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COHOMOLOGY RINGS OF SCHUBERT VARIETIES IN GRASSMANN MANIFOLDS

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Let G be a complex reductive linear algebraic group, P a parabolic subgroup of G, and Y a Schubert subvariety of $X = G/P$. The purpose of this paper is to verify the following conjecture for Schubert varieties in a Grassmann manifold: The coordinate ring $A(Y \cap Z)$ of the scheme theoretic intersection of Y and Z has a structure of a graded algebra which is isomorphic to the complex cohomology algebra $H^*(Y, \mathbb{C})$ of Y, where Z is the fixed point scheme of the holomorphic $\mathbb{C}$ action on X associated to a regular nilpotent element in the Lie algebra of G.

1. Introduction

Let V be a holomorphic vector field on a smooth complex projective variety X with isolated nonempty zero set Z. Let Ω_X^p (resp. $\mathcal{O}_X$) denote the sheaf of germs of holomorphic p-forms (resp. functions) on X and $i(V)\colon \Omega_X^p \to \Omega_X^{p-1}$ be the contraction operator associated to V. The structure sheaf $\mathcal{O}_Z$ of Z is, by definition, $\mathcal{O}_X/i(V)\Omega_X^1$. That is, Z is the variety, possibly unreduced, defined by the sheaf of ideals $\mathcal{J}(Z) = i(V)\Omega_X^1$ in $\mathcal{O}_X$. It is known that the ring $A(Z) = H^\circ(X, \mathcal{O}_Z)$ of the global sections of $\mathcal{O}_Z$ admits an increasing filtration F_p with $F_p F_q \subseteq F_{p+q}$ so that the cohomology ring $H^*(X, \mathbb{C})$ is isomorphic to the graded algebra $\operatorname{Gr} A(Z) = \oplus F_p/F_{p-1}$ ([6, 7]). We shall denote this isomorphism by $m\colon \operatorname{Gr} A(Z) \xrightarrow{\sim} H^*(X, \mathbb{C})$.

It is difficult to realize the cohomology of X on Z by means of the isomorphism m. In general, one does not know either the filtration F_p or any explicit description of m. However, when $X = G/P$ is an algebraic homogeneous space and V is the vector field induced from the holomorphic $\mathbb{C}$ action associated to a regular nilpotent element in the Lie algebra $\mathfrak{g}$ of G, then it is known that $A(Z)$ has

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a canonical structure of a graded algebra making it isomorphic to $H^*(X, \mathbb{C})$. That is, $\mathrm{Gr}\, A(Z) = A(Z)$ and $m: A(Z) \xrightarrow{\sim} H^*(X, \mathbb{C})$ is a graded algebra isomorphism ([4, 5]). Let Y be a Schubert variety in $X = G/P$, and $A(Y \cap Z)$ be the coordinate ring of scheme theoretic intersection of Y and Z. Recently, it is shown that $A(Y \cap Z)$ has also the structure of a graded algebra such that the natural map $\alpha: A(Z) \longrightarrow A(Y \cap Z)$ is a graded algebra homomorphism, and moreover, there exists a graded algebra homomorphism $\bar{m}: A(Y \cap Z) \rightarrow H^*(Y; \mathbb{C})$ making the following diagram commutative:

$$\begin{array}{ccc} A(Z) & \xrightarrow[\sim]{m} & H^*(X, \mathbb{C}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A(Y \cap Z) & \xrightarrow{\bar{m}} & H^*(Y, \mathbb{C}), \end{array}$$

where i^* is the cohomology map of the inclusion $i: Y \rightarrow X$ ([5]). It was conjectured in [4] that this surjective map $\bar{m}: A(Y \cap Z) \rightarrow H^*(Y, \mathbb{C})$ is, in fact, an isomorphism.

In this paper, we obtain the complete description of the isomorphism $m: A(Z) \xrightarrow{\sim} H^*(X, \mathbb{C})$ when $X = G/P$ is a Grassmann manifold, and show that the Plücker coordinates around the unique zero of V corresponds exactly to the Schubert cycles in $H^*(X, \mathbb{C})$. As a consequence of this, we prove that $\bar{m}: A(Y \cap Z) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ is an isomorphism when Y is a Schubert variety in a Grassmann manifold X.

