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SUPERADDITIVE FUNCTIONS ON RECTANGLES

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INTRODUCTION. The Perron integral, Pfeffer's integral ([P]), and the variational integral ([H1]) and [H2]) all have definitions intimately involving additive or superadditive functions of intervals. For example, a somewhat imprecise and simplified version of either one of Pfeffer's definitions ([P, pp.682-684]) or Henstock's ([H2, pp.50-51]) would say that a function $f(x)$ has integral 0 on an interval I if for each positive ϵ there is a superadditive function M on the interval I such that $|f(x)||B| < M(B)$ when $x \in B$ and B is a sufficiently small interval. Here $|B|$ denotes the Lebesgue measure of the interval B.

Marik [M] proved that for a function f that is Perron and absolutely Perron integrable, the function M above can be taken to be finitely additive. He then posed the following question, repeated by Henstock in [H2] and in a slightly different way by Pfeffer in [P]:

Does there exist a function f of two variables that is Perron-integrable using a finitely superadditive function M but that is not Perron-integrable using a finitely additive function M?

The corresponding question in one dimension has an easy negative answer, for suppose S is a superadditive function of subintervals of the interval $I = [a,b]$. Then defining $A([x,y]) = S([a,y]) - S([a,x])$ yields an additive function A which is at least as large as S and agrees with S on I.

Henstock [H2, pp.58-63] begins to investigate this question in two dimensions by giving examples which show that natural ways of constructing an additive function from a superadditive function S on rectangles do not always work. We completely close off this avenue by giving an example of a superadditive function S of subrectangles of a rectangle I for which there does not exist an additive function A satisfying $A(I) = S(I)$ and $A(K) \geq S(K)$ for each subrectangle K of I.

Of course our example does not answer the question posed above, but it does show that mimicking the easy one-dimensional solution does not work. It may even be that examples such as ours can be used to settle the problem. It is also easy to see that our two-dimensional example trivially gives such examples in every higher dimension.

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DEFINITIONS, ELEMENTARY RESULTS AND MOTIVATION. We are interested only in the two-dimensional case, so by an interval we mean $[a,b] \times [c,d]$, where a, b, c and d are real numbers with $a < b$ and $c < d$. If K denotes this interval, then the interior of K is $(a,b) \times (c,d)$. Two intervals overlap if their interiors have nonempty intersection. A division of an interval K is a finite family of nonoverlapping intervals whose union is K.

A real-valued function S defined on the set of subintervals of an interval I is said to be superadditive on I if $S(K) \geq \Sigma\{S(L) | L \in \mathcal{D}\}$, where K is any subinterval of I and $\mathcal{D}$ is a division of K. S would be called additive on I if "$\geq$" could be replaced by "=" in this definition.

We require just one result before motivating and presenting our example. It says that if a superadditive function S is already additive on certain rectangles, then any additive function $A \geq S$ must agree with S on those rectangles.

Theorem. Suppose that S and A are superadditive and additive functions respectively on an interval I, that $\{K_1, K_2, \ldots, K_n\}$ is a division of I such that $K_1 \cup K_2 \cup \ldots \cup K_i$ is an interval and $S(K_1 \cup K_2 \cup \ldots \cup K_i) = S(K_1 \cup K_2 \cup \ldots \cup K_{i-1}) + S(K_i)$ for each $i = 2,3,\ldots,n$, that $S(L) \leq A(L)$

for each subinterval L of I, and that $S(I) = A(I)$. Then $S(K_i) = A(K_i)$ for each $i = 1,2,\ldots,n$.

Proof. Induct on n, starting with $n = 2$. We then have $K_1 \cup K_2 = I$. By the hypothesis on S and the fact that A is additive we obtain $S(K_1) + S(K_2) = S(I) = A(I) = A(K_1) + A(K_2)$. Since $A(L) \geq S(L)$ for every interval L, we must have $A(K_i) = S(K_i)$ for $i = 1,2$.

If the theorem is true for intervals satisfying the hypotheses when $n = p$ and we have intervals satisfying the hypotheses with $n = p+1$, then the above argument shows that A and S agree on K_{p+1} and $K_1 \cup K_2 \cup \ldots \cup K_p$. The induction hypothesis then implies that A and S agree on each K_i.

The counterexample to follow grew from an attempt to prove that every superadditive function S on I admitted an additive function A on I such that $A(K) \geq S(K)$ for each subinterval K and $A(I) = S(I)$. The idea was to show that such an A existed if we required that $A(I) = S(I)$ and that $A(K) \geq S(K)$ only for intervals K formed from a nice division of I. Then a limiting process would be used to show that the result was true in general.

If $I = [a,b] \times [c,d]$, $a = x_0 < x_1 < \ldots < x_m = b$, and $c = y_0 < y_1 < \ldots < y_n = d$, then we call $\{[x_i, x_{i+1}] \times [y_j, y_{j+1}] \mid i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1\}$ an m-by-n division of I. If S is superadditive on I and D is a 2-by-2 division of I, then one can show that there is an additive function A on I with $A(I) = S(I)$ and $A(K) \geq S(K)$ for each interval K formed from intervals of D. One could hope that some sort of inductive process would then yield the same result for any m-by-n division of I.

