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Monoids Characterized by Their Quasi Injective S-Systems

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1. Introduction.

In this paper, S will denote a monoid, that is, a semigroup with identity 1, which contains a zero 0. Each S-system is assumed to be right unitary (i.e. $M1 = M$) and centered (i.e. $m0 = 0s = 0$ for the zero 0 of M). A right S-system M_S is injective if and only if, for every S-monomorphism $g: P_S \to Q_S$ and for every S-homomorphism $h: P_S \to M_S$, there exists an S-homomorphism $h^*: Q_S \to M_S$ such that $h^*g = h$. Injective S-systems were originally introduced by Berthiaume [1], and later studied by many authors. Further references to this subject can be found in [6]. An S-system M_S is quasi-injective if for $N_S \subseteq M_S$ and every S-homomorphism $f: N_S \to M_S$, there exists an S-homomorphism $g: M_S \to M_S$ such that $g|_N = f$. Every injective S-system is quasi-injective, but the converse is false. Quasi-injective S-systems have been studied by Lopez and Luedeman [8], and Satyanarayana [11]. Skornjakov ([13], Theorem 2) has proved that each direct sum of injective S-systems is injective if and only if S satisfies the ascending chain condition on its ideals. This result is not true if injectives are replaced by quasi-injectives. One object of this paper is to give a characterization of monoids whose class of quasi-injective S-systems is closed under the formation of direct sums. We shall also consider monoids all of whose S-systems are quasi-injective.

2. Preliminaries.

A right centered S-system M, denoted by M_S, is a set M, a monoid S with zero, and a function $M \times S \to M$ such that if ms denotes the image of (m, s) for $m \in M$ and $s \in S$, then the following properties hold:

(i) $(ms)t = m(st)$ for $m \in M$, and $s, t \in S$.

(ii) M contains a fixed element 0 such that $0s = 0$ for all $s \in S$.

(iii) for all $m \in M$, $m0 = 0$, and $m1 = m$, where 0 denotes the zero of S and 1 is the identity element of S.

A subsystem N of M_S, denoted by $N_S \subseteq M_S$, is a subset of M such that $ns \in N$ for all $n \in N$ and $s \in S$. A mapping $f: A_S \to B_S$ is an S-homomorphism if for any $a \in A$ and $s \in S$, $f(as) = f(a)s$. Let M be a fixed S-system. An S-system Q is called M-injective if given an S-monomorphism g of an S-system N_S into M_S, every S-homomorphism $h: N_S \to Q_S$ can be extended to an S-homomorphism $h^*: M_S \to Q_S$ with respect to g i.e. $h^*g = h$. Thus, Q is injective if and only if Q is M-injective for all S-systems M. Similarly, Q is quasi-injective if and only if Q is Q-injective. In ring theory, it is well known that "R-injective" R-modules are exactly injective. "S-injective" S-systems, however, need not be injective in the usual sense ([1], p. 272). These are called weakly injective ([7], p. 33). An S-system A is finitely injective if for every S-monomorphism $f: X_S \to Y_S$, where X is a finitely generated S-system, and for every S-homomorphism $g: X_S \to A_S$, there exists an S-homomorphism $h: Y_S \to A_S$ such that $hf = g$. If $\{M_i : I \in I\}$

is a family of objects in the category of S-systems then the product $\prod_{i \in I} M_i$ and the coproduct $\coprod_{i \in I} M_i$ exist and are isomorphic, respectively, to the cartesian product and the disjoint union of the sets M_i with the suitable action of S. Let $\{M_i : i \in I\}$ be a family of right S-systems. By their direct sum, $\oplus M_i$, we mean the subset of ΠM_i, consisting of all $(m_i) \in \Pi M_i$ for which $\{i: m_i \neq 0\}$ is finite. Then $\oplus M_i$ is a right S-system under componentwise multiplication. Recall that in the category of S-systems, epimorphisms are surjective and monomorphisms are injective. Moreover, direct products of injectives are injective, and the retract of an injective object is injective. An S-subsystem N_S of M_S is <u>essential</u> in M_S if every S-congruence on M whose restriction to N is the identity, is itself the identity on M. Note that if N_S is essential in M_S, then $N_S \cap K_S \neq 0$ for all nonzero S-subsystems K_S of M_S. It was shown by Berthiaume [1] that each S-system M_S has a unique (up to isomorphism) essential extension, called the <u>injective hull</u> of M_S. A right S-system M_S is <u>totally irreducible</u> if the only right S-congruences are the universal congruence ω_M and the identity congruence i_M, and $M_S \neq 0$. Thus, if M_S is totally irreducible, then M_S has no proper S-subsystems. An S-system M_S is <u>noetherian</u> if every subsystem of M is finitely generated. A monoid S is called <u>right noetherian</u> if S_S is noetherian. Evidently, S is right noetherian if and only if S satisfies the ascending chain condition for right ideals. Finally, a right S-system M will be called <u>Σ-countably injective</u> if direct sum of countably many copies of M is injective.