2. Graded algebra A(Z)

In this section we give the complete description of the graded algebra of $A(Z)$ when $X = GL_n/B$ is the full flag manifold or $X = G_{k,n}$ is the Grassmann manifold of k-planes in $\mathbb{C}^n$.

We shall use the following notation: G denotes the group of $n \times n$ complex invertible matrices, B the group of upper triangular matrices in G, P the parabolic subgroup consisting of all matrices in G of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where 0 is the $(n-k)\times k$ zero matrix, e_{ij} the $n \times n$ matrix having 1 in the (i,j)th entry and zero everywhere else, $r = \sum_{i=1}^{n-1} e_{i\,i+1}$, e the $n \times n$ identity matrix, x_0

(resp. $\pi(x_0)$) the element eB (resp. eP) in G/B (resp. G/P), $\tilde{V}$ (resp. V) the vector field on G/B (resp. G/P) induced from the one parameter subgroup $\exp(tr)$ associated to the regular nilpotent matrix r, $\tilde{Z}$ (resp. Z) the zero scheme of $\tilde{V}$ (resp. V), and $\pi: G/B \to G/P$ is the natural projection map.

The algebraic homogeneous space G/B is the full flag manifold and $G/P = G_{k,n}$ is the Grassmann manifold of k-planes in $\mathbb{C}^n$. Let z_{ij}, $1 \le i, j \le n$, be the natural coordinate functions on G. That is, for $x = (x_{ij}) \in G$ $z_{ij}(x) = x_{ij}$. Then by the Bruhat decomposition we know that $\{z_{ij}: 1 \le j \le i \le n\}$ (resp. $\{z_{k+i\,j}: 1 \le i \le n-k, 1 \le j \le k\}$) is a holomorphic local coordinate system at x_0 (resp. $\pi(x_0)$) in G/B (resp. G/P). We consider the polynomial algebras

$$\tilde{R} = \mathbb{C}[z_{ij}: 1 \le j < i \le n] \quad \text{and} \quad R = \mathbb{C}[z_{k+i\,j}: 1 \le i \le n-k,\ 1 \le j \le k]$$

with the grading determined by taking degree$(z_{pq}) = p-q$. In the rest of the paper, we take $z_{ij} = 0$ if either $i > n$, or $j < 1$, or $j > i$, and $z_{ii} = 1$ for $1 \le i \le n$.

<u>PROPOSITION 2.1</u>. (i) The graded algebra $A(\tilde{Z})$ is isomorphic to $\tilde{R}/I(\tilde{Z})$, where $I(\tilde{Z})$ is the homogeneous ideal generated by $a_{ij}(z) = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j})$.

(ii) Let $x_1 = z_{21}$, $x_2 = z_{32} - z_{21}$, ..., $x_j = z_{j+1\,j} - z_{j\,j-1}$, ..., $x_n = -z_{n\,n-1}$, and let $h_m(y_1, \ldots, y_s)$ be the m-th complete symmetric homogeneous function in $y_1, \ldots, y_s$. For any i, j, the following identity holds in $\tilde{R}/I(\tilde{Z})$: $z_{ij} = h_{i-j}(x_1, x_2, \ldots, x_j)$. Moreover, $\mathbb{C}[x\ , \ldots, x_n]/(\sigma_1, \ldots, \sigma_n) = \tilde{R}/I(\tilde{Z})$, where σ_i is the i-th elementary symmetric function in $x_1, \ldots, x_n$.

(iii) Under the isomorphism $\tilde{m}: \tilde{R}/I(\tilde{Z}) \cong A(\tilde{Z}) \xrightarrow{\sim} H^*(G/B, \mathbb{C})$, $\tilde{m}(z_{ij} \bmod I(\tilde{Z})) = c_{i-j}(Q_j)$, $(i-j)$-th Chern class of the universal quotient bundle Q_j of rank $n-j$ on G/B.

