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Asymptotically Balanced Functions and the Asymptotic Behaviour of the Complementary Function and the Laplace Transform

J.L. Geluk

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Introduction

Asymptotically balanced functions are introduced in a paper by de Haan and Resnick [7] concerning stochastic compactness of sample extremes. In a more recent paper (see [8]) a theorem concerning the asymptotic behaviour of the Laplace transform is proved for this class of functions. The above mentioned papers concern non-decreasing asymptotically balanced functions. Our next definition coincides with the one given in [7] and [8] in case the function f is non-decreasing.

Definition 1

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable. Then f is asymptotically balanced if there exists a function $a: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\overline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} < \infty \quad \text{for all} \quad x > 1, \tag{1}$$

$$\underline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} > -\infty \quad \text{for all} \quad x > 0 \tag{2}$$

and

there exists $x_0 > 1$ such that

$$\underline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} > 0 \quad \text{for all} \quad x \geq x_0 \tag{3}$$

Notation: $f \in AB$ or $f \in AB(a)$.

Moreover we say $f \in AB^0$ if $f(1/t) \in AB$.

Examples

$f(t) = (\log t)^\alpha + O(\log t)^{\alpha-1} (t \to \infty)$, $\alpha > 0$ is in $AB((\log t)^{\alpha-1})$.

$f(t) = c - t^{-\alpha}$, $c \in \mathbb{R}$, $\alpha > 0$ is in $AB(t^{-\alpha})$.

$f(t) \asymp t^\alpha (\log t)^\beta \ (t \to \infty)$, $\alpha > 0$, $\beta \in \mathbb{R}$ is in $AB(t^\alpha (\log t)^\beta)$.

(the notation $f(t) \asymp g(t)$ means: there exist $0 < c_1, c_2, t_0 < \infty$ such that $c_1 \le f(t)/g(t) \le c_2$ for all $t \ge t_0$.)

For properties of asymptotically balanced functions the reader is referred to Geluk and de Haan [4]. In this paper we study the asymptotic behaviour of the complementary function in the sense of Young for AB functions. This function (and the inverse complementary function) is defined as follows.

Definition 2

(i) Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is bounded on finite intervals of $\mathbb{R}^+$, $f(\infty) = \infty$ and $f(t) = o(t)$ $(t \to \infty)$. Then the transform f^*, the complementary function, is for $s > 0$ defined by

$$f^*(s) = \sup_{t>0} \{f(t) - st\}. \tag{4}$$

(ii) Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is bounded in every interval (a, ∞) for $a > 0$

and $f(0+) = \infty$. Then the <u>inverse complementary function</u> f_* is defined by

$$f_*(u) = \inf_{t>0} \{f(t) + ut\}, \quad u > 0 . \tag{5}$$

The class of functions f satisfying (i) is denoted by D.

Note that in case $f(t) = \int_0^t s(x)dx < \infty$ for $t > 0$ with $s: \mathbb{R}^+ \to \mathbb{R}^+$ continuous and strictly decreasing, we have

$$f^*(u) = f(s^{\leftarrow}(u)) - us^{\leftarrow}(u) = \int_u^\infty s^{\leftarrow}(x)dx, \tag{6}$$

where $s^{\leftarrow}$ is the inverse function of s.

Remark that in general the function f and its non-decreasing concave upper hull have the same transform f^*. The

The asymptotic behaviour as $s \to 0+$ of the complementary function $f^*(s)$ for some classes of functions is treated in several papers. For details the interested reader is referred to Bingham and Teugels [3], Matuszewska [9], Balkema et al. [2] and Geluk, de Haan and Stadtmüller [5]. In this paper we consider the asymptotic behaviour of f^* for asymptotically balanced functions which satisfy suitable growth conditions.

<u>Results</u>

From definition 1 above it follows that if $f \in AB(a)$, then the function a satisfies

$$\overline{\lim_{t\to\infty}}\, a(tx)/a(t) < \infty \quad \text{for all} \quad x > 0. \tag{7}$$

In the sequel we need the following result concerning 0-regularly varying functions, i.e., functions which satisfy (7).

Lemma 3.

Suppose $a: \mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive. Then (7) is equivalent to the following statements.

(i) $-\infty < \underline{i}(a) \leq \overline{i}(a) < \infty$, where $\underline{i}(a)$ and $\overline{i}(a)$ are defined by

$$\underline{i}(a) = \lim_{x\to\infty} \frac{\log \underline{\lim}_{t\to\infty} a(tx)/a(t)}{\log x}$$

and

$$\overline{i}(a) = \lim_{x\to\infty} \frac{\log \overline{\lim}_{t\to\infty} a(tx)/a(t)}{\log x}.$$

(ii) There exist $\alpha, \beta \in \mathbb{R}$, t_0 and $c > 1$ such that

$$c^{-1} x^{\beta} \leq \frac{a(tx)}{a(t)} \leq cx^{\alpha} \quad \text{for all} \quad x \geq 1, \quad t \geq t_0.$$

(iii) There exist $t_0 \geq 0$ and $\sigma \in \mathbb{R}$ such that

$$\int_{t_0}^{t} s^{\sigma-1} a(s)ds \asymp t^{\sigma} a(t) \qquad (t \to \infty).$$

(iv) There exists $\tau \in \mathbb{R}$ such that

$$\int_t^{\infty} s^{\tau-1}\, a(s)ds \asymp t^{\tau}\, a(t) \qquad\qquad (t \to \infty).$$

If a function a satisfies the assumptions of Lemma 3 we use the notation $a \in RO$.

