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ESTIMATION OF THE DERIVATIVES OF PROBABILITY FUNCTION AND MODE FOR UNIFORM MIXING PROCESS

by

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Abstract

This paper is concerned with the nonparametric estimation of the derivatives of the probability density function (p.d.f.) and mode. Using the kernel estimates and their derivatives proposed estimates of the derivatives $f^{(r)}(x)$, $r = 1,\dots,s$, of p.d.f. $f(x)$ given by

$$f_n^{(r)}(x) = \frac{1}{n\, a_n^{r+1}} \sum_{j=1}^{n} K^{(r)} \left(\frac{x - X_j}{a_n}\right)$$

where $K^{(r)}(y)$ is the r^{th} derivative of the known probability function $K(y)$. These estimates have been extended to the uniform mixing condition (u.m.c.). It is shown that they are consistent and asymptotically normal. It is also shown that the mode is consistent and asymptotically normal.

AMS Subject Classification: 62G05

Keywords: Fourier Transform, Consisting, Nonparametric Estimation, Derivatives, Mode and Uniform Mixing.

1. Introduction

The problem of nonparametric estimation of the stationary density function has been considered by several authors, e.g. Rousas (1969a,1969b), Roselblatt (1970,1971). These Parzen kernel type estimates are proved to be consistent and asymptotically normal under certain conditions, especially the Doeblin's condition [see Rousas (1969a)], G_2 condition, Rosenblatt or uniform mixing condition (u.m.c.) [see Abdulal and Siddiqui (1986)].

The kernel estimates of the derivatives of the p.d.f. have been investigated by Bhattacharayya (1967), Schuster (1971) and Samanta and Mugisha (1981) in the case of independent identically distributed (iid) observations. The estimates of the mode have been studied by Parzen (1962) and Samanta and Mugisha (1981). They proved that they are consistent (we k as well as strong) and normally distributed based on a sample of iid observation.

In this paper, we are mainly concerned with the estimation from the point of view of consistency and asymptotic normality. We shall look at the consistency of $f_n^{(r)}(x)$, $r = 0,1,\ldots,$ and the asymptotic normality of $f_n^{(1)}(x)$ then apply these results to study the asymptotic properties of the mode.

The asymptotic properties of the kernel estimates of p.d.f. alone under the u.m.c. have been studied by Abdulal and Siddiqui (1986).

2. Estimation of $f_n^{(r)}(x)$, $r = 1,\dots,s$

Let $\{X_n\}$ be a stationary sequence of random variables (r.v.'s) defined on the probability space (Ω, B, P). For $m \le n$ define $\sigma(m,n)$ as the σ-field generated by the r.v.'s $X_m,\dots,X_n$ and define $\sigma(m,\infty)$ as the σ-field generated by $X_m, X_{m+1}, \dots$. We shall say that the sequence $\{X_n\}$ is uniform mixing if, for each m $(m \ge 1)$ and for each n $(n \ge 1)$, $A \in \sigma(1,m)$ and $B \in \sigma(m+n, \infty)$ together imply that

$$|P(AB) - P(A)\,P(B)| \le \alpha(n)\,P(A)$$

where $\alpha(n)$, $n=1,2,\dots,$ is a nonnegative function of integers such that $\alpha(n) \downarrow 0$ as $n \to \infty$.

In this paper we consider the estimates $f_n^{(r)}(x)$ of the r^{th} derivative $f^{(r)}(x)$, $r = 0,1,\dots,$ based on the first n observations and given by

$$f_n^{(r)}(x) = \sum_{i=1}^{n} \frac{1}{n\,a_n} K^{(r)}\left(\frac{x - X_i}{a_n}\right) \tag{1}$$

where $K^{(r)}(u)$ is the r^{th} derivative of p.d.f. $K(u)$ and a_n is a monotonically decreasing sequence of positive numbers converging to zero. We assume that for some integer $s \ge 0$ the functions $K(u)$ and $K^{(r)}(u)$ satisfy the following conditions:

$$\begin{aligned}
&\text{(i)} && \lim_{|u|\to\infty} |u\,K^{(r)}(u)| = 0 , && r = 0,1,\dots,s \\
&\text{(ii)} && \lim_{|u|\to\infty} |K^{(m)}(u)| = 0 , && m = 0,\dots,r \\
&\text{(iii)} && \sup_{-\infty<u<\infty} |K^{(m)}(u)| < \infty , && m = 0,\dots,r+1
\end{aligned} \tag{2}$$

(iv) $K(u)$ and its first $(r+1)$ derivatives are bounded.

