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UNIFORM DISTRIBUTIONAL CONVERGENCE OF PROBABILITY DENSITY FUNCTION AND SOME OF ITS FUNCTIONALS

by

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Abstract

Uniform distributional convergence are proved for kernel-type estimators of probability density, distribution function, p-th order qualtile and failure rate function in the case where the obervations are from identically distributed random sequence.

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1. Introduction

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (iid) random variables with continuous probability density function (p.d.f.) f and distribution function (d.f.) F and defined on the probability space (Ω, β, P). Let $\{f_n(x)\}_{n=1}^{\infty}$ be the sequence of estimators of the density function given by:

$$f_n(x) = \frac{1}{na_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{a_n}\right) \tag{1.1}$$

where K is a known symmetric p.d.f. satisfying the following condition:

(A1) $\sup_{-\infty<u<\infty} K(u) < \infty$ and $\lim_{|u|\to\infty} |u|K(u) = 0$

and $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive and real numbers such that $a_n \to 0$ as $n \to \infty$.

The main object of this paper is to establish the uniform convergence in distribution of the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of estimators and some of its functionals on the interval [0,1]. Kernel-type estimators of probability density and its functionals have been studied extensively by several authors (see Prakasa Rao [7]). We have established the uniform convergence in distribution of the estimators and functionals which is a natural extension of the work done in this area.

2. Preliminaries

Let us now combine the results of this with those established by Parzen [6], Nadarya [5] and Watson and Leadbetter [10] to obtain concrete criteria for convergence in distribution.

Let X_n and X be random elements of D space (see Billingsly [3]). Write T_X for T_P, where P is the distribution of X. Thus T_X contains 0 and 1, and, if $0 < t < 1$, $t \in T_X$ if and only if $P\{X(t) \neq X(t-)\} = 0$.

Theorem 1. Suppose that

$$(X_n(t_1),\ldots,X_n(t_r)) \xrightarrow{\mathcal{D}} (X(t_1),\ldots,X(t_r))$$

holds whenever $t_1,\ldots,t_r$ all lie in T_X, that

$$P\{X(1) \neq P(X(1-)\} = 0; \quad \text{and}$$

$$E\{|X_n(t) - X_n(t_1)|^\gamma |X_n(t_2) - X_n(t)|^\gamma\} \leq [F(t_2) - F(t_1)]^{2\alpha}$$

for $t_1 \leq t \leq t_2$ and $n \geq 1$, where $\gamma \geq 0$, $\alpha > \frac{1}{2}$ and F is a non-decreasing continuous function on $[0,1]$. Then

$$X_n \xrightarrow{\mathcal{D}} X$$

Proof. Can be found in Billingsly [3], pp.128-130.

3. Density Function.

We define $f_n(x)$ by (1.1) and assume that the following conditions on the functions f and K and the sequence $\{a_n\}$ are satisfied.

(A2) $\int u K(u)du = 0$ and $\int u^m K(u)du < \infty;$ $m = 2,3,$

(A3) $K(u) \in Lip_1(R),$

(A4) f has uniformly bounded derivative up to the third order

(A5) f is a nondecreasing function on $[0,1]$, and

(A6) $na_n \to \infty,$ $na_n^3 \geq 1$ and $na_n^5 \to 0$ as $n \to \infty.$

Parzen [6] has shown that

$$\sqrt{na_n}\,[f_n(x) - EF_n(x)] \xrightarrow{\mathcal{D}} (0,\sigma^2) \qquad (\text{as } n \to \infty) \tag{3.1}$$

where

$$\sigma^2 = f(x) \int K^2(u)du \tag{3.3}$$

In this section it is shown that

$$Y_{1n} \xrightarrow{\mathcal{D}} N \qquad (\text{as } n \to \infty)$$

where N is a normal random variable and

$$Y_{1n}(t) = \sqrt{na_n}\,[f_n(t) - f(t)] \qquad (0 \le t \le 1) \tag{3.3}$$

First, we have to prove the following two lemmas.

