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# On Smoothed Probability Density Estimation

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# ON SMOOTHED PROBABILITY DENSITY ESTIMATION

by

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Abstract

The main object of this paper is to study properties of the estimator

$$\hat{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i} K\left(\frac{x - X_i}{a_i}\right)$$

under the assumption of stationarity of the sequence $(X_n)$.

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## 0. Introduction

Suppose a sample of observations $X_1, X_2, \ldots, X_n$ is identically distributed with density function $f$. Much research in recent years is concentrated on studying properties of the kernel estimator

$$\tilde{f}_n(x) = \frac{1}{na_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{a_n}\right) , \tag{0.1}$$

where $\{a_n\}$, $n = 1,2,\ldots$ is a given sequence of positive numbers such that $a_n \to 0$ $(n \to \infty)$ and $K$ is a given kernel. Recently properties of the estimator $\tilde{f}_n$ are studied under the assumption of stationarity of the sample. See Masry [10] for the case of a stationary continuous-time process and Castellana and Leadbetter [4] for an approach using $\delta$-sequences. In case of dependence it can be expected that properties of the estimator $\tilde{f}_n$ can be improved if the window width is not necessarily the same for each observation, that means the estimator

$$\hat{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i} K\left(\frac{x - X_i}{a_i}\right) \tag{0.2}$$

is considered. Earlier research concerning $\hat{f}_n$ in case of independent observations is done by Devroye [7], Samanta and Mugisha [11], who extended results of Yamato [13], and Davies [5].

One of the results (Theorem 1.3) in this paper is that (in case of dependence) for a suitable choice of the sequence $\{a_n\}$ the variance of the estimator $\hat{f}_n$ is smaller than the variance of the usual estimator $\tilde{f}_n$.

Moreover, under suitable assumptions, the estimator $\hat{f}_n$ is asymptotically normal. This is shown in Theorem 2.4 which contains a result about an estimator for $f'$ as well. Finally Section 3 contains uniform convergence results, weakly as well as strongly.

## 1. Pointwise Consistency

In this section we determine the asymptotic behaviour of the bias and the variance of the estimator $\hat{f}_n$. In order to obtain the asymptotic estimates the sequence $\{a_n\}$ has to be sufficiently smooth. It turns out that regular variation in this case is an appropriate property for the sequence $\{a_n\}$. Lemma 1.1 below contains the basic ingredient for the proof of Theorem 1.3.

Lemma 1.1 (See Bojanic and Seneta [2])

Suppose $\{a_n\}$ $n = 1,2,\ldots$ is a sequence of positive numbers. Then

$$\lim_{n \to \infty} a_{[nx]}/a_n = x^{-\alpha} \quad (x > 0) \quad \text{for some} \quad \alpha > -1 \tag{1.1}$$

if and only if

$$\lim_{n \to \infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{1}{a_j} = \frac{1}{\alpha + 1} \quad \text{with} \quad \alpha > -1 \tag{1.2}$$

A sequence $\{a_n\}$ satisfying the assumptions of the above lemma is called a regularly varying sequence. For more properties the reader is referred to [9].

Corollary 1.2.

If $c_n > 0$ for $n = 1,2,\ldots$, $c_n \sim a_n$ $(n \to \infty)$ and the sequence $\{a_n\}$ satisfies the assumptions of the above lemma, then

$$\lim_{n\to\infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{1}{c_j} = \frac{1}{\alpha + 1}.$$

Theorem 1.3

Suppose $\{X_n\}$ is a stationary sequence of random variables. Let $\hat{f}_n^{(r)}$ be defined by

$$\hat{f}_n^{(r)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j^{r+1}} K^{(r)}\left(\frac{x - X_j}{a_j}\right), \qquad r = 0,1. \tag{1.3}$$

Suppose that the following conditions hold

(i) the sequence $\{a_n\}$ is non-increasing and satisfies (1.1) for some $\alpha > 0$.

