A_1 and A_2 are contained in $\{(x, y): y > 0\}$, and that $s_1(\theta) \leq r_2(\theta)$ for $\alpha \leq \theta \leq \beta$.

<u>LEMMA 2</u>. If $y_0 < 0$, then $\mu_k(A_1) < \mu_k(A_2)$ if $k > 2$, and $\mu_k(A_1) > \mu_k(A_2)$ if $k < 2$.

<u>Proof</u>. Note that $(r \sin\theta + y_0) = y > 0$. Let $f(\rho) = (\rho^{1/k} \sin\theta + y_0)^{k-2} \cdot \rho^{(2-k)/k}$. Then $\frac{\partial f}{\partial \rho} = -\, y_0 \cdot (\frac{k-2}{k})\, (\rho^{1/k} \sin\theta + y_0)^{k-3} \cdot \rho^{(2-k)/k}$. Therefore the integrand in (1) increases with ρ for fixed θ, if $k > 2$, so $\mu_k(A_1) < \mu_k(A_2)$. Also, if $k = 1$, the integrand in (1) decreases with ρ

for fixed θ, giving $\mu_1(A_1) > \mu_1(A_2)$.

For $k < 0$, we need to rewrite (1) as

$$\mu_k(A_i) = \frac{1}{k}\int_\alpha^\beta \int_{s_i^k}^{r_i^k} -(\rho^{1/k} \sin\theta + y_0)^{k-2} \rho^{(2-k)/k} \, d\rho \, d\theta,$$

since $s_i^k \le r_i^k$ in this case. So again the integrand decreases in ρ for fixed θ.

Finally, if $k = 0$, let $g(\rho) = (e^\rho \sin\theta + y_0)^{-2} e^{2\rho}$; then $\frac{\partial g}{\partial \rho} = 2y_0 \, e^{2\rho}(e^\rho \sin\theta + y_0)^{-3}$, which is negative, as required.

Our assumptions for the next lemma remain the same, namely, that A_1 and A_2 have equal k-chord functions for some k.

LEMMA 3. If $y_0 = 0$, $\mu_{k+1}(A_1) < \mu_{k+1}(A_2)$.

Proof. With the same substitutions as before we obtain

$$\mu_{k+1}(A_i) = \frac{1}{k}\int_\alpha^\beta \int_{r_i^k}^{s_i^k} \rho^{1/k} |\sin^{k-1}\theta| \, d\rho \, d\theta \quad (k > 0)$$

$$\mu_{k+1}(A_i) = \frac{1}{k}\int_\alpha^\beta \int_{s_i^k}^{r_i^k} -\rho^{1/k} |\sin^{k-1}\theta| \, d\rho \, d\theta \quad (k < 0)$$

and

$$\mu_1(A_i) = \int_\alpha^\beta \int_{\ln r_i}^{\ln s_i} e^\rho |\sin^{-1}\theta| \, d\rho \, d\theta.$$

In each case the integrand increases with ρ for fixed u, and the lemma follows.

LEMMA 4. Let $T = \{(x, y): a|x - x_0| \le |y| \le b, \ (a > 0)\}$. Then $\mu_k(T) < \infty$ for $k > 0$.

We omit the easy proof. The above lemmas are generalizations of Lemmas 3.1 and 3.2 of [V]. Note that Lemma 2 is not available for $k = 2$, since we have $\mu_2(A_1) = \mu_2(A_2)$ in this case. For this reason, we shall also need the following lemma (cf. [GM], Lemma 2).

LEMMA 5. Suppose $A_i \in L$, $i = 1, 2$, are contained in $\{(x, y): y > 0\}$, and $0 < \mu_{k-1}(A_i) < \infty$ for some integer k and $i = 1, 2$. For this k, suppose that A_1 and A_2 have the same k-chord functions at a point p_0 on the x-axis. Let m_i be the center of gravity of A_i with respect to μ_{k-1}, that is, with respect to the density $|y|^{k-3}\,dx\,dy$. Then m_1 and m_2 lie on the same line through p_0.