3. Results.

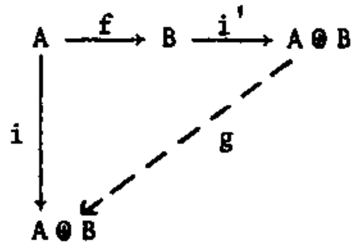
We begin with the following lemmas.

Lemma 1. Let S be a monoid. If each direct sum of quasi-injective S-systems is quasi-injective, then S satisfies the ascending chain condition on right ideals.

Proof. Let $(0) = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$ be an ascending chain of right ideals of S. Consider the S-systems S/I_i, the Rees factors of S, for $i = 1, 2, \ldots$. Let Q_i denote the injective hull of S/I_i and write $Q = \oplus Q_i$. Then Q is quasi-injective by the hypothesis. Let $I = \cup I_k$, and let $f_k\colon I \to S/I_k$ be the natural homomorphism. Thus f_k maps I into Q_k. Note that, if $a \in I$, then $a \in I_t$ for some t and then $f_k(a) = 0$, for all $k \geq t$. Define $f\colon I \to Q$ by $f(a) = (f_1(a), \ldots, f_i(a), \ldots)$, where $a \in I$. Then $f(a) \in Q$, since only a finite number of terms are nonzero. Clearly, f is an S-homomorphism from I into Q. Since $I \subseteq S \subseteq Q_1 \subseteq Q$ and Q is quasi-injective, there exists an S-homomorphism $\lambda\colon Q \to Q$ which extends f. Suppose $\lambda(1) = h$, where 1 is the identity of S. Then for all $x \in S$, $\lambda(x) = \lambda(1)\cdot x = hx$. Since we can write $h = (h_1, h_2, \ldots, h_k, \ldots)$, so there is t such that $h_k = 0$ for all $k \geq t$. Thus, since for any $x \in I$, $f(x) = hx$, and since $(hx)_t = h_t x = 0$, it follows that $I \subseteq I_t$. Hence $I_{t+1} = I_{t+2} = \cdots = I$. Thus S satisfies the ascending chain condition on right ideals.

Lemma 2. If $f\colon A_S \to B_S$ is a monomorphism, and $A \oplus B$ is quasi-injective, then A is a retract of B.

<u>Proof</u>. Using the canonical injection and projection maps, we write $A \xrightarrow{i} A \oplus B \xrightarrow{j} A = 1_A$ and $B \xrightarrow{i'} A \oplus B \xrightarrow{j'} B = 1_B$. Now consider the diagram:



Since $A \oplus B$ is quasi-injective, there is an S-homomorphism $g: A \oplus B \to A \oplus B$ such that $gi'f = i$. Hence $1_A = ji = j(gi'f) = (jgi')f$. Thus A is a retract of B.

<u>Theorem 1</u>. For a monoid S, the following conditions are equivalent:

(1) Each direct sum of quasi-injective S-systems is quasi-injective.

(2) S is noetherian and each quasi-injective S-system is injective.

<u>Proof</u>. Let us assume that each direct sum of quasi-injective S-systems is quasi-injective. Then S is notherian by Lemma 1. Let M be a quasi-injective S-system and let E be the injective hull of M. Then, by the hypothesis, $M \oplus E$ is quasi-injective. Hence, by Lemma 2, M is a retract of E. Therefore, M is injective.

Conversely, assume that S is noetherian and each quasi-injective is injective. If $M = \oplus M_i$ such that each M_i is quasi-injective, then each M_i is injective by the assumption, and since S is noetherian, M is injective by Theorem 2 of Skornjakov [13].

Corollary 1. Let S be a monoid all of whose one-sided ideals are two-sided. If each direct sum of quasi-injective S-systems is quasi-injective, then S is a semi-lattice of groups ([7], Cor. 2.3, p. 34).

Corollary 2. If S is a commutative monoid such that each direct sum of quasi-injective S-systems is quasi-injective, then S is regular.

Proof. Let A_S be a totally irreducible S-system. Then A_S is obviously quasi-injective. Hence A_S is injective. Therefore S is regular by ([7], Theorem 1.2, p. 31).