(ii) $K$ is a symmetric differentiable density such that $|K^{(r)}(u)| \le c$, $\int K^{(r)}(u)^2 du < \infty$ for $r = 0,1$, $\int |K'(u)| du < \infty$ and there exists $c_1$ such that for all $z \in \mathbb{R}$, $t > 0$ we have

$$t|K^{(r)}(tz)| \le c_1 |K^{(r)}(z)| \quad \text{for } r = 0,1. \tag{1.4}$$

(iii) The joint density $f_{1j}(x,y)$ of $X_1$ and $X_j$ $(j = 2,3,\ldots)$ is uniformly bounded in $x$, $y$ and $j$ and satisfies

$$\sup_{x,y} \sum_{j=2}^{\infty} |f_{1j}(x,y) - f(x)f(y)| \le M < \infty, \tag{1.5}$$

where $f$ is the marginal density of $X_j$. Then if $x$, $y$ are continuity points of $f$ we have for $r = 0,1$

$$\lim_{n\to\infty} na_n^{1+2r} \operatorname{cov}(f_n^{(r)}(x),\ f_n^{(r)}(y)) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{1+(2r+1)\alpha} f(x) \int K^{(r)}(u)^2 du & \text{if } x = y \end{cases} \tag{1.6}$$

<u>Proof.</u>

Since $\alpha > 0$, we have $a_n \to 0$ as $n \to \infty$.

$$\begin{aligned} na_n^{1+2r} \operatorname{cov}(\hat{f}_n^{(r)}(x),\ \hat{f}_n^{(r)}(y)) &= \frac{1}{n} \sum_{j=1}^{n} \frac{a_n^{1+2r}}{a_j^{2+2r}} \operatorname{cov}(K^{(r)}(\frac{x - X_j}{a_j}), K^{(r)}(\frac{y - X_j}{a_j})) \\ &+ \frac{a_n^{1+2r}}{n} \sum_{\substack{j\neq k \\ 1\le j,k\le n}} \frac{1}{(a_j a_k)^{1+r}} \operatorname{cov}(K^{(r)}(\frac{x - X_j}{a_j}),\ K^{(r)}(\frac{y - X_k}{a_k})) \\ &=: I_1 + I_2 \end{aligned} \tag{1.7}$$

First we estimate $I_1$. We have by stationarity

$$\operatorname{cov}_1 := \operatorname{cov}[K^{(r)}(\frac{x - X_j}{a_j}),\ K^{(r)}(\frac{y - X_j}{a_j})] =$$

$$= a_j \int K^{(r)}(z) K^{(r)}(\frac{y - x}{a_j} + z)\ f(x - a_j z) dz -$$

$$a_j^2 \int K^{(r)}(z)f(x - a_jz)dz \int K^{(r)}(z)f(y - a_jz)dz. \tag{1.8}$$

In case $x = y$ we apply Lebesgue's dominated convergence theorem to find

$$\text{var } K^{(r)}\left(\frac{x - X_j}{a_j}\right) \sim a_jf(x) \int K^{(r)}(z)^2dz \quad (j \to \infty) \tag{1.9}$$

for $r = 0,1$ at each continuity point $x$ of $f$.

Application of Lemma 1.1 and Corollary 1.2 now gives

$$I_1 = \frac{1}{1 + (2r + 1)\alpha} f(x) \int K^{(r)}(z)^2dz + o(1) \quad \text{as} \quad n \to \infty$$

at continuity points $x$ of $f$.

In case $x \neq y$ we can estimate the first term on the right-hand side in (1.8) as follows. For $j$ sufficiently large we have

$$\left|a_j \int K^{(r)}(z)K^{(r)}\left(\frac{y - x + a_jz}{a_j}\right)f(x - a_jz)dz\right|$$

$$\leq c_1\left|a_j^2 \int K^{(r)}(z)K^{(r)}(y - x + a_jz)dz\right|$$

$$\sim c_1a_j^2\, K^{(r)}(y - x)f(x) \int K^{(r)}(z)dz \quad (j \to \infty),$$

hence the first term on the right-hand side in (1.8) is $O(a_j^2)$ $(j \to \infty)$. We can estimate the second term by using Lebesgue's dominated convergence theorem. As a consequence $\text{cov}_1 = O(a_j^2)$ as $j \to \infty$ in case $x \neq y$.