Lemma 1. Assume conditions A2, A4 and A6 are satisfied, then

$$Y_{1n} \xrightarrow{\mathcal{D}} N(0,\sigma^2) \qquad (\text{as} \quad n \to \infty) \tag{3.4}$$

Proof. Using Taylor's expansion and by A2

$$\begin{aligned} Ef_n(t) - f(t) &= \frac{1}{a_n}\int f(x)\, K(\frac{t - x}{a_n})dx - f(t) \\ &= \int [f(t - a_n z) - f(t)]K(z)dz \qquad (3.5) \\ &= \frac{a_n^2\, f''(t)}{2} \int z^2 K(z)dz + f'''(\xi)a_n^3 \int u^3 K(u)du + O(a_n^3) \end{aligned}$$

where ξ is random point between $t - a_n z$ and t.

Thus, by A4 and A6

$$\sqrt{na_n}\,[Ef_n(t) - f(t)] = \frac{\sqrt{na_n^5}}{2} f''(t) \int z^2 K(z)dz \to 0 \qquad \text{as} \quad n \to \infty.$$

From (3.1) and (3.5), we can say that

$$Y_{1n} = \sqrt{na_n}\,[f_n(t) - f(t)] \xrightarrow{\mathcal{D}} N(0,\sigma^2) \quad \text{as} \quad n \to \infty .$$

Lemma 2. Assume that conditions A1, A2, A4, and A6 are satisfied, then

$$\mathbf{Y_1} = [Y_{1n}(t_1), \ldots, Y_{1n}(t_r)]' \longrightarrow \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}) \quad (\text{as} \quad n \to \infty) \tag{3.6}$$

where $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$ is r-variate normal distribution

$$\mathbf{\Sigma} = \text{diag}\left(f(t_\alpha)\int K^2(z)dz\right); \qquad \alpha = 1,\ldots,r, \tag{3.7}$$

and $t_1,\ldots,t_r$ all lie in T_X.

Proof. It is enough to show that $\mathbf{Z_1} = [Z_{1n}(t_1),\ldots,Z_{1n}(t_r)]'$ has r-dimensional normal as a limiting distribution, where

$$Z_{1n}(t_\alpha) = \sqrt{na_n}[f_n(t_\alpha) - E\, f_n(t_\alpha)]; \qquad \alpha = 1,\ldots,r. \tag{3.8}$$

Define, for $j = 1,\ldots,n$,

$$V_{nj}(t_\alpha) = \frac{1}{\sqrt{na_n}}\left[K\left(\frac{t_\alpha - X_j}{a_n}\right) - EK\left(\frac{t_\alpha - X_j}{a_n}\right)\right] \tag{3.9}$$

Then it follows from (3.9) that; for $\alpha = 1,\ldots,r$;

$$Z_{1n}(t_\alpha) = \sum_{j=1}^{n} V_{nj}(t_\alpha) \tag{3.10}$$

The lemma can be proved if we can show that for any nonzero real constants c_i, $i = 1,2$; the linear combination

$$B_{1n} = c_1 Z_{1n}(t_1) + c_2 Z_{1n}(t_2)$$

has a limiting normal distribution. B_{1n} may be rewritten as

$$B_{1n} = \sum_{j=1}^{n} \sum_{=1}^{2} c_\alpha V_{nj}(t_\alpha) = \sum_{j=1}^{n} T_{nj} \tag{3.11}$$

Since, for a fixed t_1, t_2 and n, the random variables $T_{n1}, \ldots, T_{nn}$ are iid, and the sufficient condition under which B_{1n} converges to normal in distribution is that

$$\frac{n^{-\frac{1}{2}} E|nT_{n1}|^3}{(\mathrm{Var}\; nT_{n1})^{3/2}} \longrightarrow 0 \quad \text{as } n \to \infty,$$

and this is an application to Lyapounov's Theorem (see Loeve [4]).

