King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 086

October 1986

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THE HILBERT TRANSFORM OF SCHWARTZ DISTRIBUTIONS II

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Received 9th June 1986

ABSTRACT

Let D(R) be the Schwartz space of C^∞ functions with compact support on R and let H(D) be the space of all C^∞ functions defined on R for which every element is the Hilbert transform of an element in D(R) i.e., $H(D) = \{\psi:\psi(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\, dt = H[\phi](x);\ \phi \in D(R)\}$, where the integral is defined in the Cauchy Principal-Value sense. Introducing an appropriate topology in H(D), Pandey [10] defined the Hilbert transform Hf of $f \in (D(R))'$ as an element of $(H(D))'$ by the relation:

$$\langle Hf,\phi\rangle = \langle f,-H\phi\rangle \quad \forall\ \phi \in H(D),$$

and then with an appropriate interpretation he proved that

$$\left(-\frac{1}{\pi^2}\right)H^2 f = f \quad \forall\ f \in (D(R))'$$

However, he did not describe the space H(D) and its topology in an intrinsic way. In this paper we give an intrinsic description of the space H(D) and its topology, thereby providing a solution to an open problem posed by Pandey [11, p. 90].

1980 AMS Subject Classification

Primary: 46 F12

Secondary: 44 A15

Keywords and Phrases: The Hilbert transform of generalized functions, Schwartz distributions.

INTRODUCTION

The space H(D) consists of C^∞ functions on R such that $\psi \in H(D)$ iff there exists $\phi \in D(R)$, the space of C^∞-functions with compact support, satisfying

$$\psi(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)dt}{t-x} . \tag{1}$$

It is shown in [10, p. 482] that

$$\psi^{(k)}(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-x}\, dt = \text{p.v.} \int_{-a}^{a} \frac{\phi^{(k)}(t)}{t-x}\, dt \tag{2}$$

where, the support of ϕ is contained in [-a,a]. When $x = \pm a$, the integral on the right hand side of (2) is interpreted as an improper integral. Clearly

$$\psi^{(k)}(x) = O(\ell n|\tfrac{x-a}{x+a}|) \quad \text{as} \quad |x| \to \infty. \tag{3}$$

If we assume ψ to be a C^∞ function defined on R satisfying the asymptotic order (3), does it necessarily imply that $\psi \in H(D)$? The answer is no, as it is very simple to construct counter examples. However, perhaps by adding some other conditions on ψ, we may be able to show it. But it is quite a subtle question. We will give an intrinsic description of the space H(D) by using the distributional representation of analytic functions.

An infinitely differentiable function $\phi(x)$ $(-\infty < x < \infty)$ is said to belong to the testing functions space $D_{L_p}(R)$ iff

$$\gamma_m(\phi) = \left(\int_{-\infty}^{\infty} |\phi^{(m)}(x)|^p \, dx\right)^{\frac{1}{p}} < \infty, \quad m = 0,1,2,3,\ldots \quad . \tag{4}$$

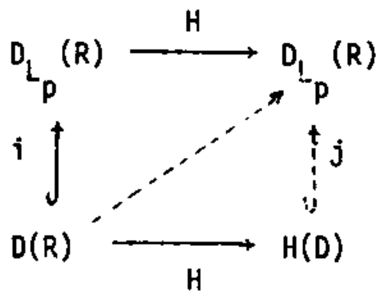
Since γ_0 is a norm, the sequence of semi-norms $\{\gamma_m\}_{m=0}^{\infty}$ is separating [15, p. 8]. The space $D_{L_p}(R)$ is a complete countably multi-normed space and $D(R)$ is dense in it [12, p. 199]. It is proved in [10, Th. 1] that the Hilbert transform $H: D_{L_p}(R) \to D_{L_p}(R)$ defined by

$$\phi \stackrel{H}{\longmapsto} \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} \, dt \tag{5}$$

is a linear homeomorphism with its inverse given by

$$H^{-1}\phi = \left(-\frac{1}{\pi^2}\right)H\phi \quad \forall \quad \phi \in D_{L_p}(R) \ . \tag{6}$$

Consider the following diagram:



Since $D(R)$ is dense in $D_{L_p}(R)$, it follows that the space $H(D)$, with the subspace topology on it, is dense in $D_{L_p}(R)$. We have not yet described the space $H(D)$ and its topology in an intrinsic way.