We define

$$\phi(t) = \int e^{itx} f(x)\, dx$$

$$\phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} \tag{3}$$

$$k(t) = \int e^{itx} K(x)\, dx$$

We now assume for some integer $r \geq 0$ the following conditions on $\phi(t)$, $k(u)$, $k(t)$ and the sequence $\{a_n\}$ are satisfied

$$\int |t^r||k(t)|dt < \infty \tag{4}$$

$$\int |t^{m+1}||\phi(t)|dt < \infty, \quad m = 0,1,\ldots,r \tag{5}$$

$$\lim_{n\to\infty} n\, a_n^{2(r+1)} = \infty \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n\, a_n^{2+2r})^2} < \infty \tag{7}$$

We note that condition (5) implies that the first $(r+1)$ derivatives of $f(x)$ are bounded.

The following Lemma will play a central role in studying the asymptotic properties of $f_n^{(r)}(x)$.

<u>Lemma 1</u>. Let $\{X_n\}$ be a uniform mixing sequence of r.v.'s. Let η_1 and η_2

be two r.v.'s measurable with respect to $\sigma(1,m)$ and $\sigma(m+n,\infty)$, respectively.

(a) If $p > 1$ and $q > 1$, are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $E|\eta_1|^p < \infty$ and $E|\eta_2|^q < \infty$, then

$$|E[\eta_1\eta_2] - E[\eta_1]E[\eta_2]| \leq 2\{\alpha(n)\ E|\eta_1|^p\}^{1/p}\ E|\eta_2|^q\}^{1/q}$$

(b) If $|\eta_i| \leq c_i$ a.s. $i = 1,2,$ then

$$|E[\eta_1\eta_2] - E[\eta_1]E[\eta_2]| \leq 2\alpha(n)c_1c_2$$

Proof. Can be found in Billingsley (1968), pp.170-171.

Remark 1. If the r.v.'s η_1 and η_2 are complex, then separating the real and imaginary parts, we again arrive at part (b) with 2 replaced by 4.

First we examine, for some integer $s \geq 0$, the convergence of the random variables $W_n^{(r)}(x) = \sup_{-\infty<X<\infty} |f_n^{(r)}(x) - f^{(r)}(x)|$ $r = 0,1,\ldots,s.$

Lemma 2. Assume the following conditions hold

(i) Condition in (2)

(ii) Conditions in (4) and (6)

(iii) $f^{(r)}(x)$ is uniformly continuous, $r = 0,1,\ldots,s$ and

(iv) $\sum_{j=1}^{\infty} \alpha(j) < \infty$

then

$$W_n^{(r)}(x) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty \tag{8}$$

Proof. For $r = 0$, the proof has been given by Abdulal and Siddiqui (1986).

Since $K^{(r)}$ and $t^r k(t)$ are absolutely integrable, we have

$$K^{(r)}(y) = \frac{(-i)^r}{2\pi} \int t^r \, e^{-ity} \, k(t) \, dt.$$

In terms of $k(t)$, we have

$$f_n^{(r)}(x) = \frac{(-i)^r}{2\pi} \int t^r \, e^{-itx} \, \{\phi_n(t) \, k(a_n t)\} dt$$

Hence,

$$|f_n^{(r)}(x) - Ef_n^{(r)}(x)| = \left|\frac{(-i)^r}{2\pi} \int t^r \{\phi_n(t) - \phi(t)\} k(a_n t) e^{-itx} \, dt\right|$$

Therefore,

$$\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)| \le \frac{1}{2\pi} \int |t^r| \, |\phi_n(t) - \phi(t)| \, |k(a_n t)| dt$$