Proof. To prove (i), it is enough to show that $i_{\tilde{V}}(dz_{ij}) = \tilde{V}(z_{ij}) = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j})$. For this we need to compute the local expression of $\tilde{V}$ in the local coordinates z_{ij}. Let $\tilde{M} = (z_{ij})$ be the $n \times n$ lower triangular unipotent matrix having z_{ij} as its entries. The change of the local coordinates z_{ij} by the action $\exp(tr)$ around x_0 is given by the holomorphic functions $z_{ij}(t)$, $1 \le j < i \le n$, which satisfy the following identity:

$$(*)\quad \exp(tr)\tilde{M}B_t = (z_{ij}(t)) \text{ for some } n\times n \text{ upper triangular matrix } B_t,$$

where $(z_{ij}(t))$ is an $n\times n$ lower triangular unipotent matrix. We note that $\exp(tr)\tilde{M} = W(f_1, \ldots, f_n) =$

$$\begin{pmatrix} f_1 & f_2 \cdots\cdots f_n \\ f_1' & f_2' \cdots\cdots f_n' \\ \vdots & \vdots \qquad \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} \cdots f_n^{(n-1)} \end{pmatrix}$$ is a Wronskian matrix, where

$f_k = \sum_{i=0}^{n-k} \frac{t^{k+i-1}}{(k+i-1)!} z_{k+i\,k}$, $1 \le k \le n$. Let $|W(g_1, \ldots, g_\ell)|$ denote the determinant of the Wronskian matrix $W(g_1, \ldots, g_\ell)$, and $W(g_1, \ldots, \hat{g}_i, \ldots, g_\ell) = W(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_\ell)$, $W(\hat{g}_1) = 1$.. By using the standard formulas involving derivatives of determinants, one can check that the following matrix B_t, defined by

$$(B_t)_{ij} = \begin{cases} (-1)^{i+j} \dfrac{|W(f_1, \ldots, \hat{f}_i, \ldots, f_j)|}{|W(f_1, \ldots, f_j)|} & \text{if } i \le j, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the identity (*). From this, one obtains

$$z_{ij}(t) = \frac{\begin{vmatrix} f_1 & \cdots\cdots\cdots & f_j \\ \vdots & & \vdots \\ f_1^{(j-2)} & \cdots\cdots & f_j^{(j-2)} \\ f_1^{(i-1)} & \cdots\cdots & f_j^{(i-1)} \end{vmatrix}}{|W(f_1, \ldots, f_j)|}, \quad 1 \le j < i \le n.$$

By using, again, the formulas involving derivatives of determinants one obtains

$$\tilde{V}(z_{ij}) = \frac{d}{dt}(z_{ij}(t))\Big|_{t=0} = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j}).$$

For (ii), let $S = \{(i,j)\colon 1 \le j < i \le n\}$. Consider the partial order on S defined by: for (i, j) and (k, ℓ) in S, $(i,j) \le (k,\ell)$

if $i \leq k$, $j \leq \ell$. We prove the identity $z_{ij} = h_{i-j}(x_1, \ldots, x_j)$ in $\tilde{R}/I(\tilde{Z})$ by induction on $(i,j) \in S$ relative to $\leq$. For the minimal element $(2,1)$ in S, we have, by definition, $z_{21} = h_1(x_1) = x_1$. For a given $(i+1, j)$ in S, by using the defining relation $a_{ij} = 0$ in $\tilde{R}/I(\tilde{Z})$, one obtains $z_{i+1\,j} = z_{i\,j-1} + x_j z_{ij}$ where $x_j = z_{j+1\,j} - z_{j\,j-1}$. Thus, by the induction hypothesis we have $z_{i+1\,j} = h_{i+1-j}(x_1, \ldots, x_{j-1}) + x_j h_{i-j}(x_1, \ldots, x_j)$. Since $h_{i+1-j}(x_1, \ldots, x_{j-1}) + x_j h_{i-j}(x_1, \ldots, x_j) = h_{i+1-j}(x_1, \ldots, x_j)$ we get $z_{ij} = h_{i-j}(x\ , \ldots, x_j)$ in $\tilde{R}/I(\tilde{Z})$. This, in particular, implies that $z_{n+i\,n} = h_i(x_1, \ldots, x_n) = 0$ in $\tilde{R}/I(\tilde{Z})$, $i = 1, 2, \ldots, n$. Thus the ideal $(h_1(x_1, \ldots, x_n), \ldots\ldots, h_n(x_1, \ldots, x_n)) = (\sigma_1, \ldots, \sigma_n)$ lies in $I(\tilde{Z})$. By comparing the dimensions we get $\mathbb{C}[x_1, \ldots, x_n]/(\sigma_1, \ldots, \sigma_n) = \tilde{R}/I(\tilde{Z})$.