To evaluate the asymptotic moments of T_{n1}, as it is necessary to obtain the asymptotic moments of $V_{n1}(t_\alpha)$; $\alpha = 1,2$. Parzen has shown that

$$na_n \;\mathrm{Var}\; f_n(t_\alpha) \doteq f(t_\alpha) \int K^2(z)dz$$

which implies that

$$n \,\mathrm{Var}\; V_{n1}(t_\alpha) \doteq f(t_\alpha) \int K^2(z)dz < \infty \tag{3.12}$$

Abdulal and Siddiqui [1] have shown that

$$\text{Cov}[K(\frac{t_1 - X}{a_n}), \quad K(\frac{t_2 - X}{a_n})] = O(a_n^2)$$

which implies that

$$n \text{ Cov}[V_{n1}(t_1), V_{n1}(t_2)] = \frac{1}{a_n} \text{Cov}[K(\frac{t_1 - X}{a_n}), \quad K(\frac{t_2 - X}{a_n})] \tag{3.13}$$

(3.12) and (3.13) imply that

$$\lim_{n \to \infty} n \text{ Var } T_{n1} < \infty . \tag{3.14}$$

It remains to show that

$$n \text{ E}|T_{n1}|^3 \longrightarrow 0 \qquad \text{as} \quad n \to \infty .$$

Using (3.11) and the c_r-inequality, it follows that

$$E|T_{n1}|^3 \leq 2^3[|c_1|^3 E|V_{n1}(t_1)|^3 + |c_2|^3 E|V_{n1}(t_2)|^3] .$$

But, for $\alpha = 1,2$;

$$n \text{ E}|V_{n1}(t_\alpha)|^3 \simeq \frac{1}{n^{\frac{1}{2}} a_n^{3/2}} \int K^3(\frac{t_\alpha - z}{a_n}) f(z) dz$$

$$= \frac{1}{\sqrt{na_n}} \int K^3(z)\ f(t_\alpha - a_n z)dz$$

$$\tilde{=} \frac{1}{\sqrt{na_n}} f(t_\alpha) \int K^3(z)dz$$

(3.15) and A6 imply that

$$n\,E|T_{n1}|^3 \le \frac{8}{\sqrt{na_n}} [\sum_{\alpha=1}^{2} |c_\alpha|^3 f(t_\alpha) \int K^3(z)dz] \longrightarrow 0 \quad \text{as} \quad n \to \infty. \qquad (3.16)$$

From (3.14) and (3.16), we get

$$\frac{n^{-\frac{1}{2}}\ E|n\ T_{n1}|^3}{(\mathrm{Var}\, n\, T_{n1})^{3/2}} = \frac{n\ E|T_{n1}|^3}{(n\ \mathrm{Var}\ T_{n1})^{3/2}} \longrightarrow 0 \qquad (\text{as} \quad n \to \infty) \qquad \Box$$

In view of Theorem 1 to show that $Y_{1n} \longrightarrow N$ as $n \to \infty$, it suffices to show that; for $t_1 \le t \le t_2$;

$$E|Y_{1n}(t) - Y_{1n}(t_1)||Y_{n1}(t_2) - Y_{1n}(t)| \le [F(t_2) - F(t_1)]^2$$

where F is a nondecreasing function in $[0,1]$.

Theorem 2. If conditions A1-A6 are satisfied, then

$$Y_{1n} \longrightarrow N \qquad (\text{as} \quad n \to \infty)$$

Proof. For $t_1 \le t \le t_2$

$$E|Y_{1n}(t) - Y_{1n}(t_1)||Y_{1n}(t_2) - Y_{1n}(t)|$$

$$= na_n\, E|f_n(t) - f_n(t_1) - (f(t) - f(t_1))||f_n(t_2) - f_n(t) - (f(t_2) - f(t))|$$

$$\leq na_n\, E|f_n(t) - f_n(t_1)||f_n(t_2) - f_n(t)|$$

$$= \frac{1}{na_n} E|K(\frac{t - X}{a_n}) - K(\frac{t_1 - X}{a_n})||K(\frac{t_2 - X}{a_n}) - K(\frac{t - X}{a_n})|$$

$$\leq \frac{M}{na_n^3} |t - t_1||t_2 - t| \leq M(t_2 - t + t - t_1)^2$$

$$= M(t_2 - t_1)^2$$

The second and fourth expressions are obtained due to A5 and A3, respectively. The fifth expression is obtained since $xy \leq (x + y)^2$ and $na_n^3 \geq 1$. □