However, a careful investigation of the 2-by-3 case showed that this was not the case and led to the following counterexample. The function S defined below is generated from a function defined only on the rectangles formed from a 2-by-3 division of I. The author believes that any function which is defined and superadditive on the rectangles formed from any m-by-n division of I generates a superadditive function on I in the same way, but the proof is extremely tedious and probably not productive.

THE EXAMPLE. Let $I = [0,2] \times [0,3]$. We define a function S on the subintervals of I as follows.

(1) $S(K) = 0$ if K does not contain one of $[0,2] \times [0,1]$, $[0,2] \times [2,3]$, $[0,1] \times [0,2]$, $[0,1] \times [1,3]$ or $[1,2] \times [0,3]$.

(2) $S([0,2] \times [0,d]) = \begin{cases} 1 & \text{if } 1 \le d \le 2 \\ d-1 & \text{if } 2 \le d \le 3 \end{cases}$

(3) $S([0,2] \times [c,3]) = \begin{cases} 1 & \text{if } 1 \le c \le 2 \\ 2-c & \text{if } 0 \le c \le 1 \end{cases}$

(4) If $K = [a,b] \times [c,d]$, then $S(K) = 1$ if $b < 2$, $d-c < 3$ and K contains either $[0,1] \times [0,2]$ or $[0,1] \times [1,3]$.

(5) $S([0,b] \times [0,3]) = b$ if $b \ge 1$.

(6) $S([a,2] \times [0,3]) = 2-a$ if $a \le 1$.

It is a routine but tedious exercise to check that S is well-defined for all subintervals K of I. We must show that S is superadditive on I.

Suppose that K is a subinterval of I and that D is a division of K. For convenience we use just Σ to denote $\Sigma\{S(J) \mid J \in D\}$ and for any interval $J = [r,s] \times [t,u]$ we let $a(J) = r$, $b(J) = s$, $c(J) = t$ and $d(J) = u$.

We note that the values of S are all nonnegative. If each member of D falls into Case (1) of the definition of S, then $\Sigma = 0$, so that certainly $\Sigma \le S(K)$.

Because D is a division, at most one member of D can fall into any one of Cases (2) - (6). Further, at most two members of D can fail to fall into Case (1) and these two members must then fall into (i) Cases (2) and (3), or (ii) Cases (2) and (4), or (iii) Cases (3) and (4), or (iv) Cases (4) and (6), or (v) Cases (5) and (6). We suppose that these two members of D are J and L respectively, so that $\Sigma = S(J) + S(L)$.

In situation (i) we must have $K = I$ and $1 \leq d(J) \leq c(L) \leq 2$. Thus $S(J) = 1$ and $S(L) = 1$, so that $\Sigma = 2 = S(K)$. In situation (ii) we must have $d(J) = 1$ and L must contain $[0,1] \times [1,3]$, so that $K = I$ and $\Sigma = 1 + 1 = S(K)$. Similarly, in situation (iii) we must have $c(J) = 2$ and L must contain $[0,1] \times [0,2]$, so that $K = I$ and $\Sigma = 1 + 1 = S(K)$.

In situation (iv) we must have $1 \leq b(J) \leq a(L) \leq 1$, so that $a(L) = 1$, $S(J) = S(L) = 1$ and again $K = I$, so that $\Sigma = S(K)$. Finally, in situation (v) we must have $1 \leq b(J) \leq a(L) \leq 1$, so that $b(J) = a(L) = 1$ and $S(J) = S(L) = 1$. Again $K = I$ so that $\Sigma = S(K)$. We have established that S is superadditive on I.

Now suppose that A is an additive function on I with $A(K) \geq S(K)$ for each subinterval K of I and $A(I) = S(I)$. It is easy to check from the definition of S that S is additive on intervals of the forms $[0,2] \times [i,j]$ or $[i,j] \times [0,3]$, where i and j are integers. It follows from the theorem proved earlier that A must agree with S on intervals having either of these forms.

Set $A([0,1] \times [0,1]) = \varepsilon$ and $A([0,1] \times [1,2]) = \delta$. Since $A \geq S$, we have $\varepsilon \geq 0$ and $\delta \geq 0$. Since A is additive and $A([0,1] \times [0,3]) = S([0,1] \times [0,3]) = 1$, it must be the case that $A([0,1] \times [1,3]) = 1 - \varepsilon$, so that $1 - \varepsilon \geq S([0,1] \times [1,3]) = 1$ and we immediately conclude that $\varepsilon = 0$. Further, $0 = S([0,2] \times [1,2]) = A([0,2] \times [1,2]) = \delta + A([1,2] \times [1,2]) \geq \delta$ since the values of A must be nonnegative. We immediately conclude that $\delta = 0$.

By additivity of A we obtain $A([0,1] \times [0,2]) = \varepsilon + \delta = 0$. However, $S([0,1] \times [0,2]) = 1$. We are forced to conclude that an additive A agreeing with S on I and no smaller than S on other subintervals cannot exist.

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