In the above lemma we can take any $\beta < \underline{i}(a)$, $\alpha > \overline{i}(a)$, $\sigma > -\underline{i}(a)$ and $\tau < -\overline{i}(a)$. For a proof the reader is referred to Aljancić and Arandelović [1].

In order to formulate our results we need the following definition.

Definition 4

The functions $f, f_0 : \mathbb{R}^+ \to \mathbb{R}$ are <u>O-inversely asymptotic</u> if there exist constants $x > 1$ and $t_0 = t_0(x)$ such that

$$f(t) \le f_0(tx) \quad \text{and} \quad f_0(t) \le f(tx) \quad \text{for} \quad t \ge t_0. \tag{8}$$

Notation: $f \overset{O}{\sim} f_0$ or $f(t) \overset{O}{\sim} f_0(t) \quad (t \to \infty)$.

Moreover we write $f(s) \overset{O}{\sim} f_0(s) \quad (s \to 0+)$ if $f(1/t) \overset{O}{\sim} f_0(1/t) \quad (t \to \infty)$.

It is easy to see that if $f \in RO$ with $\underline{i}(f) > 0$, then $f \overset{O}{\sim} g$ if and only if $f(t) \asymp g(t) \quad (t \to \infty)$. (cf. [4], lemma 3.17) Also if $f(t) \overset{O}{\sim} f_0(t) \quad (t \to \infty)$, $f \in AB(a)$ and f_0 is measurable, then $f_0 \in AB(a)$. Moreover the relation $\overset{O}{\sim}$ is an equivalence relation on the class AB, which divides it into equivalence classes. Our next result gives a description of the equivalence classes in terms of the function a.

Lemma 5

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable.

If $f_0 \in AB(a)$ and $f_1(t) \overset{0}{\sim} f_0(t) \quad (t \to \infty)$, then

$$f_1(t) - f_0(t) = O(a(t)) \quad (t \to \infty). \tag{9}$$

Conversely, if (9) holds with $f_0 \in AB(a)$ and if

$$\lim_{x \to \infty} \varliminf_{t \to \infty} \frac{f_0(tx) - f_0(t)}{a(t)} = \infty \tag{10}$$

and

$$\overline{\lim_{x \to \infty}}\ \overline{\lim_{t \to \infty}} \frac{a(tx)}{a(t)} < \infty, \tag{11}$$

then $f_1(t) \overset{0}{\sim} f_0(t) \quad (t \to \infty)$, hence $f_1 \in AB(a)$.

Proof.

The first statement is proved in [4], lemma 3.19. Conversely suppose $f_1(t) = f_0(t) + c(t)a(t)$ with $|c(t)| \le c$ for $t \ge t_0$, where c is a constant.

For $t \ge t_0$ and $x \ge 1$ we have

$$\frac{f_1(tx) - f_0(t)}{a(t)} \ge \frac{f_0(tx) - f_0(t)}{a(t)} - c\,\frac{a(tx)}{a(t)}.$$

Since a satisfies (7) this implies

$$\underline{\lim}_{t\to\infty} \frac{f_1(tx) - f_0(t)}{a(t)} \geq \underline{\lim}_{t\to\infty} \frac{f_0(tx) - f_0(t)}{a(t)} - c\ \overline{\lim}_{t\to\infty} \frac{a(tx)}{a(t)} > -\infty.$$

Hence by (10) and (11) there exists $x_1 > 1$ such that

$\underline{\lim}_{t\to\infty} \frac{f_1(tx) - f_0(t)}{a(t)} > 0$ for all $x \geq x_1$, which implies $f_1(tx) \geq f_0(t)$ for $x \geq x_1$ and $t \geq t(x)$.

The proof of the inequality $f_0(tx_2) \geq f_1(t)$ is similar. Hence (8) is satisfied if we take $x = \max(x_1, x_2)$.

Remarks

From (11) it follows that $\overline{i}(a) \leq 0$.

Note that (10) does not imply $\lim_{t\to\infty} f_0(t) = \infty$ (take e.g. $f_0(t) = c - (\log t)^\alpha$ with $\alpha < 0$).

The next examples show that additional conditions like (10) and (11) are necessary for the converse part of the lemma.

If $f_0(t) = a(t) \asymp t \quad (t \to \infty)$ and $f_1(t) = O(t) \quad (t \to \infty)$, then all conditions of the converse part except (11) are satisfied and the conclusion is not correct. A similar statement holds with (11) replaced by (10) if we take $f_0(t) = c - t^{-1}$, $a(t) = t^{-1}$ and $f_1(t) = O(t^{-1}) \quad (t \to \infty)$.

Our next result is an asymptotic version of the equality (6). We omit the proof since it is analogous to the proof of theorem 5 in [2].

Theorem 6

Let $s: \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, continuous, $\lim_{t\to\infty} s(t) = 0$ and let $s^{\leftarrow}$ denote the inverse function.

Assume that $\int_0^1 s(x)dx < \infty$.

If

$$f(t) \overset{O}{\sim} \int_0^t s(x)dx \quad (t \to \infty) \tag{12}$$

and $f \in D$ then

$$f^*(u) \overset{O}{\sim} \int_u^\infty s^{\leftarrow}(y)dy \quad (u \to 0+). \tag{13}$$

Our next result shows that the property of asymptotically balancedness is preserved under the $*$-transform if we restrict ourselves to a suitable subclass of the AB-functions. A converse statement is given in theorem 11.

Theorem 7

If $f \in D \cap AB(a)$ with $\bar{i}(a) < 1$ satisfies (12) with s as in theorem 6, then $f^* \in AB^0$. More specifically $f^*(1/t) \in AB(a_0)$ with $\underline{i}(a_0) > -1$.