Proof. By definition, $m_i = (x_i, y_i)$, where

$$x_i = [\iint_{A_i} xy^{k-3}\,dx\,dy]/\mu_{k-1}(A_i) \quad \text{and} \quad y_i = [\iint_{A_i} y^{k-2}\,dx\,dy]/\mu_{k-1}(A_i).$$

Using polar coordinates at p_0 and the expressions for A_1 and A_2 above,

$$(y_i/x_i) = \mu_k(A_i)/[k\int_\alpha^\beta (s_i^k - r_i^k)\cot\theta \; d\theta].$$

Since A_1 and A_2 have equal k-chord functions, $(y_1/x_1) = (y_2/x_2)$, and the lemma follows.

LEMMA 6. Let A_i, $i = 1, 2$, have equal k-chord functions at two points p_1, p_2, on the x-axis. Let $0 < \mu_{k-1}(A_i) < \infty$, $i = 1, 2$, and let m_i be the center of gravity of A_i with respect to μ_{k-1}. Then $m_1 = m_2$.

Lemma 6 follows from Lemma 5. For chord functions $(k = 1)$, Lemma 6 was used in [V], Theorem 1.1.

4. Uniqueness theorems.

The results of the previous section enable us to generalize theorems which give conditions under which chord functions at different points determine the shape of a convex body.

THEOREM 1. Suppose K is a convex body in $\mathbb{R}^2$, and H is another convex body with the same k-chord functions as K at two points p_1, p_2. Suppose further that

(a) the line ℓ through p_1 and p_2 supports K, or

(b) $p_1, p_2 \in \text{int } K$, or

(c) $\ell \cap \text{int } K \neq \emptyset$, $p_1, p_2 \notin \text{int } K$, and H and K intersect the same component of $\ell \setminus \{p_1, p_2\}$.

Then $H = K$ if $k \geq 1$, and also if $k \leq 0$ if in addition $\mu_k(H \Delta K) < \infty$, where μ_k is the measure obtained by taking ℓ as the x-axis.

Proof. If $k \neq 2$, this follows that of Theorem 4.1 of [V], if the measure μ $(= \mu_1)$ in that paper is replaced by μ_k, and one notes that $\mu_k(H \Delta K) < \infty$ is always true for $k \geq 2$ and is assumed for $k \leq 0$.

For $k = 2$, Lemma 2 is not available. This is used only when, in case (a), $\text{int}(H \Delta K)$ has a component A_1 with $\overline{A}_1 \cap \ell \neq \emptyset$, with p_1 and p_2 lying on the same side of H and K on ℓ. Suppose that this is the case, and $\text{int } A \cap \ell \neq \emptyset$. Let ∂H (respectively, ∂K) meet ℓ at two points at distances r_1 and s_1 (respectively, r_2 and s_2) from p_1, so that

$|s_1^2 - r_1^2| = |s_2^2 - r_2^2|$. If $\|p_1 - p_2\| = c > 0$, and p_2 is further from H and K than p_1 on ℓ, then 2-chord functions at p_2 give $|(s_1 + c)^2 - (r_1 + c)^2| = |(s_2 + c)^2 - (r_2 + c)^2|$. Now for $i = 1, 2$,

$$(s_i + c)^2 - (r_i + c)^2 = \int_{r_i^2}^{s_i^2} (1 + ct^{-1/2})dt.$$

For $i = 1, 2$, the interval of integration is of the same length, and the integrand decreases with t. Therefore $s_2 = s_1$ and $r_2 = r_1$ (or $s_2 = r_1$ and $r_2 = s_1$), contradicting the existence of the component A_1.

Therefore, $\text{int}\, A_1 \cap \ell = \emptyset$, and $H \cap \ell = K \cap \ell$. Note that $p_1(A_1) = p_2(A_1) = A_2$ say, and A_1 and A_2 also have the same 2-chord functions from p_1 and p_2. If H and K have a supporting line distinct from ℓ at $q \in \partial H \cap \partial K \cap \ell$, then the convexity of H and K, together with Lemma 4, imply that $\mu_1(\bar{A}_i) < \infty$, $i = 1, 2$. Now Lemma 6 says A_1 and A_2 have a common center of gravity with respect to μ_1, which is impossible, since $A_1 \cap A_2 = \emptyset$.