For (iii), we note that G/B is naturally biholomorphic to the full flag manifold $F_n = \{F = (F_1, \ldots, F_n): F_1 \subset \ldots \subset F_n,\ F_p$ is a p-dimensional subspace of the space of $n\times 1$ matrices $\mathbb{C}^{n\times 1}\}$. Let $Q_m = F_n \times \mathbb{C}^{n\times 1}/E_m$ be the universal quotient bundle on G/B, where $E_m = \{(F, V) \in F_n \times \mathbb{C}^{n\times 1}: v \in F_m\}$ is the tautological universal bundle of m-dimensional subspaces of $\mathbb{C}^{n\times 1}$. The holomorphic $\mathbb{C}$ action on $G/B\ \ \mathbb{C}^{n\times 1}$, $\phi(t, (F,v)) = (\exp(tr)F, \exp(tr)v)$ induces a 1-parameter subgroup of Q_m commuting with the 1-parameter subgroup $\exp(tr)$ of G/B. Thus Q_m is a $\tilde{V}$-equivariant bundle ([1]). One can then use the techniques introduced in [1] to compute the matrix representation of the $\tilde{V}$ derivation associated to ϕ around x_0. When this is carried out, one gets the diagonal matrix $\mathrm{diag}(-x_{m+1}, -x_{m+2}, \ldots, -x_n)$, $x_i = z_{i+1} - z_{i\,i-1}$. Thus by [1,7], the j-th elementary symmetric function $s_j(-x_{m+1}, \ldots, -x_n) = (-1)^j s_j(x_{m+1}, \ldots, x_n)$ represents, under the isomorphism $\tilde{m}: \tilde{R}/I(\tilde{Z}) \xrightarrow{\sim} H^*(G/B; \mathbb{C})$, the j-th Chern class $c_j(Q_m)$ of Q_m. Since $(-1)^j s_j(x_{m+1}, \ldots, x_n) = h_j(x_1, \ldots, x_m)$ in $\mathbb{C}[x_1, \ldots, x_n]/(\sigma_1, \ldots, \sigma_n)$, we get from (ii) $\tilde{m}(z_{m+j\,m}) = \tilde{m}(h_j(x_1, \ldots, x_m)) = c_j(Q_m)$. Thus $\tilde{m}(z_{ij}) = c_{i-j}(Q_j)$ ∎.

The following proposition can be proved in a similar way, the details left to the reader. We would like to note that a different proof can also be found in [8].

<u>PROPOSITION 2.2.</u> (i) The graded algebra $A(Z)$ is isomorphic to $R/I(Z)$, where $I(Z)$ is the homogeneous ideal generated by $z_{k+1+i\,j} - z_{k+i\,j-1} - z_{k+i\,k} z_{k+1\,j}$, $1 \leq j \leq k$, $1 \leq i \leq n-k$.

(ii) Under the isomorphism $m: R/I(Z) \cong A(Z) \xrightarrow{\sim} H^*(G_{k,n}, \mathbb{C})$, $m(z_{k+i\,k} \bmod I(Z)) = c_i(Q_{k,n})$, i-th Chern class of the universal quotient bundle $Q_{k,n}$ on $G_{k,n}$.

Remark. Proposition 2.2 (ii) also follows from Proposition 2.1 (iii), because the pull back $\pi^*Q_{k,n}$ of $Q_{k,n}$ on G/B is isomorphic to Q_k, and the algebra homomorphism $j: A(Z) \longrightarrow A(\tilde{Z})$ induced from $\pi: G/B \longrightarrow G_{k,n}$ is injective (cf. section 3).

In the rest of the paper, we shall take $A(\tilde{Z}) = \tilde{R}/I(\tilde{Z})$, $A(Z) = R/I(Z)$, and keep the notations as before.

3. Cohomology of Schubert varieties in $G_{k,n}$

In this section, we give the explicit description of the isomorphism $m: A(Z) \xrightarrow{\sim} H^*(G_{k,n}, \mathbb{C})$ by providing the representatives of Schubert cycles in $A(Z)$, and then prove that $\bar{m}: A(Y \cap Z) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ is an isomorphism for any Schubert variety Y in $G_{k,n}$.