If f is as above and $f_0 \overset{O}{\sim} f$ then $f_0^*(u) \overset{O}{\sim} f^*(u)$ $(u \to 0+)$.

In order to prove this theorem we need the following result.

Lemma 8

Suppose $f \in AB(a)$ and

$$f(t) = f(t_0) + \int_{t_0}^{t} s(x)dx, \quad t > t_0 \tag{14}$$

where $s: \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing. Then

$$ts(t) \asymp a(t) \quad (t \to \infty). \tag{15}$$

<u>Proof</u>

For $c > 1$ and $t > t_0$ we have

$$\frac{(c - 1)s(ct)/s(t)}{\{f(ct) - f(t)\}/a(t)} \le \frac{a(t)}{ts(t)} = \int_1^c \frac{s(tx)}{s(t)}\,dx \Big/ \frac{f(ct) - f(t)}{a(t)} \tag{16}$$

$$\le (c - 1) \Big/ \frac{f(ct) - f(t)}{a(t)}.$$

If $s(t) \to c_0 > 0$ $(t \to \infty)$ then (15) follows from (16) (since $f \in AB(a)$). Next suppose $s(t) \to 0$ $(t \to \infty)$. Since $f \in AB(a)$ the right-hand inequality in (16) implies that there exist $c_1 > 0$, $c > 1$ such that

$$\overline{\lim_{t \to \infty}} \frac{a(t)}{ts(t)} \le c_1(c - 1) < \infty. \tag{17}$$

The proof is completed by contradiction. If (15) is not true then there exists a sequence $t_n \to \infty$ $(n \to \infty)$ such that

$$\frac{a(t_n)}{t_n s(t_n)} \to 0 \quad (n \to \infty). \tag{18}$$

Since $f \in AB(a)$, the left-hand inequality in (16) implies

$s(c\,t_n)/s(t_n) \to 0$ $(n \to \infty)$ for all $c > 1$. The last relation implies that for every $\alpha > 0$ there exists $t(\alpha)$ such that $s(t) \le t^{-\alpha}$ for all $t > t(\alpha)$. Hence for all n sufficiently large we have $a(t_n)/t_n\, s(t_n) \ge t_n^{\alpha-1} a(t_n)$. This contradicts (18) if we choose $\alpha > 1 + \underline{i}(a)$.

Proof of theorem 7

Since $f \in AB(a)$ we have $\int_0^t s(x)dx \in AB(a)$. Application of lemma 8 gives $a(t) \asymp ts(t)$ $(t \to \infty)$, hence $-\infty < \underline{i}(s) \le \overline{i}(s) < 0$. Hence the function b defined by $b(t) := s^+(1/t)$ satisfies $0 < \underline{i}(b) \le \overline{i}(b) < \infty$ (see [4], cor.3.7.g).

Application of lemma 3 and Fatou's lemma gives

$$\overline{\lim_{t\to\infty}} \frac{\int_{1/tx}^{\infty} s^+(y)dy - \int_{1/t}^{\infty} s^+(y)dy}{t^{-1}\, s^+(1/t)} \le \int_1^x \overline{\lim_{t\to\infty}} \frac{s^+(1/tu)}{s^+(1/t)} \frac{du}{u^2} \le c\int_1^x u^{\alpha} du < \infty,$$

where $c > 1$ and $\alpha \in \mathbb{R}$. Also a similar lower inequality holds, which shows that the right-hand side in (13) is a function in AB^0, hence $f^* \in AB^0$. Moreover the auxiliary function a_0 satisfies $a_0(t) \asymp t^{-1}s^+(1/t)$ $(t \to \infty)$, hence $\underline{i}(a_0) > -1$.

Our next two results provide sufficient conditions in order to ensure that an asymptotically balanced function satisfies the conditions of theorem 6.

Lemma 9

Suppose $f_1 : \mathbb{R}^+ \to \mathbb{R}$ is measurable.

The following are equivalent.

(i) $f_1 \in RO$ with $0 < \underline{i}(f_1) \leq \overline{i}(f_1) < 1$.

(ii) There exists a decreasing, continuous function $s: \mathbb{R}^+ \to \mathbb{R}^+$ such that $-1 < \underline{i}(s) \leq \overline{i}(s) < 0$ and

$$f_1(t) \overset{O}{\sim} \int_0^t s(x)dx \quad (t \to \infty). \tag{19}$$

Proof

Suppose (i) holds true. By lemma 3 we have $f_1(t) \asymp \int_{t_0}^t f_1(s)/s \, ds$ and $f_1(t)/t \asymp \int_t^\infty f_1(s)/s^2 ds$. Hence $f_1(t) \asymp \int_{t_0}^t s(x) \, dx \quad (t \to \infty)$ with $s(x) = \int_x^\infty f_1(s)/s^2 ds$, which implies (12) since f_1 satisfies (i). Conversely, if (ii) is satisfied, then $ts(t) \asymp \int_1^t s(x)dx =: a(t) \quad (t \to \infty)$ (by lemma 3), hence $0 < \underline{i}(a) \leq \overline{i}(a) < 1$, which implies (i) (by (12)).

Remark

Note that if f satisfies the assumptions of lemma 9, then $f \in AB(a)$ with $a(t) \asymp f(t) \quad (t \to \infty)$ (use lemma 3(ii)).

Lemma 10

If $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable, $\lim_{t \to \infty} f(t) = \infty$ and if there exists a function $\ell: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{t\to\infty} \frac{f(at) - f(t)}{\ell(t)} = \log a \quad \text{for} \quad a > 0 \tag{20}$$

and $f_1(t) = f(t) + O(\ell(t))$ $(t \to \infty)$, then there exists a decreasing, continuous function $s: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (19).