In [10] Pandey and Chaudhry developed the theory of the Hilbert transform of Schwartz distribution space $(D_{L_p})'$, $p > 1$, which coincides with the corresponding theory for the Hilbert transform developed by Schwartz [12] by using the technique of convolution. However, the technique used by Pandey and Chaudhry in [10] is much simpler and can easily be used by applied scientists. In [11] Pandey extended the Hilbert transform to Schwartz distribution space D' but he did not describe the space H(D) and its topology in an intrinsic way. The object of this paper is to describe the space H(D) and its topology in an intrinsic way by a method anaolgous to that used by Ehrenpreis [7] for the extension of the Fourier transform to the Schwartz distribution space D'. It may however be noted that the inverse Fourier transform of $\phi \in D$ can be extended as an entire function, whereas the Hilbert transform of $\phi \in D$ cannot be extended as an entire function. This is due to the singularity of its kernel, but it can be extended as a holomorphic function $\psi(z)$ which is analytic outside the support of ϕ.

Before we prove the main theorem we will prove some lemmas which will be used in the sequel.

Lemma 1. Let $\{\phi_\nu\}_{\nu=1}^{\infty}$ be a sequence of functions tending to zero in $D_{L_p}(R)$ as $\nu \to \infty$, i.e.,

$$\gamma_k(\phi_\nu) \to 0 \text{ as } \nu \to \infty \text{ for each } k = 0,1,2,\ldots$$

then for each $k = 0,1,2,3,\ldots$

$$\phi_\nu^{(k)}(x) \to 0 \text{ as } \nu \to \infty \text{ uniformly } \forall\, x \in R.$$

Proof: This result is proved in [1] and [12]. A very simple proof can be given as follows. For $\delta \in (D_{L_p}(R))'$ we have

$$\phi^{(k)}(x) = \langle \delta(t), \phi^{(k)}(x-t) \rangle \ \forall\, \phi \in D_{L_p}(R) . \tag{7}$$

Now, there exists a constant $c > 0$ and a non-negative integer r satisfying

$$|\langle\delta(t),\phi^{(k)}(x-t)\rangle| \le c\gamma_r'(\phi^{(k)}(x-t)) \qquad [14, pp. 8-19]$$

or

$$|\phi^{(k)}(x)| \le c\gamma_r'(\phi^{(k)}(t)) , \tag{8}$$

where

$$\gamma_r'(\phi) = \mathrm{Max}(\gamma_0(\phi),\gamma_1(\phi),\ldots,\gamma_r(\phi)) \text{ and } \gamma_0'(\phi) = \gamma_0(\phi).$$

Therefore

$$|\phi_\nu^{(k)}(x)| \le c\gamma_r'(\phi_\nu^{(k)}) \to 0 \text{ as } \nu \to \infty$$

i.e., for each $k = 0,1,2,\ldots$

$$\phi_\nu^{(k)}(x) \to 0 \text{ as } \nu \to \infty \text{ independently of } x.$$

Lemma 2. Let $\phi(t) \in D$. Then as $y \to 0^+$

$$\text{(i)} \quad \int_{-\infty}^{\infty} \frac{\phi(t)(t-x)}{(t-x)^2+y^2}\, dt \ \to \ \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\, dt$$

and

(ii) $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t)(y-x)}{(t-x)^2+y^2}\, dt \rightarrow \phi(x)$

in $D_{L_p}(R)$, $p > 1$.

Proof: For the proof see [10, p. 487] and also [13, p. 136].

Lemma 3. Let $\phi(t) \in D$. Then; as $y \rightarrow 0^+$

(i) $\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}\, dt \rightarrow \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\, dt + i\pi\phi(x)$

uniformly $\forall\, x \in R$, and as $y \rightarrow 0^-$

(ii) $\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}\, dt \rightarrow \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\, dt - i\pi\phi(x)$

uniformly $\forall\, x \in R$.

Proof: We have

$$\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}\, dt = \int_{-\infty}^{\infty} \frac{\phi(t)(t-x)}{(t-z)^2+y^2}\, dt + i \int_{-\infty}^{\infty} \frac{y\phi(t)}{(t-x)^2+y^2}\, dt\ . \qquad (9)$$

Now the results (i) and (ii) follow using Lemmas 1 and 2.