Finally, suppose that ℓ is the only supporting line to H and K at points of $H \cap \ell = K \cap \ell$. In this situation, we may have $\mu_1(A_1) = \infty$, so that Lemma 6 is no longer applicable. As in [V], a 'chord-chasing' argument is needed. Let ℓ' be a line through p_1 which makes a small angle with ℓ, and meets ∂H (respectively, ∂K) at two points h_1, h_2 (respectively, k_1, k_2), with p_1, k_2, h_2, k_1, h_1, in that order on ℓ'. Then $\|h_2 - p_1\|^2 - \|k_2 - p_1\|^2 = \|h_1 - p_1\|^2 - \|k_1 - p_1\|^2$. Now let the line through p_2 and h_2 meet ∂H (respectively, ∂K) in points h_2, h_3 (respectively, q_2, q_3), with p_2, q_2, h_2, q_3, h_3 in that order. Then $\|h_2 - q_2\| > \|h_2 - k_2\|$, and $\|h_3 - p_2\|^2 - \|q_3 - p_2\|^2 = \|h_2 - p_2\|^2 - \|q_2 - p_2\|^2$. Next, let the line through p_1 and h_3 meet ∂H (respectively, ∂K) in points h_3, h_4 (respectively, k_3, k_4), with p_1, k_4, h_4, k_3, h_3 in that order. Then $\|h_3 - k_3\| > \|h_3 - q_3\|$. Together, this gives $\|h_3 - k_3\| > c\,\|h_1 - k_1\|$, where

$$c=(\|h_1-p_1\|+\|k_1-p_1\|)(\|h_2-p_2\|+\|q_2-p_2\|)/(\|h_2-p_1\|+\|k_2-p_1\|)(\|h_3-p_2\|+\|q_3-p_2\|).$$

Suppose $H \cap \ell = K \cap \ell$ is the line segment $[x, y]$, where p_2, p_1, x, y lie on ℓ in that order. Then, as the angle between the line ℓ' and ℓ approaches 0, c approaches

$$(\|p_1 - y\| \cdot \|p_2 - x\|)/(\|p_1 - x\| \cdot \|p_2 - y\|) > 1.$$

Consequently, we may choose ℓ' so that $\|h_3 - k_3\| > \|h_1 - k_1\|$, and continue inductively, to produce sequences $\{h_{2n+1}\}$ and $\{k_{2n+1}\}$ of points in ∂H and ∂K respectively, with $\|h_{2n+1} - k_{2n+1}\| > \|h_{2n-1} - k_{2n-1}\|$ for each n, and $\lim h_{2n+1} = \lim k_{2n+1} = y$. This is impossible, and the proof is complete.

Theorem 1 generalizes Theorem 4.1 of [V], for the case when p_1 and p_2 are finite points. Earlier versions of the latter theorem may be found in [F_1] and [G]. For $k \geq 1$, Theorem 1 (parts (b) and (c)) was proved by Falconer ([F_2]), using a version of the Stable Manifold theorem.

The condition $\mu_k(H \Delta K) < \infty$ is necessary, at least for $k = 0$ and $k = -1$; see Examples 2 and 3 in Section 6.

For $k = 1$, it is an open question whether there are different convex bodies H, K, with the same chord functions at two points p_1, p_2 such that the line ℓ through p_1 and p_2 meets H and K. According to Theorem 1, either (i) H and K are disjoint, with p_1, H, p_2, K lying on ℓ in that order (this is illustrated in Fig. 7 of [F_1]), or

(ii) $p_1 \notin (H \cup K)$ and $p_2 \in \text{int } H \cap \text{int } K$. The next example shows that such examples exist, for $k = 0$ and $k = -1$.

EXAMPLE 1. Let H and K be disjoint circular discs of different radii, with centers on the x-axis, and let p_1 and p_2 be the two points on the x-axis such that both H and K have common supporting lines through p_1 and through p_2 (the centers of similitude). Then H and K have equal 0-chord functions at p_1 and p_2. This shows (i) above can occur for $k = 0$. For the case (ii), take H and K to be intersecting circular discs of different radii, centers on the x-axis, and p_1, p_2 again the centers of similitude (where one is now at the intersection of the x-axis with the line through the points of intersection of ∂H with ∂K). Similar examples may be obtained for $k = -1$, by replacing the circles ∂H and ∂K by suitable ellipses; we omit the details.

Using Theorem 1, we can obtain generalizations of the three- and four-point uniqueness theorems of A. Volčič ([V], Theorems 1.1 and 1.2). We shall not state these in detail here. As an example, however, we could prove that if H and K are convex bodies, with equal k-chord functions for some $k \geq 1$ at points p_i $(1 \leq i \leq 4)$, no three of which are collinear, then $H = K$.