Proof

By the lemma in de Haan [6] and [2], theorem 3 there exists a decreasing continuous function s such that $f(t) = \int_0^t s(x)dx + o(\ell(t))$ $(t \to \infty)$ and $s(at)/s(t) \to a^{-1}$ $(t \to \infty)$ for $a > 0$. Then the assumptions of the converse part of lemma 5 are satisfied with $f_0(t) = \int_0^t s(x)dx$ and $a(t) = \ell(t) \sim ts(t)$. Note that (20) holds with f replaced by f_0 which implies (10). Moreover (11) follows from $\ell(at) \sim \ell(t)$, $t \to \infty$ for $a > 0$.

Remark that there is an analogue of theorem 6 (which can be proved similarly) for the transform f_* (see also [2], theorem 6). Combination of this analogue with theorem 6 gives a Tauberian result: if f is concave and nondecreasing, then (13) implies (12).

We show by an example that monotonicity of f is not a sufficient Tauberian condition.

Example

Let $s(x) = 1$ on $(0,1)$ and $s(x) = \{2^{(n+1)^2} - 2^{n^2}\}^{-1}$ on $[2^{n^2}, 2^{(n+1)^2})$ for $n = 0,1,2,\ldots$ and $f_1(t) := \int_0^t s(x)dx$.

Let $f_0(t) = t$ on $(0,1)$ and $f_0(t) = n$ on $[2^{(n-1)^2}, 2^{n^2})$, $n \geq 1$.

Then f_1 is the concave upper hull of f_0, hence $f_1^* = f_0^*$, but not $f_1 \overset{Q}{\sim} f_0$.

In our final result we replace the concavity of f by a different assumption in order to obtain a converse of theorem 7.

Theorem 11

If $f^*(1/t) \in AB(a_0)$ with $\underline{i}(a_0) > -1$ and f is non-decreasing and unbounded, then (13) implies (12).

For the proof of theorem 11 we need the following lemma.

Lemma 12

Suppose $f(t) = c + \int_0^t s(x)dx < \infty$, $t > 0$, where $s: \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing, $s(\infty) = 0$ and c is a constant.

Then there exists $a > 1$ with

$$\overline{\lim_{t \to \infty}} \; s(at)/s(t) < 1 \tag{21}$$

if and only if there exists $\varepsilon \in (0,1)$ such that

$$\overline{\lim_{t \to \infty}} \; \frac{f(t(1 + \varepsilon)) - 2f(t) + f(t(1 - \varepsilon))}{ts(t)} < 0 \, . \tag{22}$$

Proof

Suppose (21) holds true. Take $\varepsilon \in (1 - a^{-1}, 1)$. Since

$$\frac{f((1+\varepsilon)t - 2f(t) + f((1-\varepsilon)t)}{ts(t)} = \int_1^{1+\varepsilon} \frac{s(tx)}{s(t)}\,dx - \int_{1-\varepsilon}^{1} \frac{s(tx)}{s(t)}\,dx$$

$\leq \varepsilon - \int_{1-\varepsilon}^{a^{-1}} \frac{s(tx)}{s(t)}\,dx - (1 - a^{-1})$, we have $\overline{\lim}_{t\to\infty} \frac{f((1+\varepsilon)t) - 2f(t) + f((1-\varepsilon)t)}{ts(t)}$

$$\leq \varepsilon - \int_{1-\varepsilon}^{a^{-1}} \underline{\lim}_{t\to\infty} \frac{s(tx)}{s(t)}\,dx - (1 - a^{-1}) < \varepsilon - \int_{1-\varepsilon}^{a^{-1}} dx - (1 - a_0^{-1}) = 0.$$

If $\overline{\lim}_{t\to\infty} \frac{s(at)}{s(t)} = 1$ for all $a > 1$, then for $\varepsilon \in (0,1)$ arbitrary

$$\overline{\lim}_{t\to\infty} \left\{\int_1^{1+\varepsilon} \frac{s(tx)}{s(t)}\,dx - \int_{1-\varepsilon}^{1} \frac{s(tx)}{s(t)}\,dx\right\} \geq$$

$$\varepsilon\left[\overline{\lim}_{t\to\infty} \frac{s(t(1+\varepsilon))}{s(t)} - 1\Big/\overline{\lim}_{t\to\infty} \frac{s(t)}{s(t(1-\varepsilon))}\right] = 0,$$ contradicting (22).

<u>Proof of theorem 11</u>

From (13) it follows that $f^*(1/t) \overset{O}{\sim} \int_0^t s^+(1/x)/x^2dx$, which implies $0 < \underline{i}(s^+(1/x)) \leq \overline{i}(s^+(1/x)) < \infty$ by lemma 3 in [8]. Hence $-\infty < \underline{i}(s) \leq \overline{i}(s) < 0$ by [4], cor.3.7g which implies (21). (See [4], theorem 3.5.) As a consequence $\int_0^t s(x)dx \in AB(a)$ with $a(t) \asymp ts(t)$ $(t \to \infty)$. Now the function f_0 defined by $f_0(t) := (f^*)_*(t)$, the concave upper hull of f, satisfies the relation $f_0(t) \overset{O}{\sim} \int_0^t s(x)dx$ by the analogue of theorem 6 for the transform (5). Hence $f_0 \in AB(a)$. Writing $f_0(t) = c + \int_0^t s_0(x)dx$, we obtain $s_0(t) \asymp s(t)$ $(t \to \infty)$ by lemma 8. As a consequence the function s_0 satisfies (21) and f_0 is strictly concave in the sense of (22). Hence there exists $\varepsilon \in (0,1)$

such that for t sufficiently large any interval $(t-\varepsilon t, t+\varepsilon t)$ will contain a point x with $f(x) = f_0(x)$. Hence $f_0(t) \le f_0(x) = f(x) \le$ $\le f(t+\varepsilon t)$ by monotonicity of f. The inequality $f(t) \le f_0(t)$ follows from definition 2. Hence $f_0 \overset{O}{\sim} f$, which implies that f satisfies (12).