AN INTRINSIC DEFINITION OF THE SPACE H(D) AND ITS TOPOLOGY

Definition: A function $\psi(z)$ defined on the complex plane belongs to the space Ψ iff the following four properties hold

(P_1): $\psi(z)$ is analytic outside some closed interval [a,b], depending upon ψ;

(P_2): $\psi^{(k)}(z) = \mathcal{O}(\frac{1}{|z|})$, $|z| \rightarrow \infty$, for each fixed $k = 0,1,2,\ldots$;

(P_3): (a) For each fixed $k = 0,1,2,\ldots$, $\psi^{(k)}(x+iy)$ converges uniformly $\forall\, x \in R$ as $y \to 0^+$;

(b) For each fixed $k = 0,1,2,\ldots$, $\psi^{(k)}(x+iy)$ converges uniformly $\forall\, x \in R$ as $y \to 0^-$;

(P_4):

$$\psi(x) = \frac{\psi_+(x)+\psi_-(x)}{2},$$

where

$$\psi_+(x) = \lim_{y\to 0^+}\psi(x+iy), \qquad \psi_-(x) = \lim_{y\to 0^-}\psi(x+iy).$$

Theorem 1. A necessary and sufficient condition that a function $\psi(z)$ defined on the complex plane belongs to the space Ψ is that there exists a function $\phi(t) \in D$ satisfying

$$\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}\,dt, \quad \text{Im } z \neq 0,$$

$$= \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\,dt, \quad \text{Im } z = 0.$$

Proof: Necessity: If $\psi(z) \in \Psi$, then in view of the properties (P_1) and (P_2), $\psi(x+iy)$ as a function of x belongs to $D_{L_p}(R)$ for a fixed $y \neq 0$. In view of the properties (P_1) and (P_2) it follows that if $\{y_n\}_{n=1}^{\infty}$ is an arbitrary sequence of positive real numbers tending to zero then

$$||\psi^{(k)}(x+iy_m) - \psi^{(k)}(x+iy_n)||_p \to 0 \quad \text{as} \quad m,n \to \infty$$

independently of each other.

Therefore, $\{\psi(x+iy_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $D_{L_p}(R)$, $p > 1$.

Since $D_{L_p}(R)$ is complete it follows that there exist a function $\psi_+(x) \in D_{L_p}(R)$ such that $\lim_{n\to\infty} \psi(x+iy_n) = \psi_+(x)$ in $D_{L_p}(R)$, $p > 1$. Since $\{y_n\}$ is an arbitrary sequence of positive numbers tending to zero, it follows that there exists a function $\psi_+(x) \in D_{L_p}(R)$ such that

$$\lim_{y\to 0^+} \psi(x+iy) = \psi_+(x) \text{ in } D_{L_p}. \tag{10}$$

Similarly, we can show that there exists a function $\psi_-(x) \in D_{L_p}(R)$ such that

$$\lim_{y\to 0^-} \psi(x+iy) = \psi_-(x) \text{ in } D_{L_p}. \tag{11}$$

Now using (P_4), (10) and (11) it follows that

$$\psi(x) = \frac{\psi_+(x) + \psi_-(x)}{2} \in D_{L_p},\ p > 1.$$

From Lemma 1, (10) and (11) it follows that

$$\lim_{y\to 0^+} \psi(x+iy) = \psi_+(x) \text{ uniformly } \forall\, x \in R$$

and

$$\lim_{y\to 0^-} \psi(x+iy) = \psi_-(x) \text{ uniformly } \forall\, x \in R.$$

Since $\psi(z)$ is analytic outside a closed interval $[a,b]$, it follows that $\psi_+(x) - \psi_-(x) = 0$ outside $[a,b]$ and therefore belongs to D. Using Cauchy's integral theorem and the technique used in [10], it can be shown that for $\epsilon > 0$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(t+i\varepsilon)}{t-z}\,dt = \psi(z+i\varepsilon),\ \text{Im } z > 0, \tag{12}$$
$$= 0,\ \text{Im } z < 0.$$

Letting $\varepsilon \to 0^+$ in (12), we deduce that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_+(t)}{t-z}\,dt = \psi(z),\ \text{Im } z > 0, \tag{13}$$
$$= 0,\ \text{Im } z < 0.$$

Similarly, we can show that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_-(t)}{t-z}\,dt = -\psi(z),\ \text{Im } z < 0, \tag{14}$$
$$= 0,\ \text{Im } z > 0.$$

Combining (13) and (14) we obtain

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_+(t)-\psi_-(t)}{t-z}\,dt = \psi(z),\ \text{Im } z \neq 0. \tag{15}$$

Let $\phi(t) = (\psi_+(t) - \psi_-(t))/2\pi i$. Clearly $\phi(t) \in D$ and thus

$$\psi(z) = \int_{-\infty}^{\infty}\frac{\phi(t)}{t-z}\,dt,\ \text{Im } z \neq 0. \tag{16}$$

In view of Lemmas 2 and 3 and property (P_4), it follows that

$$\psi(x) = \text{p.v.}\int_{-\infty}^{\infty}\frac{\phi(t)}{t-x}\,dt,$$

that is,

$$\psi(z) = \text{p.v.}\int_{-\infty}^{\infty}\frac{\phi(t)}{t-x}\,dt,\ \text{Im } z = 0. \tag{17}$$

The proof for necessity follows from (16) and (17).