4. Distribution function.

In analogy with the kernel method of density estimation described in Section 3, we shall now study the uniform convergence in distribution of the sequence $\{F_n(x)\}_{n=1}^{\infty}$ on the interval $[0,1]$, where $F_n(X)$ are the estimators of the d.f. F and defined by

$$F_n(x) = \int_{-\infty}^{x} f_n(z)dz, \tag{4.1}$$

Nadarya [5] has shown that

$$\sqrt{n}(F_n(x) - E\, F_n(x)) \xrightarrow{\mathcal{D}} (0,\sigma_1^2) \quad (\text{as} \quad n \to \infty) \tag{4.2}$$

where

$$\sigma_1^2 = F(x)[1 - F(x)] \tag{4.3}$$

We now assume that the d.f. F and the sequence $\{a_n\}$ satisfy the following conditions:

(A7) F has bounded first and second derivatives,

(A8) $na_n \to \infty$, $na_n^2 \geq 1$ and $na_n^4 \to 0$ as $n \to \infty$.

In this section, we will show that

$$Y_{2n} \xrightarrow{\mathcal{D}} N \qquad (\text{as} \quad n \to \infty) \tag{4.4}$$

where

$$Y_{2n}(t) = \sqrt{n}\,[F_n(t) - F(t)] \qquad (0 \leq t \leq 1)$$

To show (4.4), we need to prove the following two lemmas and their proofs resemble that of the proofs of Lemma 1 and Lemma 2.

Lemma 3. Assume condition A2, A7 and A8 are satisfied then

$$Y_{2n}(t) \xrightarrow{\mathcal{D}} N(0,\sigma_1^2) \qquad (\text{as} \quad n \to \infty) \tag{4.5}$$

Proof. It is analogous to the proof of Lemma 1. The details are left to the reader.

Lemma 4. Assume that conditions A1, A2, A7 and A8 are satisfied, then

$$\mathbf{Y}_2 = [Y_{2n}(t_1),\ldots,Y_{2n}(t_r)]' \longrightarrow N(0, \Sigma_1) \quad (\text{as } n \to \infty)$$

where

$$\Sigma_1 = (\sigma_{ij}) \tag{4.6}$$

with

$$\sigma_{ij} = \begin{cases} F(t_i)[1 - F(t_i)] & \text{if } i = j, \ i = 1,\ldots,r \\ F[\min(t_i, t_{i+1})] - F(t_i)F(t_{i+1}) & \text{if } i \neq j \end{cases}$$

and $t_1,\ldots,t_r$ all lie in T_X.

Proof. Again it is enough to show that $\mathbf{Z}_2 = [Z_{2n}(t_1),\ldots,Z_{2n}(t_r)]$ has r-dimensional normal as limiting distribution, where

$$Z_{2n}(t_\alpha) = \sqrt{n}\,[F_n(t_\alpha) - E\,F_n(t_\alpha)]; \quad \alpha = 1,\ldots,r_j \tag{4.7}$$

We define; for $j = 1,\ldots,n$;

$$V^*_{nj}(t_\alpha) = \frac{1}{a_n\sqrt{n}}\left[\int_{-\infty}^{t_\alpha} K\left(\frac{s - X_j}{a_n}\right)ds - E\int_{-\infty}^{t_\alpha} K\left(\frac{s - X_j}{a_n}\right)ds\right]$$

