King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 100

July 1987

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On Rate of Convergence of Recursive Kernel Estimates of Probability Densities

by

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Abstract

Recursive estimates $f_n^{(r)}(x)$ of the r-th derivative $f^{(r)}(x)$ $(r = 0, 1)$ of the univariate probability density $f(x)$ for strictly stationary processes $\{X_j\}$ is considered. Asymptotic variance/covariance of $f_n^{(r)}(x)$ are established for stationary triangular arrays of random variables satisfying various asymptotic independence-uncorrelatedness conditions.

AMS Subject Classification: Primary 62G05, G2H2

Key-words: Density estimation, Recursive estimation, Stationary sequence, Asymptotic uncorrelatedness/Independence

1. Introduction.

Suppose $\{X_j\}$ be a sequence of stationary random variables (r.v.'s) with common probability density function (p.d.f.) $f(x)$. Much research in recent years is concentrated on studying properties of the kernel estimators (based on the first n observations)

$$\hat{f}_n(x) = \frac{1}{na_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{a_n}\right) \tag{1.1}$$

where $\{a_n\}$, $n \geq 1$ is a given sequence of positive numbers such that $a_n \downarrow 0$ as $n \to \infty$ and K is a given kernel. Roussas [16] and Rosenblatt [18] have studied the asymptotic properties of the estimators $\hat{f}_n$ for stationary processes satisfying Doeblin's and G_2 conditions, respectively. Masry [13], Abdul-al and Siddiqui [2], Castellana and Leadbetter [5] have studied the asymptotic properties of $\hat{f}_n$ for stationary uniform and strong mixing proccesses. In case of dependence it can be expected that asymptotic properties of the estimator $\hat{f}_n$ can be improved if the window width is not necessarily the same for each observation, that means the estimator

$$f_n(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j} K\left(\frac{x - X_j}{a_j}\right) \tag{1.2}$$

is considered. Earlier research concerning f_n in case of independent observations is done by Devroye [7], Samanta and Mugisha [20], who extended results of Yamato [22], and Davies [6]. While recently Abdul-Al [1], Masry [14], Abdul-Al and Geluk [3] have studied the properties of f_n under the uniform and strong mixing processes.

Using the first n observations from a sequence of strictly stationary random variables, consider the estimator

$$f_n^{(r)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j^{r+1}} K^{(r)} \left(\frac{x-X_j}{a_j}\right) \tag{1.3}$$

of the r-th derivative $f^{(r)}(x)$ $(r=0,1,\ldots)$, where $K^{(r)}$ is the r-th derivative of K.

The aim of this paper is to study and compare the asymptotic behavior of the variance/covariance of the estimator $f_n^{(r)}$ for strictly stationary sequence of r.v's satisfying various independence-uncorrelatedness conditions such as uniform and strong mixing, maximal correlation, G_2 and Doeblin's conditions.

2. Preliminaries and Notations

Let M_a^b denote the σ-algebra of events generated by the r.v.'s $\{X_j;\ a \le j \le b\}$, $-\infty \le a \le b \le \infty$. The concept of strong mixing is due to Rosenblatt [17]. The stationary $\{X_j\}$ process is strong mixing if for $k \ge 0$

$$\sup_{\substack{A \in M_{-\infty}^{i} \\ B \in M_{i+k}^{\infty}}} |P(AB) - P(A)\,P(B)| = \alpha(k) \downarrow 0 \quad \text{as} \quad k \to \infty \tag{2.1}$$

The value $\alpha(k)$ characterizes the mixing rate and is referred to as the strong mixing coefficient. Equation (2.1) implies that the random variables X_i and X_{i+k} becomes asymptotically independent as the lag k tends to infinity.