Using (1.8), Lemma 1.1 and similar arguments as above, we find $I_1 = O(a_n)$ as $n \to \infty$ in case $x \neq y$. Finally we give an estimate for $I_2$.

Since the sequence $\{a_n\}$ is nonincreasing, we have

$$|I_2| \leq \frac{a_n}{n} \sum_{\substack{j \neq k \\ 1 \leq j,k \leq n}} \frac{1}{a_j a_k} \left|\operatorname{cov}\left(K^{(r)}\left(\frac{x - X_j}{a_j}\right), K^{(r)}\left(\frac{y - X_k}{a_k}\right)\right)\right|$$

$$\leq \frac{a_n}{n} \sum_{\substack{j \neq k \\ 1 \leq j,k \leq n}} \iint \frac{1}{a_j a_k} \left|K^{(r)}\left(\frac{z_1}{a_j}\right) K^{(r)}\left(\frac{z_2}{a_k}\right)\right| |f_{jk}(x - z_1, y - z_2)$$

$$- f(x - z_1) f(y - z_2)| dz_1 dz_2 \tag{1.10}$$

From stationarity we have

$$\frac{1}{n} \sum_{1 \leq j < k \leq n} |f_{jk}(x - z_1, y - z_2) - f(x - z_1) f(y - z_2)|$$

$$= \sum_{k=2}^{n} \frac{n - k + 1}{n} |f_{1k}(x - z_1, y - z_2) - f(x - z_1) f(y - z_2)| \leq M,$$

the last inequality being true by assumption (iii). Moreover, a similar inequality holds in case $j > k$.

Combination with (1.4) and (1.10) gives

$$|I_2| \leq 2\, a_n M c_1^2 \iint |K^{(r)}(z_1) K^{(r)}(z_2)| dz_1 dz_2$$

$$= 2\, a_n M c_1^2 \{\int |K^{(r)}(z)| dz\}^2 . \tag{1.11}$$

Remark.

Note that if $\alpha \in (0,1)$, then $a_n \to 0$ and $na_n \to \infty$ $(n \to \infty)$. Hence in this case we have $\text{var}\ \hat{f}_n(x) \to 0$ $(n \to \infty)$ at all continuity points of $f$ (under the assumptions of Theorem 1.3)

Theorem 1.4

Suppose the sequence $\{a_n\}$ satisfies the assumptions of Lemma 1 with $0 < \alpha < \frac{1}{2}$ and $K$ is a symmetric density with $\int |z|^3 K(z)dz < \infty$. Assume that $X_1, X_2, \ldots$ are identically distributed with density $f$.

If $\hat{f}_n(x)$ is as defined before, $f$ is 3 times differentiable in a neighbourhood of $x$ and $f''(x) \neq 0$, then

$$E\hat{f}_n(x) - f(x) \sim \frac{1}{2(1-2\alpha)} f''(x) a_n^2 \int z^2 K(z)dz \quad (n \to \infty) \tag{1.12}$$

Proof.

Since $K$ is symmetric, we have, using Taylor's theorem

$$E\hat{f}_n(x) - f(x) = \frac{1}{n}\sum_{j=1}^{n}\int K(z)\{f(x - a_j z) - f(x)\}dz$$

$$= \frac{1}{n}\sum_{j=1}^{n}\int K(z)\{-a_j z f'(x) + \frac{a_j^2 z^2}{2} f''(x) + O(a_j^3 z^3)\}dz$$

$$= \frac{1}{2} f''(x)(\int z^2 K(z)dz)\ \frac{1}{n}\sum_{j=1}^{n} a_j^2 (1 + o(1)) \quad (n \to \infty)$$

Application of Lemma 1.1 finally gives (1.12).