Let W be the symmetric group in $1, 2, \ldots, n$. For any permutation $\tau = (a_1, \ldots, a_n)$ in W, let $\tau(e)$ be the $n \times n$ permutation matrix obtained from the identity matrix e by permuting the rows relative to $(a_1, \ldots, a_n)$. Let $S = \{(i) = (i_1, \ldots, i_k): 1 \le i_1 < i_2 < \ldots < i_k \le n\}$. For any (i) in S there exists a unique permutation $(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)$ in W with the property $i_{k+1} < \ldots < i_n$. We denote this permutation by $\sigma(i) = \sigma(i_1, \ldots, i_k)$. For $(i) = (i_1, \ldots, i_k)$ in S, let $Y_{(i)} = \overline{B\sigma(i)(e)\pi(x_0)}$ be the Schubert subvariety of $G_{k,n}$ associated to $1 \le i_1 < \ldots < i_k \le n$, and let $\Omega(i_1, \ldots, i_k)$ be the Poincaré dual of the cycle class of the Schubert variety $Y_{(n-i_k+1, \ldots, n-i_1+1)}$ in $H^*(G_{k,n}, \mathbb{C})$. Let $\tilde{U} = B^-$ denote the affine space of all $n \times n$ lower triangular unipotent matrices, and let $U = \pi(\tilde{U})$. $\tilde{U}$ is naturally biholomorphic to the open big cell in the Bruhat decomposition of $G/B = \cup B\tau(e)x_0$, $\tau \in W$. Thus $\tilde{U}$ (resp. U) is an open affine neighborhood of x_0 (resp. $\pi(x_0)$) in G/B (resp. $G_{k,n}$). We also note that for $x = (x_{ij})$ in $\tilde{U}$, $z_{ij}(x) = x_{ij}$.

THEOREM 3.1. For any $1 \le i_1 < i_2 < \ldots < i_k \le n$, we have $m(P_{(i_1, \ldots, i_k)} \bmod I(Z)) = \Omega(i_1, \ldots, i_k)$, where $P_{(i_1, \ldots, i_k)}$ is the Plücker coordinate of $G_{k,n}$ associated to $1 \le i_1 < i_2 < \ldots < i_k \le n$.

Proof. Let $j: A(Z) \longrightarrow A(\tilde{Z})$ be the natural map induced from the $\mathbb{C}$-equivariant map $\pi: G/B \longrightarrow G_{k,n}$. It follows from [2] that j is a graded algebra homomorphism and the following diagram is commutative:

$$\begin{array}{ccc} \tilde{m}: A(\tilde{Z}) & \xrightarrow{\ \sim\ } & H^*(G/B, \mathbb{C}) \\ j \Big\uparrow & & \Big\uparrow \pi^* \\ m: A(Z) & \xrightarrow{\ \sim\ } & H^*(G_{k,n}, \mathbb{C}) , \end{array}$$

where π^* is the cohomology map of π. Since π^* is injective, j is also an injective map ([2, 3]). Thus, to prove the theorem, it is enough to show that $\tilde{m}(j(P_{i_1,\ldots,i_k})) = \pi^*(\Omega_{(i_1,\ldots,i_k)})$. For any $x = (x_{ij})$ in $\tilde{U}$, since $j(P_{(i_1,\ldots,i_k)})(x) =$

$$\det \begin{pmatrix} x_{i_1 1} & \cdots & x_{i_1 k} \\ \vdots & & \vdots \\ x_{i_k 1} & \cdots & x_{i_k k} \end{pmatrix} = \begin{vmatrix} x_{i_1 1} & \cdots & x_{i_1 k} \\ \vdots & & \vdots \\ x_{i_k 1} & \cdots & x_{i_k k} \end{vmatrix} = \begin{vmatrix} z_{i_1 1}(x) & \cdots & z_{i_1 k}(x) \\ \vdots & & \vdots \\ z_{i_k 1}(x) & \cdots & z_{i_k k}(x) \end{vmatrix} ,$$

we get

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} z_{i_1 1} & \cdots & z_{i_1 k} \\ \vdots & & \vdots \\ z_{i_k 1} & \cdots & z_{i_k k} \end{vmatrix} \quad \text{on } \tilde{U}.$$

Thus, by Proposition 2.1, in $A(\tilde{Z})$ we have the identity

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1) \cdots & h_{i_1-k}(x_1,\ldots,x_k) \\ \vdots & \vdots \\ h_{i_k-1}(x_1) \cdots & h_{i_k-k}(x_1,\ldots,x_k) \end{vmatrix} .$$