It follows from Fubinis Theorem and Schwartz' inequality that

$$\begin{aligned} E[\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)|] &\le \frac{1}{2\pi} \int \{E|\phi_n(t) - \phi(t)|^2 \, |t^r k(a_n t)|^2\}^{\frac{1}{2}} dt \\ &= \frac{1}{2\pi} \int \{\sigma^2[\phi_n(t)]\}^{\frac{1}{2}} \{t^r |k(a_n t)|^2\}^{\frac{1}{2}} dt \end{aligned} \tag{9}$$

By stationarity, the fact that $|e^{itX}| = 1$, and Remark 1 of Lemma 1, we have

$$\sigma^2[\phi_n(t)] = \frac{1}{n} E|e^{itX} - \phi(t)|^2 + \frac{2}{n^2} \sum_{j=2}^{n} (n-j+1)\ E[e^{itX_1} - E\ e^{itX_1}] \times$$

$$\times \overline{[e^{itX_j} - e^{itX_j}]} \qquad (10)$$

$$= \frac{1}{n}[1 - |\phi(t)|^2] + \frac{2}{n^2} \sum_{j=2}^{n} (n-j+1) E[e^{it(X_1 - X_j)} - E\ e^{itX_1}\ E\ e^{-itX_j}]$$

$$\leq \frac{1}{n} + \frac{8}{n} \sum_{j=1}^{\infty} \alpha(j) = \frac{1}{n}[1 + 8 \sum_{j=1}^{\infty} \alpha(j)]$$

From (9) and (10), we have

$$E[\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E_n^{(r)}(x)|]$$

$$\leq \frac{1}{2\pi\sqrt{n}} [1 + 8 \sum_{j=1}^{\infty} \alpha(j)][\int |t^r\ k(a_n t)| dt$$

$$= \frac{1}{2\pi\sqrt{n}\ a_n^{r+1}} [1 + 8 \sum_{j=1}^{\infty} \alpha(j)][\int |t^r\ k(t)| dt$$

By applying (4), (6) and condition (iv), we have

$$\lim_{n\to\infty} E[\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\ f_n^{(r)}(x)|] = 0$$

Markov's inequality implies that

$$\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\ f_n^{(r)}(x)| \xrightarrow{P} 0 \qquad (n \to \infty) \qquad (11)$$

Since $f(x)$ and its first $(r+1)$ derivatives are bounded and

$$K^{(m)}\left(\frac{x-u}{a_n}\right) \to 0 \quad \text{for } m = 0,\ldots,r$$

as $u \to \infty$ or $-\infty$, Bhattacharayya (1967) has proved integrating by parts that

$$\frac{1}{a_n^{r+1}} \int K^{(r)}\left(\frac{x-u}{a_n}\right) f(u)du = \frac{1}{a_n} \int K\left(\frac{x-u}{a_n}\right) f^{(r)}(u)du$$

Then from Lemma 3.3 in Abdulal (1986), we have

$$\begin{aligned} &\sup_{-\infty<x<\infty} |E\, f_n^{(r)}(x) - f^{(r)}(x)| \\ &= \sup_{-\infty<x<\infty} \left|\int \frac{1}{a_n^{r+1}} K^{(r)}\left(\frac{x-u}{a_n}\right) f(u)du - f^{(r)}(x)\right| \qquad (12) \\ &= \sup_{-\infty<x<\infty} \left| \frac{1}{a_n} \int K\left(\frac{x-u}{a_n}\right) f^{(r)}(u)du - f^{(r)}(x)\right| \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

and $f^{(r)}(x)$ is uniformly continuous.

In the inequality

$$W_n^{(r)}(x) \le \sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)| + \sup_{-\infty<x<\infty} |E\, f_n^{(r)}(x) - f^{(r)}(x)|$$

the right-hand side converges to 0 as $n \to \infty$.

Lemma 3. Assume the following conditions hold

(a) condition (i) and (iii) of Lemma 2,

(b) conditions (4), (5) and (6),

(c) $\sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j) < \infty$

Then

$$W_n^{(r)}(x) \xrightarrow{W.P.1} 0 \qquad \text{as} \quad n \to \infty \tag{13}$$

Proof. For $r = 0$, the proof has been given in Abdulal and Siddiqui (1986).