Under the assumptions of Theorem 1.3 and Theorem 1.4 we find that the mean square error is equal to

$$E\{\hat{f}_n(x) - f(x)\}^2 = \frac{1}{1+\alpha} f(x)\int K^2(z)dz \frac{1}{na_n}$$

$$+ \frac{1}{4(1-2\ )^2} f''(x)^2 \left(\int z^2K(z)dz\right)^2 a_n^4 + o\left(\frac{1}{na_n}\right) + o(a_n^4) \qquad (n \to \infty)$$

It is easily seen that in case $a_n = cn^{-\alpha}$, the optimal choice for $\alpha$ is $1/5$. In that case we have

$$E\{\hat{f}_n(x) - f(x)\}^2 = \{\tfrac{5}{6}f(x)\left(\int K^2(z)dz\right)c^{-1} + \tfrac{25}{36}f''(x)^2\left(\int z^2K(z)dz\right)^2c^4\}n^{-4/5}$$

The optimal choice of $c$ then finally depends upon the kernel $K$. Depending upon the kernel and the values of $f(x)$ and $f''(x)$ the result may be better or worse than the usual estimate.

## 2. Asymptotic normality

In this section we prove asymptotic normality of the estimator $\hat{f}_n$ in case the sequence of random variables $\{X_n\}$ satisfies an array form of the strong mixing condition.

### Definition 2.1

The stationary sequence $\{X_n\}$ has strong mixing coefficients $\alpha_{n,\ell}$ if

$$\alpha_{n,\ell} := \max_{1 \le i \le n-\ell} \sup_{\substack{A \in \sigma(0,i) \\ B \in \sigma(i+\ell,n)}} |P(AB) - P(A)P(B)| \tag{2.1}$$

for $1 \le \ell \le n-1$, where $\sigma(i,j)$ is the $\sigma$-field generated by $\{X_k;\ 1 < k \le j\}$.

We need two lemmas. The first is due to Volkonskii and Rozanov [12].

<u>Lemma 2.2</u>

Suppose $X_1, X_2, \ldots, X_m$ are random variables measurable with respect to $\sigma(i_1,j_1), \sigma(i_2,j_2), \ldots, \sigma(i_m,j_m)$ respectively, where $0 \le i_1 < j_1 < i_2 < \ldots < i_m < j_m \le n$, $i_{k+1} - j_k \ge \ell \ge 1$ and $|X_k| \le 1$, $1 \le k \le m$. Then

$$|E(\prod_{k=1}^{m} X_k) - \prod_{k=1}^{m} EX_k| \le 16(m-1)\alpha_{n,\ell}$$

where $\alpha_{n,\ell}$ is as in (2.1).

<u>Lemma 2.3</u>

If the sequence $\{a_n\}$ satisfies (1.1) and $\tau = \tau(n)$, $m = m(n)$ are such that $\tau(n)m(n) \sim n$ $(n \to \infty)$, then $\sum_{k=1}^{m} 1/a_{k\tau} \sim n/(1+\alpha)\tau a_n$ $(n \to \infty)$.

<u>Proof</u>

Define $f(x) := 1/a_{[x]}$. Since the function $f$ satisfies $f(tx)/f(t) \to x^{+\alpha}$ $(t \to \infty)$ for $x > 0$, convergence is uniform on compact intervals of

$(0, \infty)$ (see de Bruijn [3]). Moreover $\int_1^x f(s)ds \sim xf(x)/(1 + \alpha)$ $(x \to \infty)$. As a consequence

$$\sum_{k=1}^{m} 1/a_{k\tau} = \int_1^{m+1} f([s]\tau)ds \sim \int_1^{m+1} f(s\tau)ds \sim \int_0^{n/\tau} f(s\tau)ds$$

$$= \frac{1}{\tau}\int_0^n f(u)du \sim n/(1 + \alpha)\tau a_n \quad (n \to \infty).$$