Our next aim is to obtain 'point-source' versions of the main theorem of [GM]. First, we need a definition. If $S = \{s_1, \ldots, s_n\}$ is a set of directions in $\mathbb{R}^2$, regarded as points at infinity, we say that a set $\{p_1, \ldots, p_n\}$ of points in $\mathbb{R}^2$ is projectively equivalent to S if there is a projective transformation Φ such that $\Phi(s_i) = p_i$, $i = 1, \ldots, n$.

<u>THEOREM 2</u>. Suppose p_i, $1 \le i \le n$, are distinct points on a line ℓ in $\mathbb{R}^2$, such that $\{p_i: 1 \le i \le n\}$ is not projectively equivalent to a subset of the directions of the diagonals of a regular polygon. Let H and K be convex bodies with the same k-chord functions at p_i, $1 \le i \le n$. Then $H = K$ if $k \ge 1$, and also if $k \le 0$ if in addition $\mu_k(H \Delta K) < \infty$, where μ_k is the measure obtained by taking ℓ as the x-axis.

<u>Proof</u>. Suppose initially that $n = 3$, and that the line ℓ meets H or K. Then we can apply Theorem 1 to show that $H = K$, unless $H \cap K = \emptyset$ and two of the points, p_1 and p_2 say, lie between H and K on ℓ. In this case, however, it is easy to see that H and K cannot have common supporting lines through p_1 and p_2, unless $H = K$.

Next, note that any set of three points on ℓ is projectively equivalent to the directions of the edges of an equilateral triangle. Therefore $n \ge 4$, and from the previous paragraph we conclude that ℓ does not meet H or K.

Let ℓ be the x-axis and μ_k the measure defined in Section 3. Assume $H \ne K$, and let A be a component of $\mathrm{int}(H \Delta K)$, with center of gravity m with respect to μ_{k-1}. For any $i = 1, \ldots, n$, the components $p_i(A)$ (or $p_i^{-1}(A)$, as appropriate) have the same μ_k-measure as A, by Lemma 1, and are disjoint from A. Further, the center of gravity m_i of $p_i(A)$ (or $p_i^{-1}(A)$) with respect to μ_{k-1} lies on the line through p_i and m, by Lemma 5. Iterating through any sequence of points from $\{p_1, \ldots, p_n\}$, each appearing infinitely often, leads to only a finite number of components $A = A_1, \ldots, A_s$, because $H \cup K$ has finite μ_k-measure, since it is disjoint from ℓ. The centers of gravity $\{m = m_1, m_2, \ldots, m_s\}$ of these components form the vertices of a convex polygon P, contained in $H \cup K$.

The polygon P is non-degenerate, since $\text{int}(H\Delta K)$ consists of at least three components. For, suppose on the contrary that $\text{int}(H\Delta K)$ has exactly two components, B_1 and B_2. Then Lemma 6 can be applied to show that $B_1 \cap B_2 \neq \emptyset$, a contradiction. Also, P possesses the following property: if v is a vertex of P, and $i = 1, \ldots, n$, the line through v and p_i contains another vertex of P.

Let Φ be a nonsingular projective transformation taking ℓ onto the line at infinity. If $s_i = \phi(p_i)$ and $S = \{s_1, \ldots, s_n\}$, the non-degenerate convex polygon $\phi(P)$ is an 'S-polygon' in the sense of [GM]. By Lemma 5 of [GM], $\phi(P)$ is an affinely regular polygon, that is, $\phi(P) = \Psi(Q)$, where Q is a regular polygon and Ψ is an affine transformation. Then $P = \phi^{-1}\Psi(Q)$, and $\phi^{-1}\psi$ is a projective transformation. It is clear that $\{p_1, \ldots, p_n\}$ is equivalent via this transformation to a subset of the directions of the diagonals of Q, which proves the theorem.

The heart of the proof is the same as that of the main theorem of [GM], with Lebesgue measure replaced by μ_k. As in [GM], Theorem 4, any set $\{p_i, 1 \leq i \leq 4\}$ of collinear points whose cross-ratio is a transcendental number will satisfy the hypothesis of Theorem 2. Theorem 2 is best possible, at least for $k = -1$; see Example 5 in Section 7.