Sufficiency: If there exists a function $\phi \in D$, which vanishes outside a closed interval, satisfying

$$\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}\, dt, \ \text{Im } z \neq 0,$$

$$= \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}\, dt, \ \text{Im } z = 0,$$

then, as proved in [10],

$$\psi^{(k)}(z) = \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-z}\, dt, \ \text{Im } z \neq 0,$$

$$= \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-x}\, dt, \ \text{Im } z = 0.$$

Clearly ψ satisfies (P_1), (P_2). The conditions (P_3) and (P_4) are also satisfied in view of Lemma 3. This completes the proof of Theorem 1.

Theorem 1 shows that there is a one to one correspondence between the space Ψ and the space H(D). We can therefore define the space H(D) in a genuinely intrinsic way as follows:

A C^{∞} function $\psi(x)$ belongs to H(D) iff there exists a holomorphic function $\psi(z)$ satisfying (P_1), (P_2), (P_3) and (P_4). In other words $\psi(x) \in H(D)$ iff it is the average of the upper and lower limit of a holomorphic function satisfying conditions (P_1), (P_2) and (P_3). That is, $\psi(x)$ can be extended uniquely as a holomorphic function satisfying (P_1), (P_2), (P_3) and (P_4).

The convergence of a sequence $\{\psi_\mu(x)\}_{\mu=1}^{\infty}$ to zero in H(D) can be defined in an intrinsic way as follows:

A sequence $\{\psi_\mu\}_{\mu=1}^{\infty}$ in H(D) converges to zero in H(D) iff

(i) the associated functions $\psi_\mu(z)$ in accordance with Theorem 1 are analytic outside a closed interval [a,b] or else $\psi_\mu(x)$ is analytic outside a fixed closed interval [a,b],

(ii) $\psi_\mu(x) \to 0$ in D_{L_p} as $\mu \to \infty$.

Clearly if $\{\phi_\mu(x)\}_{\mu=1}^{\infty}$ is a sequence in D tending to zero in D as $\mu \to \infty$ and

$$\psi_\mu(x) = \text{p.v.} \int_{\infty}^{\infty} \frac{\phi_\mu(t)}{t-x}\, dt, \tag{18}$$

and

$$\psi_\mu(z) = \int_{-\infty}^{\infty} \frac{\phi_\mu(t)}{t-z}\, dt, \ \text{Im } z \neq 0, \tag{19}$$

then $\psi_\mu(z)$ is analytic outside the closed interval [a,b] [see [6]] and

$$||\psi_\mu^{(k)}||_p \leq C_p ||\phi_\mu^{(k)}||_p \to 0 \text{ as } \mu \to \infty.$$

Therefore, (i) and (ii) are satisfied.

If (i) and (ii) are assumed then there exists a closed interval [a,b] containing the supports of all $\phi_\mu(x)$. From (18), (19) and the fact that $-\frac{1}{\pi^2} H^2 = I$, it follows that

$$\phi_\mu(x) = -\frac{1}{\pi^2}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{\psi_\mu(t)}{t-x}\, dt.$$

Therefore we have

$$||\phi_\mu^{(k)}(x)||_p \leq \frac{1}{\pi^2} C_p||\psi_\mu^{(k)}||_p \to 0 \text{ as } \mu \to \infty,$$

i.e., $\phi_\mu(x) \to 0$ in D_{L_p} as $\mu \to \infty$.
Therefore, by Lemma 1, $\phi_\mu(x) \to 0$ uniformly $\forall\, x \in R$ as $\mu \to \infty$. By (i) all $\phi_\mu(x)$ have supports contained in a fixed interval [a,b]. Therefore if $\{\psi_\mu(x)\}_{\mu=1}^{\infty} \to 0$ in H(D) as $\mu \to \infty$ then the associated sequence $\{\phi_\mu\}_{\mu=1}^{\infty}$ tends to zero in D as $\mu \to \infty$. Thus we have proved that

$$\phi_\mu \to 0 \text{ in } D \text{ as } \mu \to \infty \iff \psi_\mu \to 0 \text{ in } H(D) \text{ as } \mu \to \infty.$$

Thus the conditions (i) and (ii) together describe intrinsically the convergence of a sequence $\{\psi_\mu\}_{\mu=1}^{\infty}$ to zero in H(D) as $\mu \to \infty$.

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