A second notation of asymptotic independence is the uniform mixing due to Ibragimov [10]. The stationary process $\{X_j\}$ is uniform mixing if for $k \geq 0$ and $P(A) > 0$

$$\sup_{\substack{i \\ A \in M_{-\infty}^{i} \\ B \in M_{i+k}^{\infty}}} |P(AB) - P(A)\,P(B)| = P(A)\,\phi(k) \downarrow 0 \quad \text{as} \quad k \to \infty. \tag{2.2}$$

It is clear that $\alpha(k) \leq \phi(k)$, that means a uniform mixing process is always strong mixing.

A third notation is the maximal correlation due to Kolmogorov and Rozanov [11]. The stationary process $\{X_j\}$ satisfies the maximal correlation if for $k \geq 0$

$$\sup_{\substack{i \\ U \in L_2(M_{-}^{i}) \\ V \in L_2(M_{i+k}^{\infty})}} \frac{|\operatorname{cov}(U, V)|}{\sqrt{\operatorname{var}(U)\operatorname{var}(V)}} = \rho(k) \downarrow 0 \quad \text{as} \quad k \to \infty \tag{2.3}$$

where $L_2(M_a^b)$ denotes the collection of all second-order random variables measurable with respect to M_a^b.

It is known (see [21]) that $4\alpha(k) \leq \rho(k) \leq 2\phi^{1/2}(k)$. Thus a stationary process $\{X_j\}$ satisfying the maximal correlation is uniform mixing and a uniform mixing process is strong mixing.

It will be assumed that the joint probability density $f_{1j}(x, y)$ of the r.v.'s X_1, X_j (j=2,3,...) are absolutely continuous. Define the primary measure of dependence of the sequence $\{X_j\}$ as (see [5])

$$\beta_n = \sup_{x,y} \sum_{j=2}^{n} |f_j(x,y) - f(x)\,f(y)| \tag{2.4}$$

which is finite for each n. The sequence $\{\beta_n\}$ is called the dependence index sequence for the process $\{X_j\}$. Clearly for i.i.d. sequence $\beta_n = 0$ for all n.

Throughout this paper the functions $K^{(r)}$, $r = 0, 1$, and the constants a_n are assumed to satisfy the following conditions:

Condition 1:

(i) $\sup_x |K^{(r)}(x)| < \infty$,

(ii) $\int |K^{(r)}(x)|\, dx < \infty$ and (2.5)

(iii) $\lim_{|x|\to\infty} |xK^{(r)}(x)| = 0$.

Condition 2:

$$a_n \to 0 \text{ and } na_n \to \infty \quad \text{as } n \to \infty. \tag{2.6}$$

Define $K_j^{(r)}(x)$ by

$$K_j^{(r)}(x) = K^{(r)}\left(\frac{x}{a_j}\right)$$

and write the estimator $f_n^{(r)}(x)$ as

$$f_n^{(r)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j^r} K_j(x - X_j) . \tag{2.7}$$

In order to obtain the asymptotic behavior of the covariance of the estimators, the sequence $\{a_n\}$ has to be sufficiently smooth. It turns out that regular variation in this case is an appropriate property for the sequence $\{a_n\}$. The lemma below contains the basic ingredients for the proofs.

Lemma 2.1. (see Bojonic and Seneta [ ]).

Suppose $\{a_n\}$, $n \geq 1$, is a sequence of positive numbers, then

$$\lim_{n\to\infty} a_{[nx]}/a_n = x^{-\alpha} (x > 0) \quad \text{for some } \alpha > -1 \text{ if and only if}$$

$$\lim_{n\to\infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{1}{a_j} = \frac{1}{\alpha + 1} .$$

A sequence $\{a_n\}$ satisfying the assumptions of Lemma 2.1 is called a regularly varying sequence. For more properties the reader is referred to [9].

Corollary 2.2. If $c_n > 0$ for $n \geq 1$, $c_n \sim a_n$ $(n \to \infty)$ and the sequence $\{a_n\}$ satisfies the assumptions of Lemma 2.1, then

$$\lim_{n\to\infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{1}{c_j} = \frac{1}{\alpha + 1} .$$

Main Result:

In this section we establish precise asymptotic expression for the covariance of the estimators $f_n^{(r)}$. These results will be held under a suitable asymptotic-independence, or uncorrelatedness assumptions on the process $\{X_j\}$.