Examples

1. Suppose

$$f(t) = (\log t)^\alpha + O(\log t)^{\alpha-1} \quad (t \to \infty), \quad \alpha \ge 1 \,. \tag{23}$$

By lemma 5 we have $f(t) \overset{O}{\sim} (\log t)^\alpha$ $(t \to \infty)$, hence $f(t) \overset{O}{\sim} \int_0^t s(x)dx$ where $s(x) = \alpha(\log x)^{\alpha-1}/x$, $x > e^{\alpha-1}$, $s(x)$ is decreasing for $x > 0$ and $\int_0^{e^{\alpha-1}} s(x)dx < \infty$. Hence $f^*(u) \overset{O}{\sim} \int_u^{s(0+)} s^{+}(x)dx =$ $(\log \frac{1}{u})^\alpha + \alpha(\alpha-1)(\log\log \frac{1}{u})(\log \frac{1}{u})^{\alpha-1} + O(\log \frac{1}{u})^{\alpha-1}$ $(u \to 0+)$, which is (by lemma 5) equivalent to

$$f^*(u) = (\log \frac{1}{u})^\alpha + \alpha(\alpha-1)(\log\log \frac{1}{u})(\log \frac{1}{u})^{\alpha-1} + O(\log \frac{1}{u})^{\alpha-1} \quad (u \to 0+) \,. \tag{24}$$

Conversely, (23) can be obtained from (24) under the assumption that f is non-decreasing (since the assumptions of theorem 11 are satisfied).

2. If $f(t) \asymp t^\alpha (\log t)^\beta$ $(t \to \infty)$, $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, then $f^*(u) \asymp u^{-\alpha/(1-\alpha)}(-\log u)^{\beta/(1-\alpha)}$ $(u \to 0+)$ and a converse statement holds under the assumption that f is non-decreasing.

Our next result (theorem 14) entails the asymptotic behaviour of $\tilde{f}$, defined by

$$\tilde{f}(s) := \log[s\int_0^\infty \exp\{f(t) - st\}dt], \quad s > 0 \tag{25}$$

in case the function f is asymptotically balanced. It turns out that the asymptotic behaviour of this transform is similar to the behaviour of the complementary function. However in order to obtain a converse statement I had to impose a stronger Tauberian condition. In order to prove the Abelian result we need a lemma. The proof can easily be adapted from [2], theorem 9.

__Lemma 13__

Suppose $s:\mathbb{R}^+ \to \mathbb{R}^+$ is decreasing, $s(\infty) = 0$, $\int_0^1 s(x)dx < \infty$, $ts(t) \to \infty$ $(t \to \infty)$ and let $\exp(f(t))$ be locally integrable. Define the function f_0 by $f_0(t) := \int_0^t s(x)dx$, $t > 0$. Then

$$f(t) \overset{0}{\sim} f_0(t) \quad (t \to \infty) \tag{26}$$

implies

$$\tilde{f}(u) \overset{0}{\sim} \tilde{f}_0(u) \quad (u \to 0+). \tag{27}$$

__Theorem 14__

Suppose $f:\mathbb{R}^+ \to \mathbb{R}$ is locally integrable and $\tilde{f}(s)$ is finite for $s > 0$, where $\tilde{f}$ is defined by (25).

Let $s:\mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, continuous, $\int_0^1 s(x)dx < \infty$, $ts(t) \to \infty$

$(t \to \infty)$ and suppose there exists $c > 1$ such that

$$\overline{\lim_{t\to\infty}}\, s(ct)/s(t) < 1. \tag{28}$$

Then

$$f(t) \overset{O}{\sim} \int_0^t s(x)dx \quad (t \to \infty) \tag{29}$$

implies

$$\tilde{f}(u) \overset{O}{\sim} \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+), \tag{30}$$

where $s^{\leftarrow}$ is the inverse function of s.

Proof of theorem 14

In view of lemma 13 we may assume that $f(t) = \int_0^t s(x)dx$, where s satisfies the assumptions of theorem 14.

Hence the function Δ defined by

$$\Delta(u) := us(t) - f(t + u) + f(t) \tag{31}$$

in convex, positive for $u > -t$, $u \neq 0$ and $\Delta(0) = 0$. By assumption there exists $\alpha < 1$ such that $s(ct) \leq \alpha s(t)$ for $t \geq t_0$. Hence for $y > (c - 1)t$ and $t > t_0$ we have $\Delta'(y) = s(t) - s(t + y) \geq s(t) - s(ct) \geq$ $\geq (1 - \alpha)s(t)$ which implies

$$\Delta(u + ct) = \int_0^{u+ct} \Delta'(x)dx \geq \int_{ct}^{u+ct} \Delta'(x)dx \geq (1 - \alpha)us(t) \tag{32}$$

for all $u > 0$, $t > t_0$.