5. Starshaped sets and the case $k \leq 1$.

We now wish to study the condition $\mu_k(H\Delta K) < \infty$ assumed in our theorems, in order to relate our work to that on equichordal problems. To do this satisfactorily, we must allow non-convex bodies into the discussion. For a straightforward exposition, we limit this to chord functions at interior points.

We have already defined k-chord functions f_p^k for sets starshaped at p, in Section 2.

THEOREM 3. Suppose D and E are sets which are starshaped at common interior points p_1 and p_2, and D and E have equal k-chord functions at p_1 and p_2. Then $D = E$ if $k > 1$, and also if $k \leq 1$ if in addition $\mu_k(D \Delta E) < \infty$, where μ_k is the measure obtained by taking the line ℓ through p_1 and p_2 as the x-axis.

Proof. This is the same as that of the relevant part of Theorem 1. However, the extra assumption $\mu_k(D\Delta E) < \infty$ may now be required for $k = 1$, if the curves ∂D and ∂E are tangent to the line through p_1 and p_2 (which is impossible if D and E are convex).

The restriction $\mu_k(D\Delta E) < \infty$ is unintuitive, and our next task is to replace it by more familiar conditions.

LEMMA 7. Under the hypotheses of Theorem 3, D and E meet the line ℓ through p_1 and p_2 in common points a and b.

Proof. If $\partial D \cap \partial E = \emptyset$, then D and E cannot have equal k-chord functions at p_1 or p_2. Let $x_1 \in \partial D \cap \partial E$. Then there is another point y_1 on the line through x_1 and p_1 with $y_1 \in \partial D \cap \partial E$, since D and E have equal k-chord functions at p_1. Similarly, there is a point $x_2 \neq y_1$ on the line through y_1 and p_2 with $x_2 \in \partial D \cap \partial E$. Continuing, we obtain a sequence of points $x_n \in \partial D \cap \partial E$. Clearly $x_n \to a$, where $a \in \partial D \cap \partial E \cap \ell$.

It follows that there is a $b \neq a$ with $b \in \partial D \cap \partial E \cap \ell$.

THEOREM 4. Suppose D and E are sets which are starshaped at common interior points p_1 and p_2, and D and E have equal k-chord functions at p_1 and p_2. Then $D = E$ if in addition

(i) $k = 1$, and there exist curves through a and through b, analytic at a or b, respectively, which lie between $(\partial D \cup \partial E)$ and ℓ near a or b, respectively;

(ii) $k \leq 0$, and ∂D and ∂E are $C^{(1-k)}$ functions at a and b, $(2-k)$ times differentiable at a and b, with respect to polar coordinates centered at p_1, and with equal i-th derivatives at a and also at b, for $i = 1, \ldots, (1-k)$.

Proof. (i) For $k = 1$, Lemma 4 holds with $|x - x_0|$ replaced by $|x - x_0|^c$, for any $c > 0$, as was also noted in [V], 4.2, Remark 2. Consequently if ∂D and ∂E are bounded away from ℓ by analytic curves, $\mu_1(D \Delta E) < \infty$, and the result follows from Theorem 3.

(ii) Let ℓ be the x-axis, a the origin, and p_1 the point (1, 0). Let (r, θ) be polar coordinates centered at p_1, and let ∂D and ∂E have parametric equations $r = f(\theta)$ and $r = g(\theta)$, respectively, near a, say for $\theta_0 \leq \theta \leq \pi$. Finally, let $A = (D \Delta E) \cap \{(r, \theta): \theta_0 \leq \theta \leq \pi\}$. It will suffice to show that $\mu_k(A) < \infty$, by Theorem 3.

Then we have

$$\mu_k(A) = \left| \int_{\theta_0}^{\pi} \int_{f(\theta)}^{g(\theta)} r^{k-1} \sin^{k-2}(\theta)\, dr d\theta \right|$$

$$= \left| \int_{\theta_0}^{\pi} (v(\theta) - w(\theta)) \sin^{k-2}(\theta)\, d\theta \right|,$$

where $v(\theta) = [f(\theta)]^k/k$, $w(\theta) = [g(\theta)]^k/k$ if $k \neq 0$, and $v(\theta) = \ln f(\theta)$, $w = \ln g(\theta)$ if $k = 0$.