The following lemma can be found in [20], and will be used in the sequel.

Lemma 3.1. Let $g(x)$ be a real-valued Borel measurable function. If $x \in C(f)$ [$C(f)$ denote the set of continuity points of $f(x)$] and if

$$\sup_x |g(x)| < \infty, \int |g(x)| dx < \infty \text{ and } \lim_{|x|\to\infty} |xg(x)| = 0$$

then for every $c \geq 0$

$$\lim_{n\to\infty} \{\frac{1}{a_n} \int |g(\frac{y}{a_n})|^{1+c} f(x-y)dy\} = f(x) \int |g(y)|^{1+c} dy.$$

We first consider the variance/covariance of the estimators $f_n^{(r)}$, r=0,1, under the assumption that $\beta_n = o(\frac{1}{a_n})$ as $n \to \infty$.

<u>Theorem 3.2</u>. Suppose the stationary sequence $\{X_j\}$ has dependence sequence $\{\beta_n\}$ such that $\beta_n = o(\frac{1}{a_n})$ as $n \to \infty$.

Suppose that the following conditions hold:

(a) the sequence $\{a_n\}$ satisfies Condition 2 and assumptions of Lemma 2.1 for some $\alpha > 0$,

(b) K is a symmetric differentiable density and satisfies the following conditions

(i) Condition 1,

(ii) $K^{(r)}(u)| \leq c$ and $\int K^{(r)}(u)^2 \, du < \infty$ for $r = 0, 1$ and c is finite positive constant,

(iii) $\int |K'(u)| du < \infty$ and there exists c_1 such that for all $z \in \mathbb{R}$, $t > 0$

We have

$$t|K^{(r)}(tz)| \leq c_1 |K^{(r)}(z)| \quad \text{for } r = 0, 1.$$

Then if x and y are in $C(f)$, we have for $r = 0, 1$,

$$\lim_{n\to\infty} na_n^{1+2r}\{\text{cov}[f_n^{(r)}(x), f_n^{(r)}(y)]\} = \begin{cases} \theta_r f(x) \int K^{(r)}(u)^2 du & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad (3.1)$$

where

$$\theta_r = \frac{1}{1 + (2r+1)\alpha} .$$

Proof.

$$na_n^{2r+1}\mathrm{cov}(f_n^{(r)}(x), f_n^{(r)}(y)) = \frac{1}{n}\sum_{j=1}^{n} \frac{a_n^{1+2r}}{a_j^{2r}} \mathrm{cov}(K_j^{(r)}(x-X_j),\ K_j^{(r)}(y-X_j))$$

$$+ \frac{a_n^{1+2r}}{n} \sum_{\substack{j\neq i \\ 1\leq i,j\leq n}} \frac{1}{(a_i a_j)^r} \mathrm{cov}(K_i^{(r)}(x-X_i),\ K^{(r)}(y-X_j))$$

$$= I_1 + I_2 . \tag{3.2}$$

Considering I_1, we have

$$\mathrm{cov}_1 := \mathrm{cov}[K_j^{(r)}(x-X_j),\ K_j^{(r)}(y-X_j)]$$

$$= \frac{1}{a_j}\int K^{(r)}(z)K^{(r)}\left(\frac{x-y}{a_j} + z\right) f(x-a_j z)dz$$

$$- \int K^{(r)}(z)f(x-a_j z)dz \int K^{(r)}(z)f(y-a_j z)dz . \tag{3.3}$$

In case $x = y$, we have by Lemma 3.1

$$\mathrm{var}[K_j(x-X_j)] \sim \frac{1}{a_j} f(x) \int K^{(r)}(z)^2 dz + o\left(\frac{1}{a_j}\right) \quad (j \to \infty)$$

for $r = 0, 1$, at each x and y in $C(f)$ by Lebesgue's dominated convergence theorem.