<u>Theorem 2.4</u>

Suppose the assumptions of Theorem 1.3 are satisfied. Assume that there exists a sequence of integers $k_n$ $(n \geq 1)$ for which

$$(n/a_n)^{\frac{1}{2}}\alpha_{n,k_n} \to 0 \quad \text{and} \quad k_n = o(na_n) \quad (n \to \infty) \tag{2.3}$$

Then if $\alpha \in (0, 1)$ (see (1.1)) at each continuity point $x$ of $f^{(r)}$

$$(\hat{f}_n^{(r)}(x) - E\hat{f}_n^{(r)}(x))\ c_0(na_n^{2r+1})^{\frac{1}{2}}, \tag{2.4}$$

where

$$c_0 = c_{0,r} = \{\frac{1}{1 + (2r + 1)\alpha} f(x) \int K^{(r)}(z)^2 dz\}^{-\frac{1}{2}} \tag{2.5}$$

has the standard normal limiting distribution for $r = 0$ and $r = 1$.

<u>Proof.</u>

From (2.3) it follows that there exists a sequence $\lambda_n$ such that $\lambda_n \to \infty$ $(n \to \infty)$, $\lambda_n k_n = o(na_n)^{\frac{1}{2}}$ and $\lambda_n (n/a_n)^{\frac{1}{2}}\alpha_{n,k_n} \to 0$ $(n \to \infty)$.

We write the expression in (2.4) as a sum of $m$ blocks of length $\tau$ (the large blocks $(S_1)$) and $m$ blocks of length $\tau'$ (the small blocks $(S_2)$), where $m = [n/(\tau+\tau')]$, and a resulting block. Define $\tau$, $\tau'$ by $\tau = [\lambda_n^{-1}(na_n)^{\frac{1}{2}}]$ and $\tau' = k_n$. Note that $\tau$, $\tau'$ depend on $n$, $na_n \to \infty$ since $\alpha \in (0, 1)$ and

$$\tau = o(na_n) \ , \quad \tau' = o(\tau) \quad \text{and} \quad \frac{n}{\tau}\alpha_{n,\tau'} \to 0 \quad (n \to \infty) \tag{2.6}$$

Now for

$$(f_n^{(r)}(x) - Ef_n^{(r)}(x))c_0\sqrt{a_n^{2r+1}n} = \sum_{i=1}^{n} \frac{a_n^{(2r+1)/2}}{n} \frac{c_0}{a_i^{r+1}} \{K^{(r)}(\frac{x - X_i}{a_i}) - EK^{(r)}(\frac{x - X_i}{a_i})\} = \sum_{i=1}^{n} Y_i \tag{2.7}$$

We write

$$\sum_{i=1}^{n} Y_i = \sum_{k=1}^{m} \sum_{j\in\Lambda_{k,n}} Y_j + \sum_{k=1}^{m} \sum_{j\in\Lambda'_{k,n}} Y_j + \sum_{j\in\Lambda''_n} Y_j =: S_1 + S_2 + S_3$$

where

$$\Lambda_{k,n} = \{(k-1)(\tau+\tau') + 1, \ . \ . \ . \ (k-1)(\tau+\tau') + \tau\}$$

$$\Lambda'_{k,n} = \{(k-1)(\tau+\tau') + \tau + 1, \ . \ . \ ., \ k(\tau+\tau')$$

and

$$\Lambda''_n = \{m(\tau + \tau') + 1, \ . \ . \ ., \ n\}$$

First we show that $\text{var}\ S_2 \to 0$ as $n \to \infty$. Using (1.9) we find

$$\text{var } S_2 \le \frac{c_0^2\, a_n^{2r+1}}{n} \sum_{k=1}^{m} \sum_{j\in\Lambda'_{k,n}} \frac{c_1}{a_j^{2r+1}} +$$

$$+ \frac{2c_0^2 a_n}{n} \sum_{1\le j<k\le n} \frac{1}{a_j a_k} \left|\text{cov}\left(K^{(r)}\left(\frac{x - X_j}{a_j}\right), K^{(r)}\left(\frac{x - X_k}{a_k}\right)\right)\right|$$

where $c_1 > f(x) \int K^{(r)}(z)^2 dz$ is a constant.