The next two theorems sharpen Skornjakov's result stated in Section 1.

Theorem 2. For a monoid S, the following conditions are equivalent:

(1) Each injective S-system is countably $\sum$-injective.

(2) S is noetherian.

Proof. Assume that each injective S-system is countably $\sum$-injective. Using the notations employed in the proof of Lemma 1, we write $M = \Pi Q_i$. Then, as stated before, M is injective. Hence $\oplus_{i \in I} M_i$, where each M_i is a copy of M, is injective by the hypothesis. Let $M_j = \Pi Q_i = Q_j \oplus P_j$ where $P_j = \Pi_{i \neq j} Q_i$. Then $\oplus M_j = \oplus_{j \in \mathbf{N}} Q_j \oplus \oplus_{j \in \mathbf{N}} P_j$. Thus, Q becomes a direct summand of an injective S-system. Hence Q is injective. The rest of the proof is omitted, as it is similar to that of Lemma 1. The converse is an immediate consequence of Skornjakov's result stated in Section 1.

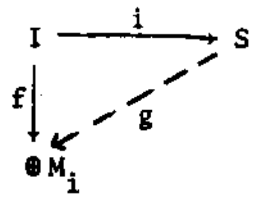
Theorem 3. For a monoid S, the following conditions are equivalent:

(1) Each direct sum of weakly injective S-systems is weakly injective.

(2) S is noetherian.

<u>Proof</u>. (1) Let us assume that each direct sum of weakly injective S-systems is weakly injective. This assumption together with the proof of Lemma 1 with necessary modifications, can be used to show that S is noetherian.

(2) Assume now that S is noetherian. Let $\{M_i : i \in I\}$ be a family of weakly injective S-systems. Consider the diagram:



where I is a right ideal of S and i is an S-monomorphism. Since S is (right) noetherian, $I = \bigcup_{i=1}^{n} (a_i S)$. Now for each j, the vector $f(a_j)$ in $\oplus M_i$ has only finitely many nonzero coordinates. Since there are only finitely many a_j, the set $\{f(a_1), \ldots, f(a_n)\}$ collectively involve only finitely many M_i, say, $M_{i_1}, \ldots, M_{i_m}$. It follows that $\mathrm{Im}\, f \subseteq M_{i_1} \oplus \ldots \oplus M_{i_m}$, which is weakly injective, being a finite direct sum of weakly injectives. Hence there is a map $g: S \to M_{i_1} \oplus \ldots \oplus M_{i_m}$ which extends f. We may regard g as a map whose image is in the larger S-system $\oplus M_i$.

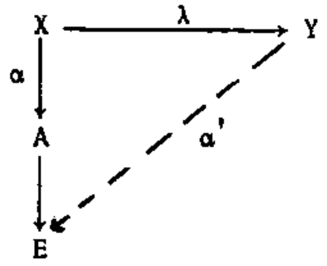
Recall that an S-subsystem A of an S-system B is <u>pure</u> if $A \cap Bx \subset Ax$ for every $x \in S$. Normak [9] called an S-system <u>absolutely pure</u> if it is pure within its injective hull. Gould ([4], Prop. 3.8, p. 259) has given a diagrammatic characterization of absolutely pure S-systems. Below we give the definition of strongly pure S-systems. This concept lies strictly between absolute purity and injectivity.

Definition: Let A_S be a subsystem of an S-system B. A is called strongly pure in B if to each element $a \in A$ (equivalently, any finite set $\{a_1, \ldots, a_n\}$ of elements of A) there exists an S-homomorphism $\alpha: B \to A$ such that $\alpha(a) = a(\alpha(a_i) = a_i$ for $i = 1, \ldots, n)$.

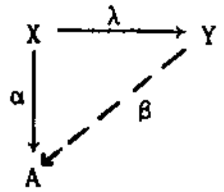
The following is an analogue for the corresponding result in module theory [10].

Theorem 4. An S-system A is strongly pure in every S-system containing A as a subsystem if and only if A is finitely injective.

Proof. (1) Suppose A is strongly pure in every S-system containing A as a subsystem. Then A is strongly pure in its injective hull E (say). Now consider the diagram:

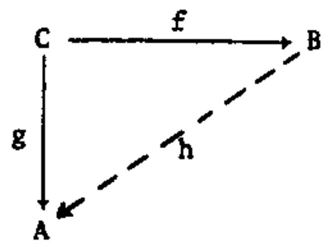


where λ is an S-monomorphism, α an S-homomorphism, and X a finitely generated S-system. Let $\{x_1, \ldots, x_n\}$ be the generating set of X. By the injectivity of E, there exists an S-homomorphism $\alpha': Y \to E$ such that $\alpha'\lambda = \alpha$. Since A is strongly pure in E, there exists $f: E \to A$ such that $f(\alpha(x_k)) = \alpha(x_k)$ for $k = 1, \ldots, n$. Let $\beta = f\alpha'$. Then we have the following commutative diagram:



Hence A is finitely injective.