Application of Lemma 2.1, Corollary 2.2, and Toeplitz Lemma, [12, p. 238] now gives

$$I_1 = \theta_r f(x) \int K^{(r)}(z)^2 dz + o(1) \quad \text{as } n \to \infty \text{ at each } x \in C(f).$$

In case $x \neq y$, the first term on the right hand side of (3.3) can be estimated as follows. For j is sufficiently large we have

$$\left|\frac{1}{a_j}\int K^{(r)}(z)K^{(r)}\left(\frac{x-y-a_jz}{a_j}\right) f(x-a_jz)dz\right|$$

$$\leq c_1 \left|\int K^{(r)}(z)K^{(r)}(x-y+a_jz)F(x-a_jz)dz\right|$$

$$\to c_1 f(x)\left|K^{(r)}(x-y)\right| \int \left|K^{(r)}(z)\right|dz \qquad (j \to \infty)$$

hence the first term on the right hand side of (3.3) is bounded as $j \to \infty$. The second term can be estimated by similar arguments using the Lebesgue's dominated convergence theorem. As a consequence cov_1 is bounded as $j \to \infty$ in case $x \neq y$. Using (3.3), Lemma 2.1 and Toeplitz Lemma, we find that $I_1 = O(a_n)$ as $n \to \infty$ in $x \neq y$. Finally we estimate I_2.

Since the constants a_n are nondecreasing, we have

$$|I_2| \leq \frac{a_n}{n} \sum_{\substack{j\neq i \\ 1\leq i,j\leq n}} \left|\mathrm{cov}(K_i^{(r)}(x-X_i),\ K_j^{(r)}(y-X_j)\right|$$

$$\leq \frac{a_n}{n} \sum_{j\neq i} \iint \left|K_i^{(r)}(z_1)K_j^{(r)}(z_2)\right| \left|f_{ij}(x-z_1,y-z_2)\right.$$

$$\left. - f(x-z_1)f(y-z_2)\right|dz_1dz_2. \tag{3.4}$$

By stationarity, we have

$$\frac{1}{n} \sum_{1<i<j\leq n} \left|f_{ij}(x-z_1,y-z_2) - f(x-z_1)f(y-z_2)\right|$$

$$= \sum_{j=2}^{n} \frac{n-j+1}{n} \left|f_{1j}(x-z_1,y-z_2) - f(x-z_1)f(y-z_2)\right|$$

$$= o\left(\frac{1}{a_n}\right).$$

Moreover, a similar equality holds in case $i > j$.

Combination with (2.4) and (3.4) gives

$$|I_2| \leq 2c_1^2 a_n \{o(\frac{1}{a_n}) \int\int |K^{(r)}(z_1)K^{(r)}(z_2)| dz_1 dz_2$$

$$= (1) \quad \text{as} \quad n \to \infty.$$

Remark 1. Note that if $\alpha \in (0,1)$, then $a_n \to 0$ and $na_n \to \infty$ $(n \to \infty)$. Hence in this case we have var $f_n^{(r)}(x) \to 0$ $(n \to \infty)$ at all $x \in C(f)$.

Second, we consider the variance/covariance of $f_n^{(r)}$ under the uniform and strong mixing condition on the process $\{X_j\}$.

Theorem 3.3. Assume Conditions 1 and 2 and assumptions of Lemma 2.1 on the function K and the sequence $\{a_n\}$ are satisfied.

(a) If $\{X_j\}$ is a stationary sequence of r.v.'s satisfying the uniform mixing condition (2.2) such that $\beta_n = o(\frac{1}{a_n})$ and $\sum_{j=1}^{\infty} [\phi(j)]^{1/2} < \infty$, then, for $r = 0, 1$, (3.1) continues to hold.

(b) If $\{X_j\}$ is a stationary sequence of r.v.'s satisfying the strong mixing condition (2.1) such that $\beta_n = o(\frac{1}{a_n})$ and $\sum_{j=1}^{\infty} [\frac{\alpha(j)}{a_j}]^q < \infty$ for some $q \in (0, 1)$, then, for $r = 0, 1$, (3.1) continues to hold.