In this determinant, by replacing the 1st column with 1st column + $x_2 \cdot$(2nd column), we obtain

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2) & h_{i_1-2}(x_1,x_2) & \ldots & h_{i_1-k}(x_1,\ldots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2) & h_{i_k-2}(x_1,x_2) & \ldots & h_{i_k-k}(x_1,\ldots,x_k) \end{vmatrix},$$

just because $h_\ell(x_1, x_2) = h_\ell(x_1) + x_2\, h_{\ell-1}(x_1, x_2)$. Now, by replacing the 2nd column with 2nd column + x_3 (3rd column) one gets

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2) & h_{i_1-2}(x_1,x_2\ x_3) & \ldots & h_{i_1-k}(x_1,\ldots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2) & h_{i_k-2}(x_1,x_2,x_3) & \ldots & h_{i_k-k}(x_1,\ldots,x_k) \end{vmatrix}.$$

This time, replace the 1st column by 1st column + x_3 (2nd column) to obtain

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2,x_3) & h_{i_1-2}(x_1,x_2,x_3) & \ldots & h_{i_1-k}(x_1,\ldots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2,x_3) & h_{i_k-2}(x_1,x_2,x_3) & \ldots & h_{i_k-k}(x_1,\ldots,x_k) \end{vmatrix}.$$

By using similar column operations and the (obvious) identity $h_\ell(x_1,\ldots,x_s) = h_\ell(x_1,\ldots,x_{s-1}) + x_s\, h_{\ell-1}(x_1,\ldots,x_s)$ one obtains in $A(\tilde{Z})$,

$$j(P_{(i_1,\ldots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1, \ldots, x_k) & \ldots & h_{i_1-k}(x_1, \ldots, x_k) \\ \vdots & & \vdots \\ h_{i_k-1}(x_1, \ldots, x_k) & \ldots & h_{i_k-k}(x_1, \ldots, x_k) \end{vmatrix}.$$

By Proposition 2.1, since $z_{i+k\,k} = h_i(x_1, \ldots, x_k)$ in $A(\tilde{Z})$ and $\tilde{m}(z_{i+k\,k}) = c_i(Q_k)$, we get

$$\tilde{m}(j(P_{(i_1,\ldots,i_k)})) = \begin{vmatrix} c_{i_1-1}(Q_k) & \ldots & c_{i_1-k}(Q_k) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_k) & \ldots & c_{i_k-k}(Q_k) \end{vmatrix} =$$

$$\pi^*\left(\begin{vmatrix} c_{i_1-1}(Q_{k,n}) & \cdots\cdots & c_{i_1-k}(Q_{k,n}) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_{k,n}) & \cdots\cdots & c_{i_k-k}(Q_{k,n}) \end{vmatrix}\right),$$

because the pull back $\pi^*Q_{k,n}$ of the bundle $Q_{k,n}$ is isomorphic to Q_k on G/B. By the determinantal formula in Schubert calculus ([3]), since

$$\Omega(i_1,\ldots,i_k) = \begin{vmatrix} c_{i_1-1}(Q_{k,n}) & \cdots\cdots & c_{i_1-k}(q_{k,n}) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_{k,n}) & \cdots\cdots & c_{i_k-k}(Q_{k,n}) \end{vmatrix},$$

we get $\tilde{m}(j(P_{(i_1,\ldots,i_k)})) = \pi^*(\Omega_{(i_1,\ldots,i_k)})$ and this completes the proof ■.

We consider the natural partial order on $S = \{(i) = (i_1,\ldots,i_k): 1 \leq i_1 < \ldots < i_k \leq n\}$ defined by: for (i) and (j) in S, $(i) \leq (j)$ if $i_1 \leq j_1, \ldots, i_k \leq j_k$. It is well known that this partial order on S is compatible with the Bruhat ordering on $G_{k,n} = \cup B\sigma(i)(e)\pi(x_0)$, $(i) \in S$. That is, for (i) and (j) in S $(i) \leq (j)$ if and only if $Y_{(i)} \subseteq Y_{(j)}$ ([9]).

<u>LEMMA</u>. For any (j) in S, we have

(i) the ideal $I(Y_{(j)})$ of the Schubert variety $Y_{(j)}$ in the neighborhood U of $\pi(x_0)$ is generated by the Plücker coordinates $P_{(\ell)}$, $(\ell) \nleq (j)$,

(ii) The Euler-Poincaré characteristic $\chi(Y_{(j)})$ of $Y_{(j)}$ is equal to the cardinality of the set $\{(\ell) \in S: (\ell) \leq (j)\}$.