From (9) we have

$$\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)| \leq \frac{1}{2\pi} \int |\phi_n(t) - \phi(t)||t^r k(a_n t)|dt$$

From Schwartz' inequality, and Fubini's theorem that

$$E \sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)|^4$$

$$\leq \frac{1}{16\pi^4} \iiiint \prod_{i=1}^{4} |t_i^r||k(a_n t_i)||E \prod_{i=1}^{4} |\phi_n(t_i) - \phi(t_i)|dt_i$$

$$\leq \frac{1}{16\pi^4} \iiiint \prod_{i=1}^{4} |t_i^r||k(a_n t_i)|E^{\frac{1}{4}}|\phi_n(t_i) - \phi(t_i)|^4 dt_i$$

From (3.35) in Abdulal and Siddiqui (1986), we have

$$E[\sup_{-\infty<x<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)|^4] \leq \frac{768}{\pi^4 n^2} [\sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j)]^2 [\int |t^r||k(a_n t)|dt]^4$$

$$= \frac{768}{\pi^4 n^2\, a_n^{(2+2r)}} [\sum_{j=1}^{\infty} (\alpha^{\frac{1}{2}}(j))]^2 [\int |t^r||k(t)|dt]^4$$

and by Markov's inequality, we have

$$P[\sup_{-\infty<X<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)|^4 > \varepsilon]$$

$$\leq \frac{768}{\pi^4 \varepsilon^4 n^2\, a_n^{2(2+2r)}} [\sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j)]^2 [\int |t^r||k(t)|dt]^4$$

It follows from condition (c) and Borel-Cantelli's Lemma that

$$\sup_{-\infty<X<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)| \xrightarrow{W.P.1} 0 \quad \text{as} \quad n \to \infty. \tag{14}$$

In the inequality

$$W_n^{(r)}(x) \leq \sup_{-\infty<X<\infty} |f_n^{(r)}(x) - E\, f_n^{(r)}(x)| + \sup_{-\infty<X<\infty} |E\, f_n^{(r)}(x) - f_n^{(r)}(x)|$$

The right-hand side converges to 0 W.P.1 as $n \to \infty$ by (12) and (14).

The asymptotic behavior of the variance of $f_n^{(1)}(x)$ will be determined before its asymptotic normality.

Theorem 1. Assume the following conditions held:

(i) Condition (2),

(ii) $f^{(1)}(x)$ is continuous and bounded,

(iii) $a_n \to 0$ as $n \to \infty$,

(iv) $\sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j) < \infty$

(v) $f_j(x,y)$ (the joint density of X_1 and X_j) are continuous $\forall\, j = 2,3,\ldots,$; and $\sum_{j\neq 1} |f_j(x,y) - f(x)f(y)| \leq M < \infty \quad \forall\, x$ and y.

Then

$$\lim_{n \to \infty} n\, a_n^3 \operatorname{Var} f_n^{(1)}(x) = f(x) \int [K^{(1)}(z)]^2 dz \tag{15}$$

<u>Proof</u>. By stationarity

$$\operatorname{Var} f_n^{(1)}(x) = \frac{1}{n^2 a_n^4} \operatorname{Var} K^{(1)}\left(\frac{x - X_1}{a_n}\right) + \frac{2}{n^2 a_n^4} \sum_{j=2}^{n} (n - j + 1)$$

$$\times \operatorname{Cov}\left[K^{(1)}\left(\frac{x - X_1}{a_n}\right), K^{(1)}\left(\frac{x - X_j}{a_n}\right)\right]$$

$$= I_{n_1} + I_{n_2} \tag{16}$$

Consider I_{n_1}. Then

$$\operatorname{Var} K^{(1)}\left(\frac{x - X_1}{a_n}\right) = a_n \int [K^{(1)}(z)]^2 f(x - a_n z) dz$$

$$- a_n^2 \left[\int K^{(1)}(z) f(x - a_n z) dz\right]^2 \tag{17}$$

$$\cong a_n\, f(x) \int [K^{(1)}(z)]^2 dz$$

as $n \to \infty$ and $f(x)$ is continuous and bounded.