Proof.

(a) It is enough to show that I_2 in (3.2) converges to zero as n tends to ∞.

First, we shall get a bound on I_2 under the assumptions that the sequence $\{X_j\}$ satisfies the uniform mixing with coefficient $\phi(u)$ such that (see Abdul-Al [1])

$$\sum_{j=1}^{\infty} [\phi(j)]^{1/2} < \infty.$$

We have by fact that $\{a_n\}$ is nonincreasing sequence, Lemma 2.1 in [1] and Lemma 3.1 that

$$|I_2| \leq \frac{2a_n}{n} \sum_{i>j} |\mathrm{cov}[K_i^{(r)}(x-X_i),\ K_j^{(r)}(x-X_j)]|$$

$$\leq \frac{4a_n}{n} \sum_{i>j} [\phi(i-j)]^{1/2} E^{1/2}|K_i^{(r)}(x-X_i)|^2 E^{1/2}|K_j^{(r)}(x-X_j)|^2$$

$$= \frac{4a_n}{n\sqrt{a_i a_j}} \sum_{i>j} [\phi(i-j)]^{1/2} f(x) \int K^{(r)}(z)^2 dz$$

$$\leq (4f(x)\int K^{(r)}(z)^2 dz)\Big(\sum_{j=1}^{\infty} [\phi(j)]^{1/2}\Big) < \infty. \tag{3.6}$$

By the fact that $\{X_j\}$ is a stationary sequence of r.v.'s, $a_n \beta_n = o(1)$ and $\{a_n\}$ is a nonincreasing sequence, we get

$$|I_2| \leq \frac{2a_n}{n} \sum_{i>j} |\mathrm{cov}[K_i^{(r)}(x-X_i),\ K_j^{(r)}(x-X_j)|$$

$$= \frac{2a_n}{n} \sum_{i>j} \int\int |K_i^{(r)}(x-u)||K_j^{(r)}(x-w)||f_{ij}(u,w) - f(u)f(w)|\,du\,dw$$

$$\leq \frac{2a_n}{n} \sum_{i>j} \sup_{u,w} |f_{ij}(u,w) - f(u)f(w)| \int K^{(r)}(z)^2 dz$$

$$= \frac{2a_n}{n} \sup_{u,w} \sum_{j=2}^{n} (u-j+1)|f_j(u,w) - f(u)f(w)| \int K^{(r)}(z)^2 dz$$

$$= o(1) \quad \text{as} \quad n \to \infty. \tag{3.7}$$

(b) Proceed exactly as in Part (a) except we use a lemma due to Doe [8] to find the bound on I_2 as in (3.6).

$$|I_2| \leq \frac{2a_n}{n} \sum_{i>j} |\mathrm{cov}[K_i^{(r)}(x-X_i), K_j^{(r)}(x-X_j)]|$$

$$\leq \frac{20a_n}{n} \sum_{i>j} [\alpha(i-j)]^{1-\gamma} \{E K_i^{(r)}(x-X_i)|^{2+\delta} E|K_j(x-X_j)|^{2+\delta}\}^{\frac{1}{2+\delta}} \tag{3.8}$$

where $\gamma = \frac{2}{2+\delta}$ for some $\delta > 0$.

Note that

$$E|K_i(x-X_i)\ ^{2+\delta} \sim \frac{1}{a_i^{1+\delta}} f(x) \int |K^{(r)}(z)|^{2+\delta}\, dz \quad \text{as} \quad i \to \infty$$

(3.7) and (3.8) imply that

$$|I_2| \leq \frac{M}{n} \sum_{i>j} \{ \frac{(a_i a_j)^{\frac{\delta}{2(2+\delta)}}}{\sqrt{a_i a_j}} [\alpha(i-j)]^{1-\gamma} \}$$

$$= \frac{M}{n} \sum_{i>j} (\frac{1}{a_i a_j})^{\frac{\delta}{2(2+\delta)}} [\alpha(i-j)]^{1-\gamma}$$

$$\leq \frac{M}{n} \sum_{i>j} (\frac{1}{a_j})^{\frac{\delta}{2+\delta}} [\alpha(i-j)]^{1-\gamma}$$

$$\leq M \sum_{k=1}^{\infty} [\frac{\alpha(k)}{a_k}]^{1-\gamma} \quad . \tag{3.9}$$

where M is finite constant.