As a consequence we have for $t > t_0$

$$\tilde{f}(s(t)) = f^*(s(t)) + \log\{s(t)\int_{-t}^{\infty} e^{-\Delta(u)}du\}$$

$$\leq f^*(s(t)) + \log s(t)\{\int_{-t}^{ct} 1\, du + \int_{0}^{\infty} e^{-\Delta(u+ct)}du\}$$

$$\leq f^*(s(t)/c) - \{f^*(s(t)/c) - f^*(s(t))\} +$$

$$+ \log\{(c+1)ts(t) + (1-\alpha)^{-1}\}.$$

Since $f^*(s(t)/c) - f^*(s(t)) = \int_{s(t)/c}^{s(t)} s^{+}(x)dx \geq (1 - c^{-1})ts(t)$ we have $\tilde{f}(s(t)) \leq f^*(s(t)/c) - (1 - c^{-1})ts(t) + \log\{(c+1)ts(t) + (1-\alpha)^{-1}\}$. Now let $t \to \infty$. Then $s(t) \to 0$ and $ts(t) \to \infty$ by assumption. From the last inequality we have $\tilde{f}(s) \leq f^*(s/c)$ for all sufficiently small s. Combination with the inequality $\tilde{f}(s) \geq f^*(s)$ $(s > 0)$ (see [2], lemma 6) finishes the proof.

Corollary 15

If $f \in AB(a)$ with $\bar{i}(a) < 1$, $\lim_{t\to\infty} a(t) = \infty$ satisfies (29) with $s:\mathbb{R}^+ \to \mathbb{R}^+$ decreasing and continuous, f is locally integrable on $\mathbb{R}^+$ and $\tilde{f}(s)$ defined by (25) is finite for $s > 0$, then $\tilde{f} \in AB^0$.

Moreover $\tilde{f}(1/t) \in AB(a_0)$ with $\underline{i}(a_0) > -1$.

Proof

Similar to the proof of theorem 7.

Next we give a converse result.

Theorem 16

Suppose f is non-decreasing, $\tilde{f}(1/t) \in AB(a_0)$ with $\underline{i}(a_0) > -1$. Let $s: \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, continuous, $\int_0^1 s(x)dx < \infty$, $ts(t) \to \infty$ $(t \to \infty)$.

Suppose there exists $M > 0$ (not depending on a) such that

$$\overline{\lim_{t\to\infty}} \frac{s(at)}{s(t)} < \frac{M}{a} \quad \text{for all} \quad a > 1. \tag{33}$$

Then (30) implies (29).

In order to prove theorem 16 we need the following lemma.

Lemma 17

Let $f(t) = \int_0^t s(x)dx$ $(t > 0)$ with s continuous, decreasing, $ts(t) \to \infty$ $(t \to \infty)$ and suppose there exists $M > 0$ (not depending on a) such that (33) holds.

Then for all $c > 1$ there exist constants $\beta_1 \in (0,1)$, β_2, $t_0 > 0$ (depending on c) such that for $t \geq t_0$

$$\log\{s(t)\int_{I^c} e^{f(u)-us(t)}\, du\} \leq \tilde{f}(cs(t)), \tag{34}$$

where I^c is the complement of $I = (t-\beta_1 t, t+\beta_2 t)$.

Proof

Suppose $c > 1$ is arbitrary.

From [2], theorem 10 it follows that

$$\int_{cu}^{\infty} s^{\leftarrow}(x)dx \le \tilde{f}(u) \le \int_{u/c}^{\infty} s^{\leftarrow}(x)dx \quad \text{for} \quad u \le u_0.$$

Hence for $u \le u_0$

$$\tilde{f}(u) \le \int_{u/c}^{\infty} s^{\leftarrow}(x)dx \le \int_{c^2u}^{\infty} s^{\leftarrow}(x)dx + (c^2 - c^{-1})us^{\leftarrow}(u/c),$$

where $s^{\leftarrow}$ is the inverse function of s. From (33) it follows that for u sufficiently small we have $s^{\leftarrow}(u/c) \le 2Mcs^{\leftarrow}(u)$. As a consequence we have

$$\tilde{f}(u) \le \tilde{f}(cu) + 2Mc(c^2 - c^{-1})us^{\leftarrow}(u) \tag{35}$$

for all u sufficiently small.

As in the proof of lemma 8 in [2] we find

$$\int_{t+\beta_2 t}^{\infty} e^{f(u)-us(t)}\, du \le e^{-\gamma_1\beta_2 ts(t)} \int_{t}^{\infty} e^{f(u)-us(t)}\, du \tag{36}$$

for t sufficiently large and $\beta_2 > 0$ (γ_1 is a constant, not depending on β_2).

Also, with Δ as defined in (31), $\Delta(-\beta t) \ge \gamma_2\beta ts(t)$ for $\beta \in (0,1)$ and t sufficiently large, where $\gamma_2 > 0$ is a constant, not depending on β.

Hence since Δ is convex and $\Delta(0) = 0$,

$$\int_0^{t/2} e^{f(u)-us(t)}\,du = e^{f(t)-ts(t)}\int_{-t}^{-t/2} e^{-\Delta(u)}\,du \le$$

$$\le e^{f(t)-ts(t)}\int_{-t}^{-t/2} e^{-\Delta(u+\frac{t}{2})-\Delta(-\frac{t}{2})}\,du \le e^{f(t)-ts(t)-\gamma_2 ts(t)/2}\int_{-t/2}^{0} e^{-\Delta(u)}\,du =$$

$$= e^{-\gamma_2 ts(t)/2}\int_{t/2}^{t} e^{-us(t)+f(u)}\,du$$ for all t sufficiently large. Hence

for $n \in \mathbb{N}$ we have for $t > t(n)$

$$\int_0^{t/2^n} e^{f(u)-us(t)}\,du \le \exp\{-\frac{\gamma_2}{2}\frac{t}{2^{n-1}}\,s(\frac{t}{2^{n-1}})\}\int_0^{t/2^{n-1}} e^{f(u)-us(t)}\,du.$$