The terms in I_{n_2} have the following asymptotic behavior

$$\operatorname{Cov}\left[K^{(1)}\left(\frac{x - X_1}{a_n}\right), K^{(1)}\left(\frac{x - X_j}{a_n}\right)\right] =$$

$$= a_n^2 \iint K^{(1)}(z_1)\, K^{(1)}(z_2) f_j(x - a_n z_1,\, x - a_n z_2)\, dz_1 dz_2$$

$$- a_n^2 \left[\int K(z) f(x - a_n z)\, dz\right]^2 \tag{18}$$

$$\cong a_n^2 \{f_j(x,x) - f^2(x)\}\left[\int K^{(1)}(z)\, dz\right]^2$$

as $n \to \infty$ and the joint density functions $f_j(x,x)$ are continuous and bounded. We shall now get a bound on (18) under the assumption that the sequence $\{X_n\}$ satisfies the uniform mixing condition

$$\mathrm{Cov}\left[K^{(1)}\left(\frac{x - X_1}{a_n}\right),\ K^{(1)}\left(\frac{x - X_j}{a_n}\right)\right] \leq 2\left\{\alpha(j-1) E\left|K\left(\frac{x - X}{a_n}\right)\right|^4\right\}^{\frac{1}{2}}$$

$$\cong 2a_n\, \alpha^{\frac{1}{2}}(j-1) f(x) \int [K^{(1)}(z)]^2 dz \tag{19}$$

for sufficiently large n and $f(x)$ is continuous and bounded.

Inequality (19) implies that

$$\frac{1}{n} \sum_{j=2}^{n} (n - j + 1)\, \mathrm{Cov}\left[K^{(1)}\left(\frac{x - X_1}{a_n}\right),\ K^{(1)}\left(\frac{x - X_j}{a_n}\right)\right]$$

$$\leq 2\, a_n \sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j) f(x) \int [K^{(1)}(z)]^2 dz$$

and (16) - (19) indicate that

$$n\, a_n^3\, \mathrm{Var}\, f^{(1)}(x) \cong f(x) \int [K^{(1)}(z)]^2 dz + 2a_n \sum_{j=2}^{n} \left(1 - \frac{j-1}{n}\right)$$

$$\times [f_j(x,x) - f^2(x)]\left[\int K^{(z)}(z)\, dz\right]^2$$

and therefore by condition (v)

$$\lim_{n \to \infty} n\, a_n^3 \operatorname{Var} f^{(1)}(x) = f(x) \int [K^{(1)}(z)]^2 dz$$

In the remainder of this section we prove the asymptotic normality of $\sqrt{n\, a_n^3}[f_n^{(1)}(x) - E\, f_n^{(1)}(x)]$. This random variable is asymptotically normal with mean zero and variance given by (15).

Theorem 2. Assume that the conditions of Theorem 1 and the following conditions hold:

(i) $\lim_{n \to \infty} n\, a_n^3 = \infty$, and

(ii) for any pair of sequence $m = m(n)$, $p = p(n)$, such that $m, p \to \infty$ as $n \to \infty$ but $m = o(n)$, $p = o(m(n))$, and

$$\lim_{n \to \infty} n\, m^{-1} \alpha(p) = 0.$$

Then at all points x

$$\sqrt{n\, a_n^3}\, [f_n^{(1)}(x) - E\, f_n^{(1)}(x)]$$

is asymptotically normal with mean 0 and variance given by (15).

Proof. $\sqrt{n\, a_n^3}\, [f_n^{(1)}(x) - E\, f_n^{(1)}(x)] = \sum_{j=1}^{n} \frac{1}{\sqrt{n\, a_n}} [K^{(1)}(\frac{x - X_j}{a_n}) -$

$$- E\, K^{(1)}(\frac{x - X_j}{a_n})] = \sum_{q=1}^{v} A_q + \sum_{q=1}^{v+1} B_q$$

With

$$A_q = \sum_{j=(q-1)(m+p)+1}^{qm+(q-1)p} U_j \qquad q = 1,\dots,v$$

$$B_q = \sum_{j=qm+(q-1)p+1}^{q(m+p)} U_j \qquad q = 1,\dots,v$$

and

$$B_{v+1} = \sum_{n=v(m+p)+1}^{n} U_j$$

where $$U_j = \frac{1}{\sqrt{n}\, a_n} [K^{(1)}(\frac{x - X_j}{a_n}) - E\, K^{(1)}(\frac{x - X_j}{a_n})]$$

then the proof follows exactly the same way as that of Theorem 4.1 in Abdulal and Siddiqui (1986).