Remark 2. If we define the triangular array $\alpha_n(j)$ of mixing coefficient as in [ ] by

$$\alpha_n(k) = \begin{cases} \max\limits_{1 \leq i \leq n-k} \ \sup\limits_{\substack{A \in M_1^i \\ B \in M_{i+k}^n}} |P(AB)-P(A)P(B)|, & k=1,\ldots,n-1 \\ 0 & , \quad k \geq n \end{cases}$$

Then

$$\alpha_n(k) = \sup_n \alpha_n(k), \qquad k \geq 1 .$$

Also replace the condition $\sum_{j=1}^{\infty} (\frac{\alpha(j)}{a_j})^q < \infty$ for some $q \in (0, 1)$ in the above theorem by $\sum_{j=1} [\alpha(j)]^q < \infty$. Let $\{C_n\}$ be a sequence of real numbers in $[1, \infty)$ such that $C_n \to \infty$ and $C_n a_n \to 0$ as $n \to \infty$. It can be proved that for some $q \in (0, 1)$

$$\frac{1}{a_n^q} \sum [\alpha_n(k)]^q \to 0 \quad \text{as} \quad n \to \infty$$

and $\quad I_2 \to 0$ as $n \to \infty$ (see [14] for the proof.

The results of Theorem 3.3 hold in particular for asymptotically uncorrelated processes, provided $\sum_{j=1}^{\infty} [\rho(j)]^q < \infty$ for some $0 < q < 1$. However, for this class of processes of Theorem 3.3 can be established under the weaker condition $\sum_{k=1}^{\infty} \rho(k) < \infty$.

<u>Theorem 3.4</u>. Let $\{X_j\}$ be a stationary sequence of r.v.'s satisfying the asymptotic uncorrelatedness condition such that $\beta_n = (o(\frac{1}{a_n})$ and $\sum_{j=1} \rho(j) < \infty$, the kernel k and its first derivative satisfy Condition 1 and the sequence $\{a_n\}$ satisfy the assumptions of Lemma 2.1 and Condition 2; <u>then</u>, for $r = 0, 1$, (3.6) <u>continues to hold</u>.

<u>Proof</u>. Proceed exactly as the proof in part (a) of Theorem 3.3 except we use inequality (2.3).

If $\{X_j\}$ is a discrete time stationary Markov process satisfying Doeblin's condition (D_0), then it can be shown that $\{X_j\}$ is uniform mixing with $\phi(j) = a\rho^j$ for $a \geq 1$ and $\rho \in (0, 1)$. Hence, the class of uniform and strong mixing processes contains the class of Markov processess satisfying the Doeblin's condition (D_0). This means that Theorem 3.3 is true if the process $\{X_j\}$ satisfies the Doeblin's Condition (D_0).

Let $\{X_j\}$ be a stationary process and define the transition operator H_n by

$$(H_n f)(x) = E[f(X_{n+1}) | X_1 = x]$$

for all x, where f is a bounded measurable function on $\mathbb{R}$. Then define

$$|H_n|_2 = \sup_{\{f:E[f(X)]=0\}} E^{1/2}(H_n f)^2 / E^{1/2}(f^2).$$

The transition operator H_n is said to satisfy the Condition $G_2(m, \alpha)$ if there exists a positive integer m such that $|H_m|_2 \leq \alpha$ with $\alpha \in (0, 1)$. If $\{X_j\}$ is a stationary Markov process, then

$$H_{m+n} = H_m H_n = H_n H_m$$

and for every $n \geq 0$,

$$|H_n|_2 < \beta^n/\alpha \tag{3.11}$$

with $\beta = \alpha^{1/m}$ in $(0, 1)$. (see [15, p. 322]).