From our assumptions, f and g are $C^{(1-k)}$ functions, $(2-k)$ times differentiable at $\theta = \pi$. Therefore v and w also have these properties, since $f(\pi) = g(\pi) = 1 \neq 0$. Also, we have $f^i(\pi) = g^i(\pi)$ for $i = 1, \ldots, (1-k)$, and it follows that $v^i(\pi) = w^i(\pi)$ for $i = 0, 1, \ldots, (1-k)$.

By Taylor's theorem,

$$v(\theta) = \sum_{i=0}^{(1-k)} v^i(\pi)\ (\theta - \pi)^i/i! + O((\theta - \pi)^{(2-k)}) \quad \text{and}$$

$$w(\theta) = \sum_{i=0}^{(1-k)} w^i(\pi)\ (\theta - \pi)^i/i! + O((\theta - \pi)^{(2-k)}),$$

for $\theta_0 \leq \theta \leq \pi$. Therefore $(v(\theta) - w(\theta)) = O((\theta - \pi)^{(2-k)})$, so

$$(v(\theta) - w(\theta)) \sin^{k-2}(\theta) = O(1).$$

The convergence of $\mu_k(A)$ follows.

6. Equichordal problems.

Suppose E is a set which is starshaped at an interior point p. If each chord $[xy]$ of E through p satisfies

(i) $\|x - p\|^k + \|y - p\|^k = \alpha \quad (k \neq 0)$, or (ii) $\|x - p\| \cdot \|y - p\| = \alpha \ (k = 0)$,

we say that p is a k-equipower point of E. The constant α is called an equipower constant.

If $k = 1, 0,$ or $-1,$ the k-equipower point p is also known as an equichordal, equiproduct (or power), or equireciprocal point of E, respectively.

Note that if p is a k-equipower point of E then the chord function f_p^k of E is constant.

The existence of a convex body K with two equichordal points p_1, p_2 is the subject of the famous unsolved equichordal problem (see, for example, [Kl]). If a starshaped E exists with two equichordal points, the associated equipower constants must clearly be equal.

THEOREM 5. There is at most one starshaped set E with given equichordal points p_1, p_2, and given equipower constant α.

This was proved for convex bodies in [D]. Theorem 5 must be attributed to E. Wirsing ([W]), although he does not explicitly state it. In [W], it is proved that ∂E must be analytic, and uniqueness follows from Theorem 4(i).

Equiproduct points are studied in [Y] and [K], for example. Every point interior to a circle is an equiproduct point. Such points have different equipower constants according to their position inside the circle. In [Y] and [K] it is shown that any set E starshaped at two equiproduct points p_1, p_2, such that ∂E is differentiable, must be a circle.

The paper $[F_3]$ of K. Falconer deals with equireciprocal points. Each focus of an ellipse is an equireciprocal point. If E is starshaped at two equireciprocal points p_1, p_2, and if E is convex or ∂E is twice differentiable, then the corresponding equipower constants are equal; also, if ∂E is twice differentiable, then ∂E must be an ellipse with foci at p_1 and p_2. Finally, Falconer exhibits non-elliptical convex bodies with once differentiable boundaries and two equireciprocal points.

<u>EXAMPLE 2</u>. Let H, K be two congruent circular discs whose boundaries ∂H, ∂K intersect at exactly two points a, b. Let p_1, p_2 be any two points on the line ℓ through a and b, which are interior to H and K. Then p_1 and p_2 are equiproduct points of H and K, and the corresponding equipower constants α_1 and α_2 are the same for both H and K. So, H and K have equal 0-chord functions at p_1 and at p_2. This shows that the condition $\mu_k(H\Delta K) < \infty$ is necessary in Theorems 1, 2 and 3, for $k = 0$. It also shows that it is necessary to specify the derivative at a or b, as in Theorem 4(ii), for uniqueness.

<u>EXAMPLE 3</u>. Let H be an ellipse (with interior) with foci p_1, p_2. A non-elliptical convex body K with equireciprocal points p_1, p_2, can be constructed as in [F_3], as mentioned above. Furthermore, the corresponding equipower constants α_1 and α_2 are the same for H and K, so H and K have equal -1-chord functions at p_1 and p_2. So, $\mu_k(H\Delta K) < \infty$ is necessary in Theorems 1 and 3 for $k = -1$; further, some higher order differentiability condition such as that in Theorem 4(ii) is necessary, since ∂H and ∂K are both differentiable.