It is easy to see that

$$Z_{2n}(t_\alpha) = \sum_{j=1}^{n} V^*_{nj}(t_\alpha) \tag{4.8}$$

$\mathbf{Z}_2$ will have r-dimensional normal as a limiting distribution, if it can be shown for any nonzero constants b_1 and b_2 the linear combination

$$B_{2n} = b_1 Z_{2n}(t_1) + b_2 Z_{2n}(t_2)$$

has a limiting normal distribution. But

$$B_{2n} = \sum_{j=1}^{n} \sum_{\alpha=1}^{2} b\, V^*_{nj}(t_\alpha) = \sum_{j=1}^{n} T^*_{nj} \tag{4.9}$$

where $T^*_{n1},\ldots,T^*_{nn}$ are iid random variables for fixed t_1, t_2 and n. We need to show that

$$\frac{n^{-\frac{1}{2}} E|nT^*_{n1}|^3}{(\text{Var}\, n\, T^*_{n1})^{3/2}} \longrightarrow 0 \qquad \text{as} \quad (n \to \infty) \tag{4.10}$$

Nadarya has shown that

$$n \text{ Var } F_n(t_\alpha) \cong F(t_\alpha)[1 - F(t_\alpha)] < \infty$$

which implies that

$$n \text{Var } V^*_{n1}(t_\alpha) \cong F(t_\alpha)[1 - F(t_\alpha)] < \infty \tag{4.11}$$

Now

$$n \text{ Cov}(V^*_{n1}(t_1),\ V^*_{n1}(t_2)) = \frac{1}{a_n^2} \iint_{-\infty}^{t_2}\int_{-\infty}^{t_1} K\left(\frac{s_1-x}{a_n}\right)K\left(\frac{s_2-x}{a_n}\right) f(x)\, ds_1 ds_2 dx$$

$$- E\, F_n(t_1)\ EF_n(t_2)$$

$$= \iint_{-\infty}^{t_2}\int_{-\infty}^{t_1} K(z)K\left(\frac{s_2-s_1}{a_n} + z\right) f(s_1 - a_n z)\, ds_1 ds_2 dz - EF_n(t_1)EF_n(t_2)$$

$$= \iint_{-\infty}^{\frac{t_2-s_1}{a_n}} \int_{-\infty}^{t_2} K(z)K(w + z) f(s_1 - a_n z)\, ds_1 dw dz - EF_n(t_1)EF_n(t_2)$$

$$= \iint_{-\infty}^{\frac{\max(t_2,t_1)-s}{a_n}} \int_{-\infty}^{\min(t_1,t_2)} K(z)K(w+z)f(s-a_n z)\,ds\,dw\,dz - EF_n(t_1)EF_n(t_2)$$

$$= F[\min(t_1,\ t_2)] - F(t_1)F(t_2) \tag{4.12}$$

From (4.11) and (4.12), we get

$$\lim_{n\to\infty} n \operatorname{Var} T^*_{n1} < \infty \tag{4.13}$$

Finally, we show that

$$n\ E|T^*_{n1}|^3 \longrightarrow 0 \qquad \text{as} \quad n \to \infty.$$

But

$$E|T^*_{n1}|^3 \le 2^3[|b_1|^3 E|V^*_{n1}(t_1)| + |b_2|^3 E|V_n(t_2)|^3]$$

and, for $\alpha = 1,2$,

$$n^{3/2}\ E\ V^*_{n1}(t_\alpha)|^3 = \int \left| \int_{-\infty}^{t_\alpha} \frac{1}{a_n} K(\frac{s-x}{a_n})ds - EF_n(t_\alpha)\right|^3 f(x)dx$$

$$= \int \left| \int_{-\infty}^{\frac{t_\alpha - x}{a_n}} K(z)dz - EF_n(t_\alpha)\right|^3 f(x)dx$$

$$\longrightarrow \begin{cases} \int |1 - F(x)|^3 f(x)dx & \text{if} \quad x < t_\alpha \\ \int |\frac{1}{2} - F(x)|^3 f(x)dx & \text{if} \quad x = t_\alpha \\ \int |F(x)|^3 f(x)dx & \text{if} \quad x > t_\alpha \end{cases}$$

i.e.

$$n\,E|T^*_{n1}|^3 \longrightarrow 0 \qquad \text{as} \quad n \to \infty . \qquad \square$$

Finally, we establish the uniform convergence of $Y_{2n}(t)$.