Repeated application of this inequality gives

$$\int_0^{t/2^n} e^{f(u)-us(t)}\,du \le \exp\{-\frac{\gamma_2}{2}\sum_{k=1}^{n}\frac{t}{2^{k-1}}\,s(\frac{t}{2^{k-1}})\}\int_0^{t} e^{f(u)-us(t)}\,du.$$

We have by assumption

$\frac{t}{2^{k-1}}\,s(\frac{t}{2^{k-1}})/ts(t) \ge \frac{1}{2M}$ for $t \ge t_n$, $k = 1,\dots,n$, hence

$$s(t)\int_0^{t/2^n} e^{f(u)-us(t)}\,du \le s(t)\; e^{-\frac{\gamma_2 n}{4M}ts(t)}\int_0^{t} e^{f(u)-us(t)}\,du. \qquad (37)$$

Now we take $\beta_1 = 1 - 2^{-n}$.

Combination of (36) and (37) then gives

$$\log\{s(t)\int_{I^c} e^{f(u)-us(t)}\,du\} \le \tilde{f}(s(t)) - \gamma ts(t), \qquad (38)$$

where $\gamma := \min\,(\beta_2\gamma_1,\ n\gamma_2/4M)$.

Then combination of (38) with (35) shows that the inequality (34) is satisfied if we choose $n \in \mathbb{N}$ and $\beta_2 > 0$ so that $\gamma > 2Mc(c^2 - c^{-1})$.

Proof of theorem 16

Define the function h by $h(t) := (\tilde{f})_*(t)$. From $\tilde{f} \geq f^*$ (see [2], lemma 6) it follows that

$$h(t) \geq (f^*)_*(t) \geq f(t) . \tag{39}$$

The latter inequality follows since $(f^*)_*$ is the concave upper hull of f.

From $\tilde{f}(u) \overset{O}{\sim} \int_u^\infty s(x)dx \quad (u \to 0+)$ it follows by the analogue of theorem 6 for the transform f_* that

$$h(t) \overset{O}{\sim} \int_0^t s(x)dx, \qquad t \to \infty . \tag{40}$$

Write $h(t) = c_0 + \int_0^t s_1(x)dx$, where s_1 is non-increasing. As in the proof of theorem 11 we find $s_1(x) \asymp s(x) \quad (x \to \infty)$. Hence s_1 satisfies (33). Let $c > 1$ be arbitrary. By lemma 17 with $f = h$, there exist $\beta_1 \in (0,1)$, $\beta_2 > 0$ such that (34) holds with f replaced by h.

It remains to prove that $f(t) \overset{O}{\sim} h(t) \quad (t \to \infty)$. The proof is by contradiction. If $f(t) \overset{O}{\sim} h(t)$ is not true, then since f and h satisfy (39) for any $c' > 1$ there exists a sequence $\tau_n \to \infty \quad (n \to \infty)$ such that $h(\tau_n/c') \geq f(\tau_n c')$. Take $c' := \max((1 + \beta_2)/(1 - \beta_1), c)$. This implies since f and h are non-decreasing) that $h(t/c') \geq f(t)$ for $\tau_n < t < c'\tau_n$.

Hence $h(t/c') \geq f(t)$ for $t \in I_n := (t_n - \beta_1 t_n,\ t_n + \beta_2 t_n)$ with $t_n := \tau_n/(1-\beta_1)$.

Together with (39) this gives for $s > 0$

$$\tilde{f}(s) = \log\{s \int_0^\infty e^{f(u)-us}\, du\}$$

$$\leq \log\{s \int_{I_n} e^{h(u/c')-us}\, du + s \int_{I_n^c} e^{h(u)-us}\, du\}. \qquad (41)$$

Since (34) holds with f replaced by h, h is non-increasing and $c' \geq c$ we have with $s_n := s(t_n)$

$$\tilde{f}(s_n) \leq \log(e^{h(c's_n)} + e^{h(cs_n)}) \leq \tilde{h}(cs_n) + 1 \qquad (42)$$

for all n sufficiently large.

Since $xs^+(x) \to \infty$ $(x \to 0+)$ by assumption, we have

$$\tilde{h}(s_n\sqrt{c}) - \tilde{h}(s_n c) \to \infty \quad (n \to \infty).$$

Hence (42) implies $\tilde{f}(s_n) \leq \tilde{h}(s_n c)$ for sufficiently large n. Since $c > 1$ is arbitrary, this implies that $\tilde{f}(u) \overset{Q}{\sim} \tilde{h}(u)$ $(u \to 0+)$ is not true.

Application of theorem 14 (note that (28) is satisfied since $\bar{l}(s) < 0$) shows that (40) implies

$$\tilde{h}(u) \overset{Q}{\sim} \int_u^\infty s^+(x)dx, \qquad u \to 0+ . \qquad (43)$$

Now from (30) and (43) it follows that $f(u) \overset{0}{\sim} h(u) \quad (u \to 0+)$, which gives a contradiction.

Remark

In example 1 following the proof of theorem 11 we may replace f^* by $\tilde{f}$ in case $\alpha > 1$, supposed f is locally integrable. The converse statement in example 2 cannot be concluded from theorem 16 since (33) is not satisfied.

Theorem 14 above gives the asymptotic behaviour of the transform $\tilde{f}$ in case the function f satisfies certain conditions; theorem 16 is a corresponding converse statement. In both cases the characterization of the asymptotic behaviour is in terms of the relation $\overset{0}{\sim}$. If one imposes additional conditions it is possible to transform these results to an O-result. This is accomplished in our last theorem.