Proof. This lemma is not new. In fact, part (i) can be found in [9], and part (ii) follows from the cellular decomposition $Y_{(j)} = \cup B\sigma(\ell)(e)\pi(x_0)$, $(\ell) \leq (j)$, of $Y_{(j)}$ ([5]) ■.

<u>THEOREM 3.2</u>. Let $Y = Y_{(i)}$, $(i) \in S$, be a Schubert subvariety of $G_{k,n}$. The graded algebra isomorphism $m: A(Z) \xrightarrow{\sim} H^*(G_{k,n}, \mathbb{C})$ induces

an isomorphism $\bar{m}: A(Y \cap Z) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ which commutes with the natural maps $\alpha: A(Z) \longrightarrow A(Y \cap Z)$ and $i^*: H^*(G_{k,n}, \mathbb{C}) \longrightarrow H^*(Y, \mathbb{C})$.

Proof. By [5], we know that m induces a graded algebra homomorphism $\bar{m}: A(Y \cap Z) \longrightarrow H^*(Y, \mathbb{C})$ which commutes with α and i^*. Since $\bar{m}$ is a surjective map, we only need to show that $\dim_{\mathbb{C}} A(Y \cap Z) \leq \dim_{\mathbb{C}} H^*(Y, \mathbb{C})$. By the basis theorem of Schubert calculus and Theorem 3.1, we know that the Plücker coordinates $P_{(j)}$, $(j) \in S$, form a basis of $A(Z)$. Thus $\{\alpha(P_{(\ell)}): (\ell) \in S\}$ spans the vector space $A(Y \cap Z)$. By the Lemma, since $P_{(j)} = 0$ on $Y = Y_{(i)}$ whenever $(j) \nleq (i)$, we get $\alpha(P_{(j)}) = 0$ in $A(Y \cap Z)$ for $(j) \nleq (i)$. This implies, $I = \{\alpha(P_{(\ell)}): (\ell) \leq (i)\}$ spans $A(Y \cap Z)$. Therefore, the cardinality of $I = \#\{(\ell) \in S: (\ell) \leq (i)\} \geq \dim_{\mathbb{C}} A(Y \cap Z)$. By the same Lemma, since $\chi(Y) = \dim_{\mathbb{C}} H^*(Y, \mathbb{C}) = \#\{(\ell) \in S: (\ell) \leq (i)\}$, we get $\dim_{\mathbb{C}} A(Y \cap Z) \leq \dim_{\mathbb{C}} H^*(Y, \mathbb{C})$ and this completes the proof ∎.

REFERENCES

[1] AKYILDIZ, E.: Vector fields and equivariant bundles, Pac. Jour. of Math. 81, 283-289 (1979).

[2] AKYILDIZ, E.: Vector fields and cohomology of G/P, Lecture Notes in Mathematics 956, Springer-Verlag, 1-9 (1982).

[3] AKYILDIZ, E.: Gysin homomorphism and Schubert calculus, Pac. Jour. of Math. 115, 257-266 (1984).

[4] AKYILDIZ, E., CARRELL, J.B.: Cohomology of projective varieties with regular SL_2 actions, to appear in Manuscripta Mathematica.

[5] AKYILDIZ, E., CARRELL, J.B., LIEBERMAN, D.I.: Zeros of holomorphic vector fields on singular spaces and intersection rings of Schubert varieties, Compositio Math. 57, 237-248 (1986).

[6] CARRELL, J.B., LIEBERMAN, D.I.: Holomorphic vector fields and compact Koehler manifolds, Inventiones Math. 21, 303-309 (1973).

[7] CARRELL, J.B., LIEBERMAN, D.I.: Vector fields and Chern numbers, Math. Ann. 225, 263-273 (1977).

[8] CARRELL, J.B., LIEBERMAN, D.I.: Vector fields, Chern classes and cohomology, Proc. Symp. Pure Math. 30, 251-254 (1977).

[9] LAKSHMIBAI, V., SESHADRI, C.S.: Geometry of G/P-II, Proc. Indian Nat. Sci. Acad. Part A 87, 1-54 (1978).

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