Theorem 3. Assume conditions of Lemma 3 and Lemma 4 are satisfied, then

$$Y_{2n} \longrightarrow N \qquad \text{as} \quad n \to \infty .$$

Proof. Again, it is enough to show; for $t_1 \le t \le t_2$

$$E|Y_{2n}(t) - Y_{2n}(t_1)||Y_{2n}(t_2) - Y_{2n}(t)| \le [G(t_2) - G(t_1)]^2$$

Now,

$$E|Y_{2n}(t) - Y_{2n}(t_1)||Y_{2n}(t_2) - Y_{2n}(t)|$$

$$\le \frac{1}{na_n^2} E\left|\int_{-\infty}^{t} K\left(\frac{s-X}{a_n}\right)ds - \int_{-\infty}^{t_1} K\left(\frac{s-X}{a_n}\right)ds\right|\left|\int_{-\infty}^{t_2} K\left(\frac{s-X}{a_n}\right)ds - \int_{-\infty}^{t} K\left(\frac{s-X}{a_n}\right)ds\right|$$

$$= \frac{1}{n} E\left|\int_{\frac{t_1-X}{a_n}}^{\frac{t-X}{a_n}} K(z)dz\right|\left|\int_{\frac{t-X}{a_n}}^{\frac{t_2-X}{a_n}} K(z)dz\right|$$

$$\le \frac{1}{na_n^2} [\sup_z K(z)]^2 |t - t_1||t_2 - t| \le M[t_2 - t_1]^2 . \qquad \square$$

5. Quantile.

Let t_p denote the p-th order quantile of the d.f. F, i.e.,

$$F(t_p) = p \qquad\qquad 0 < p < 1$$

We assume that it is unique. As an approximation to t_p, we take

$$F_n(\hat{t}_p) = p$$

where $\hat{t}_p$ is the sample quantile. Nadarya [4] has showed that

$$\sqrt{n}\ f(t_p)(\hat{t}_p - t_p) \xrightarrow{\mathcal{D}} N(0,\ \sigma_1^2) \qquad (\text{as } n \to \infty) \tag{5.1}$$

and

$$\frac{\sqrt{n}\ f(t_p)(\hat{t}_p - t_p)}{\sqrt{F(t_p)(1 - F(t_p)}} \quad \text{and} \quad \frac{\sqrt{n}\ [F_n(t_p) - EF_n(t_p)]}{\sqrt{F(t_p)(1 - F(t_p)}}$$

have the same limiting normal distribution under the conditions of the following theorem.

Theorem 4. Assume that the conditions of Lemma 4 hold. If f is uniformly continuous, then

(i) $$Y_3 = [Y_{3n}(t_{p_1}),\ldots,Y_{3n}(t_{p_r})]' \xrightarrow{\mathcal{D}} N(0,\ \Sigma_1) \tag{5.2}$$

where

$$Y_{3n}(t_{p_\alpha}) = \sqrt{n}\ f_n(t_{p_\alpha})(\hat{t}_{p_\alpha} - t_{p_\alpha}),$$

Σ_1 is defined as in (4.6), and

$t_{p_1},\ldots,t_{p_r}$ all lie in T_X.

(ii) $$Y_{3n} \xrightarrow{\mathcal{D}} N \qquad \text{as } n \to \infty .$$

Proof. For part (i), it is enough to show that for any constants d_1 and d_2

$$B_{3n} = \sqrt{n}\,[d_1 f_n(t_{p_1})(\hat{t}_{p_1} - t_{p_1}) + d_2 f_n(t_{p_2})(\hat{t}_{p_2} - t_{p_2})]$$

has a limiting normal distribution. But B_{3n} and B_{2n} have the same limiting normal distribution.