Since $\sum_{j\in\Lambda'_{k,n}} \frac{1}{a_j^{2r+1}} \sim \frac{\tau'}{a_{k\tau}^{2r+1}}$ $(n \to \infty)$, Lemma 2.3 together with (1.11) show that

$$\text{var } S_2 \le c_2 \frac{a_n^{2r+1}}{n} \cdot \tau' \cdot \frac{n}{\tau a_n^{2r+1}} + c_3 a_n = c_2\tau'/\tau + c_3 a_n \to 0$$

as $n \to \infty$, where $c_2$ and $c_3$ are constants.

Since $ES_2 = 0$ this implies $P(|S_2| > \varepsilon) \le \text{var } S_2/\varepsilon^2 \to 0$ as $n \to \infty$. Similarly we find that $S_3$ tends to zero in probability. As a consequence, the asymptotic distribution of $\sum_{i=1}^{n} Y_i$, if it exists, is the same as for $S_1$.

We claim that the asymptotic distribution (if it exists) of $S_1$ is the same as the distribution of $S_1' = \sum_{k=1}^{m} \gamma_k$ where $\gamma_k = \sum_{j\in\Lambda_{k,n}} Y_j$ and the $y_k$'s are independent. Indeed, by Lemma 2.2, $|E(\exp(it \sum_{k=1}^{m} \gamma_k)) - \prod_{k=1}^{m} \exp(it\, \gamma_k)|$ is bounded by $16(m - 1)\alpha_{n,\tau'} \sim 16\, \frac{n}{\tau}\, \alpha_{n,\tau'} \to 0$ $(n \to \infty)$ (see (2.6)).

Finally we prove that $S_1' \xrightarrow{D} N$, where $N$ is the standard normal distribution.

First we show that $\operatorname{var} S' \to 1$ as $n \to \infty$.

$$\operatorname{var} \gamma_k = \operatorname{var} \sum_{j\in\Lambda_{k,n}} Y_j = \sum_{j\in\Lambda_{k,n}} \operatorname{var} Y_j + \sum_{\substack{i\neq j\\ i,j\in\Lambda_{k,n}}} (\operatorname{cov} Y_j, Y_i) = T_{1,k} + T_{2,k}.$$

Note that by (1.9) and Lemma 2.3 we have

$$T_{1,k} \sim \sum_{j\in\Lambda_{k,n}} \frac{a_n^{2r+1}}{n} \frac{c_0^2}{a_j^{2r+2}} a_j \; f(x) \int K^{(r)}(z)^2 dz$$

$$\sim \frac{1}{1 + (2r+1)\alpha} \frac{c_0^2 a_n^{2r+1}}{n} f(x) \int K^{(r)}(z)^2 dz \left(\frac{k\tau}{a_{k\tau}^{2r+1}} - \frac{(k-1)\tau}{a_{(k-1)\tau}^{2r+1}}\right) \tag{2.8}$$

Moreover, with $I_2$ as defined in (1.7), we have
$\sum_{k=1}^{m} T_{2,k} \leq |I_2| = O(a_n)$ $(n \to \infty)$ (as in (1.11)).