(2) Conversely, suppose A is an S-subsystem of B such that A is finitely injective. Then for any finitely generated subsystem C of A, the diagram:



where f and g are the inclusion maps, yields an S-homomorphism $h: B \to A$ such that $hf = g$ i.e. $h|_C = 1_C$. Hence A is strongly pure in B.

Corollary: If A_S is strongly pure, then A_S is absolutely pure ([4], Prop. 3.8, p. 259).

Next, we state the following lemma whose proof is similar to the second part of the proof of Theorem 3.

Lemma 3. For a monoid S, each direct sum of finitely injective S-systems is finitely injective.

Theorem 5. For a monoid S, the following conditions are equivalent:

(1) Every finitely injective S-system is weakly injective.

(2) S is noetherian.

Proof (1) $\Rightarrow$ (2): Again, the arguments employed in the proof of Lemma 1 and the statement in Lemma 3 can be used to establish the desired implication.

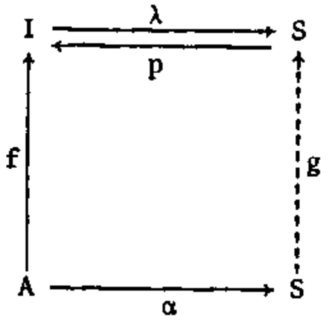
(2) ⇒ (1): Since S is noetherian, this implication is a direct consequence of the definitions of finitely and weakly injective S-systems.

A monoid S is called completely right injective if all S-systems are injective. Internal characterizations of completely right injective monoids have been given by Fountain [3], Isbell [5], and Shoji [12]. In line with the above terminology, we call a monoid S completely quasi-injective if all S-systems are quasi-injective.

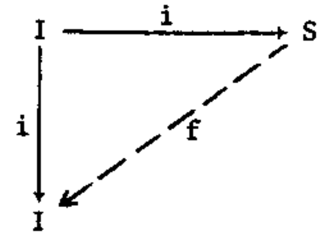
We now prove the following:

Lemma 4. If each finitely generated S-system is quasi-injective, then S is a regular monoid, each of whose finitely generated ideal is generated by an idempotent.

Proof. Since S itself is a finitely generated S-system, it follows that S_S is quasi-injective i.e. S is "S-injective". Let I be a finitely generated right ideal of S. Then $I \otimes S$ is a finitely generated S-system. Hence $I \otimes S$ is quasi-injective. Therefore by Lemma 2, I is a retract of S. So, there exists a diagram $I \underset{p}{\overset{\lambda}{\rightleftarrows}} S$ such that $p\lambda = 1_I$. Now consider the diagram:



in which A is a right ideal of S, α an S-monomorphism and f an S-homomorphism. Since S is S-injective, there is a map $g: S \to S$ such that $g\alpha = \lambda f$. Define $h: S \to I$ by $h = pg$. Then $h\alpha = pg\alpha = p\lambda f = 1_I f = f$, since $p\lambda = 1_I$. Hence I is S-injective. Hence the diagram



is commutative. It is easy to see that $f: S \to I$ is an idempotent S-epimorphism. Hence $I = f(S) = f(1)\cdot S$. But $f(1)\cdot f(1) = f(1\cdot f(1)) = f(1)$. Hence $f(1)$ is an idempotent, showing that I is generated by an idempotent. Hence S is regular by ([2], Lemma 1.13, p. 27).

Finally, we prove the following:

Theorem 6. Let S be a monoid whose idempotents are in the center of S. Then the following statements are equivalent:

(1) S is completely (right) quasi-injective.

(2) S is noetherian, all of whose finitely generated S-systems are quasi-injective.

(3) S is completely (right) injective.

Proof. (1) $\Rightarrow$ (2): If S is completely quasi-injective, then S is noetherian by Lemma 1. The second part of the implication is obvious.

(2) $\Rightarrow$ (3): Since S is noetherian, every (right) ideal of S is finitely generated. Hence by Lemma 4, every ideal of S is generated by an idempotent. Hence S is completely right injective ([9], Prop. 7, p. 167).

The implication (3) $\Rightarrow$ (1) is immediate.

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