The proof of part (ii) is much involved. Some preliminary results will facilitate it. Applying Taylor's expansion to $F_n(\hat{t}_p)$, we get

$$P = F(\hat{t}_p) = F_n(\hat{t}_p) = F_n(t_p) + (\hat{t}_p - t_p)f_n(\theta) \tag{5.3}$$

where θ is a random point between $\hat{t}_p$ and t_p. From (5.3), we obtain

$$f(t_p)(\hat{t}_p - t_p) = [F_n(t_p) - F(t_p)]\,\frac{f(t_p)}{f_n(\theta)} \tag{5.4}$$

In view of Theorem 1, we need to show, for $t_{p_1} \le t_p \le t_{p_2}$, that

$$E|Y_{3n}(t_p) - Y_{3n}(t_{p_1})||Y_{3n}(t_{p_2}) - Y_{3n}(t_p)| \le [G(t_{p_2}) - G(t_{p_1})]^2$$

where G is a nondecreasing function in $[0,1]$.

Now,

$$E|Y_{3n}(t_p) - Y_{3n}(t_{p_1})||Y_{3n}(t_{p_2}) - Y_{3n}(t_p)|$$

$$= n\,E\,|f(t_p)(\hat{t}_p - t_p) - f(t_{p_1})(\hat{t}_{p_1} - t_{p_1})||f(t_{p_2})(\hat{t}_{p_2} - t_{p_2}) - f(t_p)(\hat{t}_p - t_p)|$$

$$= n\,E\left|\frac{f(t_p)}{f_n(\theta)}\,[F_n(t_p) - F(t_p)] - \frac{f(t_{p_1})}{f_n(\theta_1)}\,[F_n(t_{p_1}) - F(t_{p_1})]\right| \qquad (5.5)$$

$$\times \left|\frac{f(t_{p_2})}{f_n(\theta_2)}\,[F_n(t_{p_2}) - F(t_{p_2})] - \frac{f(t_p)}{f_n(\theta)}\,[F_n(t_p) - F(t_p)]\right|$$

But

$$\frac{f(t_p)}{f_n(\theta)} = \frac{f(t_p)}{f(t_p) + f_n(\theta) - f(t_p)} = \frac{1}{1 - \dfrac{f(t_p) - f_n(\theta)}{f(t_p)}} \qquad (5.6)$$

$$= \sum_{j=1}^{\infty} \left|\frac{f(t_p) - f_n(\theta)}{f(t_p)}\right|^j$$

$$= 1 + O(1)$$

From (5.5) and (5.6), we have that

$$E|Y_{3n}(t_p) - Y_{3n}(t_{p_1})||Y_{3n}(t_{p_2}) - Y_{3n}(t_p)|$$

$$\le n\,E|F_n(t_p) - F(t_p) - (F_n(t_{p_1}) - F(t_{p_1}))|$$

$$\times |F_n(t_{p_2}) - F(t_{p_2}) - (F_n(t_p) - F(t_p)|$$

$$\le K(t_{p_2} - t_{p_1})^2$$

The last inequality is from the proof of Theorem 3. □

6. Failure Rate Function.

The failure is given by

$$h(x) = \frac{F(x)}{1 - F(x)}$$

for all x such that $F(x) < 1$. Using $f_n(x)$ and $F_n(x)$ given by (1.1) and (4.1), respectively, we estimate $h(x)$ by

$$h_n(x) = \frac{f_n(x)}{1 - F_n(x)} \tag{6.1}$$

Watson and Leadbetter [9] have proved that

$$\sqrt{na_n}\ [h_n(x) - h(x)] \longrightarrow N(0,\ \sigma_3^2) \qquad (\text{as} \quad n \to \infty) \tag{6.2}$$

where

$$\sigma_3^2 = \frac{f(x)}{[1 - F(x)]^2} \int K^2(z)\,dz \tag{6.3}$$

In the following Lemma the limiting distribution of $[h_n(t_1),\ldots,h_n(t_r)]'$ is derived. It is a special case of Theorem 4.1 of Ahmad and Lin [2]. Its proof is based on a generalization of Theorem (iii) of Rao [8], p.322. The proof is omitted.