Theorem 18

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is locally integrable and $\tilde{f}(s)$ is finite for $s > 0$, where $\tilde{f}$ is defined by (25). Suppose $s: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

$$s \text{ is decreasing} \tag{44}$$

$$\lim_{t \to \infty} s(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} ts(t) = \infty, \tag{45}$$

$$\underline{\lim}_{t\to\infty} s(tx)/s(t) > 0 \quad \text{for} \quad x > 1 \tag{46}$$

There exist x_0, $M > 0$ (not depending on x) such that

$$\overline{\lim}_{t\to\infty} s(tx)/s(t) < M/x \quad \text{for all} \quad x \geq x_0 \tag{47}$$

$$\lim_{x\to\infty} \underline{\lim}_{t\to\infty} \int_1^x \frac{s(tu)}{s(t)}\, du = \infty. \tag{48}$$

Then

$$f(t) = \int_0^t s(x)dx + O(ts(t)) \quad (t \to \infty) \tag{49}$$

implies

$$\tilde{f}(u) = \int_u^\infty s^{\leftarrow}(x)dx + O(us^{\leftarrow}(u)) \qquad (u \to 0+), \tag{50}$$

where $s^{\leftarrow}$ is an inverse function to s.

Conversely if $\tilde{f}$ satisfies (50), where s satisfies the assumptions (44) to (48), (47) for all $x > 0$ and f is non-decreasing, then (49) holds.

Proof

From (46), (47), (48) and (49) it follows that $f \in AB(a)$ with $a(t) \asymp ts(t)$ $(t \to \infty)$. Application of lemma 5 shows that (29) is satisfied. Hence we can apply theorem 14 to find (30). (Note that (28) follows from (47).)

Note that lemma 5 has an analogue for AB^o functions. (with $t, x \to \infty$ replaced by $t, x \to 0+$).

The proof of the implication (30) → (50) is immediate from the analogue of of the first part of lemma 5 and is omitted. Conversely, suppose (50) holds. Using the analogue of lemma 5 again, in order to prove the implication (50) $\rightarrow$ (30), we have to show that $\tilde{f} \in AB^{o}$,

$$\overline{\lim_{x\to 0+}}\ \overline{\lim_{t\to 0+}}\ xs^{+}(tx)/s^{+}(t) < \infty \tag{51}$$

and

$$\lim_{x\to 0+}\ \underline{\lim_{t\to 0+}}\ \frac{\tilde{f}(tx) - \tilde{f}(t)}{ts^{+}(t)} = \infty . \tag{52}$$

First we observe that $\tilde{f} \in AB^{o}$ is immediate from (46), (47) and (50). Suppose $x \geq x_0$. In view of (47) there exist constants c_1 (not depending on x) and $t_0 = t_0(x)$ such that

$$xs(tx)/s(t) \leq c_1 \qquad \text{for all } \ t > t_0(x),$$

or, equivalently,

$$s^{+}(c_1 t/x)/s^{+}(t) \leq x \qquad \text{for } \ t < t_1(x).$$

Hence we have for $x \geq x_0$

$$\overline{\lim_{t\to 0+}}\ \frac{c_1}{x}\, s^{+}(\frac{c_1 t}{x})/s^{+}(t) \leq c_1, \tag{53}$$

which gives (51).

Hence, in order to prove (52) it is sufficient to show that

$$\lim_{x\to 0+}\ \underline{\lim_{t\to 0+}}\ \frac{\int_{tx}^{\infty} s^{+}(u)du - \int_{t}^{\infty} s^{+}(u)du}{ts^{+}(t)} = \lim_{x\to 0+}\ \underline{\lim_{t\to 0+}} \int_{x}^{1} \frac{s^{+}(tv)}{s^{+}(t)}\, dv = \infty. \tag{54}$$

Note that

$$\int_x^1 \frac{s^+(tv)}{s^+(t)}\,dv = 1 - \frac{xs^+(tx)}{s^+(t)} + \int_1^{s^+(tx)/s^+(t)} \frac{s(\tau v)}{s(\tau)}\,dv\,, \tag{55}$$

where $\tau = s^+(t)$. Since we now assume (47) for all $x > 0$, we have $s^+(tx)/s^+(t) \geq (Mx)^{-1}$ for $x \in (0,1)$ and $t < t_0(x)$. Hence

$$\underline{\lim}_{t \to 0+} \int_1^{s^+(tx)/s^+(t)} \frac{s(\tau v)}{s(\tau)}\,dv \geq \underline{\lim}_{\tau \to \infty} \int_1^{(Mx)^{-1}} \frac{s(\tau v)}{s(\tau)}\,dv. \tag{56}$$

Combination of (48), (51), (55) and (56) shows that (54) is satisfied.

<u>Corollary 19</u>

If

$$f(t) = \int_0^t s(x)dx + O(ts(t)) \qquad (t \to \infty), \tag{57}$$

with

$$s \text{ decreasing}, \qquad ts(t) \to \infty \quad (t \to \infty) \quad \text{and}$$
$$s(tx)/s(t) \to x^{-1} \quad (t \to \infty) \quad \text{for} \quad x > 0, \tag{58}$$

then

$$\tilde{f}(u) = \int_u^\infty s^+(x)dx + O(us^+(u)) \qquad (u \to 0+)\,. \tag{59}$$

Conversely, if f is non-decreasing and $\tilde{f}$ satisfies (59) with s as in (58), then (57) holds.

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Department of Mathematical Sciences
University of Petroleum & Minerals
Dhahran, Saudi Arabia