Summation over $k \in \{1, 2, . . ., m\}$ then gives

$$\operatorname{var} S_1' \sim \frac{a_n^{2r+1}}{n} \frac{m\tau}{a_{m\tau}^{2r+1}} \to 1 \quad \text{as} \quad n \to \infty.$$

Finally we verify that the Lindeberg condition

$$\sum_{k=1}^{m} \frac{1}{S_m^2} E\{\gamma_k^2 I(|\gamma_k| \geq \varepsilon S_m)\} \to 0 \qquad \text{for each } \varepsilon > 0 \tag{2.9}$$

where $S_m^2 = \operatorname{var} S_1'$, is satisfied.

By the definition of $Y_j$ (see (2.7)) and assumption (ii) in Theorem 1.3 we have

$$|\gamma_k| \leq \sum_{j\in\Lambda_{k,n}} |Y_j| \leq \sum_{j=(k-1)\tau}^{k(\tau+\tau')} \frac{\sqrt{a_n^{2r+1}}}{\sqrt{n}} \frac{2cc_0}{a_j^{r+1}}.$$

In view of Lemma 1.1 the last expression is asymptotic to

$$c' \frac{a_n^{(2r+1)n}}{\sqrt{n}} \left\{ \frac{k\tau}{a_{k\tau}^{r+1}} - \frac{(k-1)\tau}{a_{(k-1)\tau}^{r+1}} \right\} = O\left( \frac{\tau\, a_n^{(2r+1)2}}{\sqrt{n}\;\, a_{k\tau}^{r+1}} \right) \qquad (n \to \infty)$$

where $c'$ is a constant. Since the sequence $\{a_n\}$ is non-increasing and $na_n \to \infty$, we have $O(\tau\ \ a_n^{2r+1}/\sqrt{n}\ a_{k\tau}^{r+1}) = o(1)$, so that $P(|\gamma_k| > \varepsilon) = 0$ for all $n$ sufficiently large. This finishes the proof of (2.9) since $S_m^2 \to 1$.

## 3. Uniform convergence

In this section a uniform error measure, namely

$$W_n^{(r)} := \sup_{x \in \mathbb{R}} |\hat{f}_n^{(r)}(x) - f^{(r)}(x)| \tag{3.1}$$

is discussed. Under suitable conditions regarding the strong mixing coefficients $\alpha_{n,\ell}$ and the sequence $\{a_n\}$ and restrictions on the kernel $K$, it will be shown that $W_n^{(r)}$ converges to $0$ weakly as well as strongly. We define the functions $\phi$, $k$ and $\phi_n$ by

$$\phi(t) = \int e^{itx}\, f(x)dx$$

$$k(t) = \int e^{itx}\, K(x)dx$$

and

$$\phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j}.$$

<u>Theorem 3.1</u>

Assume $\{X_n\}$ is a sequence of stationary random variables. Suppose the following conditions hold for some integer $r \geq 0$.

(i) $K$ is symmetric.

(ii) $k$ is non-increasing on $\mathbb{R}^+$ and $t^m\, k(t) \in L^1(-\infty, \infty)$ for $0 \leq m \leq r$.

(iii) the sequence $\{a_n\}$ is non-increasing, $a_n \to 0$ $(n \to \infty)$ and $na_n^{2(r+1)} \to \infty$ $(n \to \infty)$

(iv) the strong mixing coefficients $\alpha_{n,j}$ (see (2.1)) satisfy the condition

$$\overline{\lim_{n\to\infty}} \sum_{j=1}^{n-1} \alpha_{n,j} < \infty$$

(v) $f^{(m)}(x)$ is bounded and continuous for $0 \leq m \leq r$.

Then $W_n^{(r)} \xrightarrow{P} 0$ as $n \to \infty$.