3. Estimation of the Mode

Estimating the mode of a probability density function is an important practical problem which has, as yet, received relatively little attention in the statistical literature. The lack of interest may be due to the difficulty of the problem. The overall objective of this section is to develop theorems for the weak and strong convergence, and an asymptotic distribution of the sample mode.

If $f_n(x)$ is defined as

$$f_n(x) = \frac{1}{2\pi} \int e^{-itx}\, \phi_n(t) k(a_n t) dt,$$

then for every sample sequence $f_n(x)$ is continuous and tends to 0 as x tends to $\pm\infty$. Consequently, there is a random variable θ_n such that

$$f_n(\theta_n) = \max_{-\infty<X<\infty} f_n(x)$$

Similarly, if the probability density function $f(x)$ is uniformly continuous in x, then $f(x)$ possesses a mode θ defined by

$$f(\theta) = \max_{-\infty<X<\infty} f(x)$$

We consider θ_n as an estimate of θ which is assumed to be unique.

Next, we examine the convergence of the sample mode θ_n.

Lemma 4.

(i) Assume the conditions of Lemma 2 are satisfied, then $\theta_n \xrightarrow{P} \theta$ as $n \to \infty$.

(ii) Assume conditions of Lemma 3 are satisfied, then

$$\theta_n \xrightarrow{W.P.1} \theta \qquad \text{as } n \to \infty$$

if θ is unique.

Proof. The proof follows from Lemma 2 when $r = 0$ and the fact that, for any $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$|f(\theta) - f(x)| \geq \eta \quad \text{if} \quad |\theta - x| \geq \varepsilon.$$

So to prove part (i), it suffices to show that

$$P[|f(\theta_n) - f(\theta)| < \varepsilon] \to 1 \qquad \text{as } n \to \infty \tag{20}$$

Now

$$|f(\theta_n) - f(\theta)| \le |f(\theta_n) - f_n(\theta_n)| + |f_n(\theta_n) - f(\theta)|$$

$$\le \sup_{-\infty<X<\infty} |f_n(x) - f(x)| + |\sup_{-\infty<X<\infty} f_n(x) - \sup_{-\infty<X<\infty} f(x)| \tag{21}$$

$$\le \sup_{-\infty<X<\infty} |f_n(x) - f(x)| + \sup_{-\infty<X<\infty} |f_n(x) - f(x)|$$

$$= 2 \sup_{-\infty<X<\infty} |f_n(x) - f(x)|$$

From (21) and Lemma 2, one obtains (20). This implies that part (i) is true.

(ii) To prove part (ii) it suffices to show that

$$f_n(\theta) \xrightarrow{W.P.1} f(\theta) \qquad \text{as} \quad n \to \infty$$

This follows from (21) and Lemma 3 when $r = 0$.

In the remainder of this section we state conditions on the constants a_n, $f(x)$ and $K(y)$, such that the sample mode θ_n is asymptotically normal.

Consider a p.d.f. $f(x)$ with a unique mode at θ. If $f(x)$ has a continuous second derivative, then

$$f^{(1)}(\theta) = 0 \qquad \text{and} \qquad f^{(2)}(\theta) < 0$$

Similarly, if the estimate $f_n(x)$ is chosen to be twice differentiable (that is, $K(y)$ is chosen to be twice differentiable), then

$$f_n^{(1)}(\theta_n) = 0 \quad \text{and} \quad f_n^{(2)}(\theta_n) < 0 \tag{22}$$

By Taylor's expansion

$$0 = f_n^{(1)}(\theta_n) = f_n^{(1)}(\theta) + (\theta_n - \theta) f_n^{(2)}(\theta_n^*)$$

for some random variable θ^* between θ_n and θ.