The order of differentiability assumed in Theorem 4(ii) is one higher than that needed for the above uniqueness results for the equiproduct and equireciprocal points. Of course, Theorem 4(ii) applies to quite general k-chord functions, not just to constant ones; but we do not know if the extra order is necessary.

7. Chord functions in the projective plane.

It is profitable to study chord functions of convex bodies in the projective plane $\mathbb{P}^2$. Here, we regard $\mathbb{P}^2$ as $\mathbb{R}^2$ with a line at infinity adjoined, and by a convex body in $\mathbb{P}^2$ we simply mean a convex body in $\mathbb{R}^2$.

A point at infinity corresponds to a direction in $\mathbb{R}^2$, and we define the chord function at such a point to be the same as that in the corresponding direction (see [GM] and [V]). Such a chord function f_s of a set K gives the lengths of chords of K parallel to the given direction s; specifically $f_s(t)$ is the length of the chord $K \cap \{(x, y): y = t\}$ in a coordinate system where the x-axis has direction s.

THEOREM 6. Suppose that p_i, $1 \le i \le n$, are distinct collinear points in $\mathbb{P}^2$, such that $\{p_i: 1 \le i \le n\}$ is not projectively equivalent to a subset of the directions of the diagonals of a regular polygon. If H and K are convex bodies in $\mathbb{P}^2$ with the same chord functions at p_i, $1 \le i \le n$, then $H = K$.

Proof. Theorem 1 holds when $k = 1$ and $\mathbb{R}^2$ is replaced by $\mathbb{P}^2$ (so that either p_1, or p_2, or both may be points at infinity); see [V], Theorem 4.1. When all the points p_i are at infinity, the proof is given in [GM]. Otherwise, at most one point may be at infinity. Now the proof follows that of Theorem 2, once we note that $p_i(A)$ or $p_i^{-1}(A)$ have the same μ_0- or μ_1-measure as A, even if p_i is at infinity (see [V], Lemma 3.4, for $k = 1$; the proof for $k = 0$ is similar).

Theorem 6 generalizes the relevant direction of the main theorem of [GM] and complements Theorem 1.2 of [V], which applies to non-collinear sets of points. The next example is well known.

EXAMPLE 4. ([GM]) Let H be a regular n-gon, and let K be obtained by rotating H by π/n about its center. Then H and K have equal chord functions at n points p_i at infinity, corresponding to the directions of the diagonals of the convex hull of $H \cap K$. This shows Theorem 6 is best possible for points at infinity.

EXAMPLE 5. Let H and K be as in Example 4. Let ℓ be a line, not at infinity, which does not meet $H \cup K$. Let Φ be a projective transformation which takes ℓ onto the line at infinity, and vice versa. It is not hard to check that $\Phi(H)$ and $\Phi(K)$ are convex polygons which have equal -1-chord functions from the n points $\Phi(p_i)$, $i = 1, \ldots, n$, on ℓ, where p_i is as in Example 4. This shows that Theorem 2 is best possible for $k = -1$.

Example 5 results from a duality between chord functions from points at infinity and -1-chord functions from finite points, which we shall not pursue here.

8. Open questions.

The uniqueness aspect of Hammer's X-ray problems, rephrased as problems concerning chord functions, is now fairly well understood. The known results only leave open the possibility that a certain class of examples exists. We shall describe these now.

Possible examples involving two different convex bodies with the same chord functions at two points on a line which meets both bodies were detailed in the paragraph preceding Example 1. These may be viewed as degenerate cases of the following situation.

Suppose $n \geq 2$, and Q is a regular 2n-gon, and let $s_1, \ldots, s_n$ be the directions of the diagonals of Q. Let Φ be a nonsingular projective transformation taking the line at infinity onto a different line ℓ, so that $p_i = \Phi(s_i)$ is a point on ℓ for each i, and let $P = \Phi(Q)$. Now it is easy to construct distinct convex bodies H and K, such that $\partial H \cap \partial K$ is precisely the set of vertices of P. We are unable to decide whether H and K can also have the same chord functions at $p_1, \ldots, p_n$. Further, if c is the center of P, and n is odd, it is conceivable that H and K could also have the same chord functions at $p_{n+1} = \Phi(c)$.

For general k, many questions remain, including that of the existence of convex bodies with two k-equipower points for $k \neq 0, -1$.

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