Lemma 5. If conditions of Lemma 4 hold, then

$$\mathbf{Y}_4 = [Y_{4n}(t_1),\ldots,Y_{4n}(t_r)]' \xrightarrow{\mathcal{D}} N(0,\ \Sigma_2)$$

where

$$Y_{4n}(t_\alpha) = \sqrt{na_n}\,[h_n(t_\alpha) - h(t_\alpha)], \qquad \alpha = 1,\dots,r$$

and

$$\Sigma_2 = (\sigma_{\alpha\beta}) \qquad \alpha,\beta = 1,\dots,r$$

with

$$= \begin{cases} [\bar{F}(t_\alpha)]^{-2} f(t_\alpha) \int K^2(z)\,dz + h^2(t_\alpha)F(t_\alpha)[\bar{F}(t_\alpha)]^{-1} & \text{for } \alpha = \beta \\ h(t_\alpha)h(t_\beta)\gamma_{\alpha\beta}[\bar{F}(t_\alpha)]^{-1}[\bar{F}(t_\beta)]^{-1} & \text{for } \alpha \neq \beta \end{cases}$$

$$\gamma_{\alpha\beta} = F(\min(t_\alpha, t_\beta)) - F(t_\alpha)F(t_\beta),$$

$$\bar{F}(t_\alpha) = 1 - F(t_\alpha),$$

and $t_1,\dots,t_r$ all lie in T_X.

Finally, we establish the uniform distributional convergence of $Y_{4n}(t)$ under the assumption of Theorem 2 and Theorem 4, together.

Theorem 5. Assume that the conditions of Theorem 2 and Theorem 4 hold. If $\bar{F}(x) > \delta > 0$, then

$$Y_{4n} \xrightarrow{\mathcal{D}} N \qquad \text{as } n \to \infty .$$

Proof. In view of Theorem 1 and Lemma 5, it suffices to show, for $t_1 \le t \le t_2$, that

$$E|Y_{4n}(t) - Y_{4n}(t_1)|\,|Y_{4n}(t_2) - Y_{4n}(t)| \le [G^*(t_2) - G^*(t_1)]^2$$

where G^* is a nondecreasing function in $[0,1]$.

Rice and Rosenblatt [8] have shown that

$$h_n(t) = \frac{f_n(t)}{1 - F(t)}\left(1 + O_p\left(\frac{1}{\sqrt{n}}\right)\right) \tag{6.4}$$

$$= a_n(t)\left(1 + O_p\left(\frac{1}{\sqrt{n}}\right)\right), \quad \text{say.}$$

(6.4) implies that

$$E|Y_{4n}(t) - Y_{4n}(t_1)||Y_{4n}(t_2) - Y_{4n}(t)|$$

$$\leq na_n\ E|h_n(t)-h_n(t_1) - (h(t)-h(t_1))||h_n(t_2)-h_n(t) - (h(t_2)-h(t))||$$

$$\leq na_n\ E|a_n(t)-a_n(t_1) - (h(t)-h(t_1)||a_n(t_2)-a_n(t) - (h(t_2)-h(t))|$$

$$\leq na_n\ \delta E|f_n(t)-f_n(t_1) - (f(t)-f(t_1))||f_n(t_2)-f_n(t) - (f(t)-f(t_1))|$$

$$\leq \frac{\delta}{na_n} E\left|K\left(\frac{t-X}{a_n}\right) - K\left(\frac{t_1-X}{a_n}\right)\right|\left|K\left(\frac{t_2-X}{a_n}\right) - K\left(\frac{t-X}{a_n}\right)\right|$$

$$\leq \frac{\delta}{na_n^3} M|t - t_1||t_2 - t| \leq M^*(t_2 - t_1)^2 . \qquad \square$$

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