This implies that

$$\theta_n - \theta = - \frac{f_n^{(1)}(\theta)}{f_n^{(2)}(\theta_n^*)} \tag{23}$$

if the denominator does not vanish. Using (23) as a basis, we now state conditions under which the sample mode θ_n is asymptotically normal.

<u>Theorem 3</u>. Let $\{X_n\}$ be a stationary sequence of r.v.'s satisfying the uniform mixing condition. Assume that the following conditions hold:

(i) $K(y)$ is symmetric,

(ii) $f^{(r)}(x)$, $r = 1,2,3$ exist and continuous,

(iii) $\sum_{j=1}^{\infty} \alpha(j) < \infty$ and $\sum_{j=1}^{\infty} \alpha^{1/2}(j) < \infty$

iv) $\lim_{n \to \infty} na_n^6 = 0$, and $\lim_{n \to \infty} na_n^7 = 0$, e.g., $a_n = n^{-11/75}$

(v) $\int |t|^2 |k(t)| dt < \infty$ and $\int |t|^3 |\phi(t)| dt < \infty$

Then

$$na_n^3 \, (\theta_n - \theta) \xrightarrow{D} N(0,V)$$

where

$$V = \frac{f(\theta)}{f^{(2)}(\theta)} \left[\int |K^{(1)}(z)|^2 dz\right]$$

Proof. First we have to show that

$$f_n^{(2)}(\theta_n^*) \xrightarrow{P} f^{(2)}(\theta) \qquad \text{as} \quad n \to \infty \tag{24}$$

From conditions (ii), (iv), (v), the fact that $\sum_{j=1}^{\infty} \alpha(j) < \infty$, and Lemma 2, we have

$$f_n^{(2)}(\theta) \xrightarrow{P} f^{(2)}(\theta) \qquad \text{as} \quad n \to \infty \tag{25}$$

Then (24) holds from (25) and the fact that θ_n^* tends to θ, since it is between θ_n and θ and θ_n tends to θ in probability.

Second, we have to show that

$$\sqrt{na_n^3} \, Ef_n^{(1)}(\theta) = 0$$

It follows from (22) that

$$\sqrt{na_n^3} \, E \, f_n^{(1)}(\theta) = \sqrt{na_n^3} \, [E \, f_n^{(1)}(\theta) - f^{(1)}(\theta)]$$

Since $f^{(r)}(x)$, $r = 1,2,3$, exist and are continuous, and $K(y)$ is symmetric, by using condition (v) and Taylor's expansion of $f^{(1)}(\theta)$, we have

$$\sqrt{na_n^3}\,(E\, f_n^{(1)}(\theta) - f^{(1)}(\theta))$$

$$= \sqrt{na_n^3}\,[\frac{1}{a_n^2}\int K^{(1)}(\frac{\theta - y}{a_n})f(y)dy - f^{(1)}(\theta)]$$

$$= \sqrt{na_n^3}\,[\frac{1}{a_n}\int K(\frac{\theta - y}{a_n})f^{(1)}(y)dy - f^{(1)}(\theta)]$$

$$= \sqrt{na_n^3}\,[\int K(z)f^{(1)}(\theta - a_n z)dz - f^{(1)}(\theta)]$$

$$= \sqrt{na_n^3}\{\int K(z)[f^{(1)}(\theta) - a_n z f^{(2)}(\theta) + \frac{a_n^2 z^2}{2} f^{(3)}(\theta_{..})]dz - f^{(1)}(\theta)\}$$

$$= \sqrt{na_n^7}\,\{\int z^2 K(z)dz\} \longrightarrow 0 \qquad \text{as} \quad n \to \infty \tag{26}$$

From (26), the fact that $\sum_{j=1}^{\infty} \alpha^{\frac{1}{2}}(j) < \infty$, and Theorem 1, we have

$$\sqrt{na_n^3}\, f_n^{(1)}(\theta) \xrightarrow{D} N(0,\, f(\theta)\int [K^{(1)}(z)]^2 dz). \tag{27}$$

From (23), (25) and (27), we have

$$\sqrt{na_n^3}\,(\theta_n - \theta) \xrightarrow{D} N(0,\, V).$$

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