<u>Proof.</u>

Similar to the proof of Theorem 3.8 in Abdul-Al and Siddiqui [1]. Since $k \in L^1(-\infty, \infty)$ we have $K(x) = (2\pi)^{-1}\int e^{-itx}k(t)dt$ (see e.g. Feller [8] XV.4, Lemma 2). In view of assumption (ii) we may differentiate $r$ times to find

$$K^{(r)}(x) = (2\pi)^{-1}(-i)^r\int t^r\, e^{-itx}\, k(t)dt \tag{3.2}$$

By substitution in (1.3) we find

$$\hat{f}_n^{(r)}(x) = (2\pi n)^{-1}\int \sum_{j=1}^{n} (-t)^r e^{-it(x-X_j)} k(a_j t)dt. \tag{3.3}$$

Hence we have

$$V_n := \sup_{x\in\mathbb{R}} |\hat{f}_n^{(r)}(x) - E\hat{f}_n^{(r)}| \leq \frac{1}{2\pi n} \int \left| t^r \sum_{j=1}^{n} \{e^{itX_j} - \phi(t)\} k(a_j t)\right| dt.$$

Since $a_n$ is non-increasing and $k$ is non-increasing, this implies

$$V_n \leq \frac{1}{2\pi} \int |t|^r |\phi_n(t) - \phi(t)| |k(a_n t)| dt . \tag{3.4}$$

Hence, by Fubini's theorem and Schwarz' inequality,

$$EV_n \leq \frac{1}{2\pi} \int \{\sigma^2(\phi_n(t))\}^{\frac{1}{2}} |t^r k(a_n t)| dt. \tag{3.5}$$

By stationarity and Lemma 2.2 we have

$$\sigma^2(\phi_n(t)) = \frac{1}{n} E|e^{itX} - \phi(t)|^2 + \frac{2}{n^2} \sum_{j=2}^{n} (n - j + 1) E[e^{it(X_1-X_j)} -$$

$$- E e^{itX_1} E e^{itX_j}] \leq \frac{4}{n} + \frac{32}{n} \sum_{j=1}^{n-1} \alpha_{n,j}$$

Combination with (3.5) gives

$$EV_n \leq \frac{2}{\pi n^{\frac{1}{2}} a_n^{r+1}} \left(\int |t^r k(t)| dt\right) \left(1 + 8 \sum_{j=1}^{n-1} \alpha_{n,j}\right)^{\frac{1}{2}} \to 0$$

as $n \to \infty$, hence $V_n \xrightarrow{P} 0.$ (3.6)

Since $f(x)$ and its first $r$ derivatives are bounded and continuous and $\lim_{|u|\to\infty} K^{(m)}(u) = 0$ for $0 \leq m \leq r$ (this follows from (3.2) by application of the Riemann-Lebesgue Lemma), we have

$$E\hat{f}_n^{(r)}(x) = \frac{1}{a_j^{r+1}} \int K^{(r)}\left(\frac{x-u}{a_j}\right) f(u)\,du = \frac{1}{a_j} \int K\left(\frac{x-u}{a_j}\right) f^{(r)}(u)\,du$$

Application of Lemma 1 in Yamato [13] then gives

$$\sup_{x\in\mathbb{R}} |Ef_n^{(r)}(x) - f^{(r)}(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Combination with (3.6) finishes the proof.

In the remainder of this section we discuss strong uniform convergence. We omit the proof which is similar to the proof of theorem 3.9 in Abdul-Al and Siddiqui [1] (using Lemma 4.1 in Davydov [6]).

Theorem 3.3

Suppose $\{X_n\}$ is a sequence of stationary random variables satisfying the following conditions. Suppose conditions (i) and (ii) of Theorem 3.1 hold and

(i) $f$ is uniformly continuous,

(ii) the sequence $\{a_n\}$ satisfies $a_n \to 0$ $(n \to \infty)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2 a_n^{4(r+1)}} < \infty$,

(iii) $x^r K(x) \to 0$ $(x \to \infty)$, and

(iv) $\sum_{j=1}^{\infty} (\sup_n \alpha_{n,j}) < \infty.$

Then $\sup_{s\in \mathbb{R}} |\hat{f}_n^{(r)}(x) - f^{(r)}(x)| \xrightarrow{W.P.1} 0$ as $n \to \infty$.

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