<u>Theorem 3.5</u>. Let $\{X_j\}$ be a stationary Markov process satisfying $G_2(\alpha, m)$. Let K and the sequence $\{a_n\}$ satisfy Conditions 1 and 2, respectively. <u>Then</u> for $r = 0, 1$, (3.6) <u>continues to hold</u>.

<u>Proof</u>. It is sufficient to prove that I_2 in (3.2) converges to 0 as $n \to \infty$.

$$|I_2| \leq \frac{2a_n}{n} \sum_{i>j} |\mathrm{cov}[K_i^{(r)}(x-X_i), K_j^{(r)}(x-X_j)]|$$

$$\leq \frac{2a_n}{n} \sum_{i>j} \frac{\beta^{(i-j)}}{\alpha} E^{1/2}|K_i^{(r)}(x-X_i)|^2 E^{1/2}|K_j^{(r)}(x-X_j)|^2$$

$$\leq \frac{2}{\alpha} M_1 \sum_{j=1}^{\infty} \beta^j \leq 2M_1/(1-\beta) < \infty,$$

because $\beta \in (0, 1)$ and M_1 is a finite constant.

From (3.7) and (3.12), we have

$$I_2 \to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

Theorem 3.6. Suppose the sequence $\{a_n\}$ satisfies Condition 2 and the assumptions of Lemma 2.1 with $\alpha \in (0, 1/2)$ and K is a symmetric density with $\int |z|^3 K(z)\,dz < \infty$. Assume that $X_1, X_2, \ldots$ are identically distributed with density f. If f_n is defined as in (1.2), f is 3 times differentiable in a neighborhood of x and $f''(x) \neq 0$, then

$$Ef_n(x) - f(x) \sim \frac{1}{2(1-2\alpha)} f''(x) a_n^2 \int z^2 K(z)\,dz \quad (n \to \infty). \tag{3.12}$$

Proof. Since K is symmetric, we have, using Taylor's theorem

$$Ef_n(x) - f(x) = \frac{1}{n}\int \sum_{j=1}^{n} K(z)\ f(x - a_j z) - f(x)\}dz$$

$$= \frac{1}{n}\sum_{j=1}^{n} \int K(z)\{-a_j z f'(x) + \frac{a_j^2 z^2}{2} f''(x) + O(a_j^3 z^3)\}dz$$

$$= \frac{1}{2} f''(x)\left(\int z^2 K(z)dz\right) \frac{1}{n}\sum_{j=1}^{n} a_j^2(1 + o(1)) \quad (n\to\infty)$$

Application of Lemma 2.1 finally gives (3.12).

Remark 3. Under the assumptions of Theorem 3.2 - Theorem 3.5 we find that the mean square error is equal to

$$E\{f_n(x) - f(x)\}^2 = \frac{1}{1+\alpha} f(x) \int K^2(z)dz \frac{1}{na_n}$$

$$+ \frac{1}{4(1-2)^2} f''(x)^2 \left(\int z^2 K(z)dz\right)^2 a_n^4 + o\left(\frac{1}{na_n}\right) + o(a_n^4) \quad (n \to \infty).$$

It is easily seen that in case $a_n = cn^{-\alpha}$, the optimal choice for α

is 1/5. In that case we have

$$E\{f_n(x) - f(x)\}^2 = \{\tfrac{5}{6}f(x)(\int K^2(z)dz)c^{-1} + \tfrac{25}{36}f''(x)^2(\int z^2K(z)dz)^2c^4\}n^{-4/5}.$$

The optimal choice of c then finally depends upon the kernel K. Depending upon the kernel and the values of $f(x)$ and $f''(x)$ the result may be better or worse than the usual estimate.

Conclusions:

1. We have established in Theorems 3.2 - 3.6 the asymptotic expressions for the variance/covariance bias and the quadratic mean consistency r, of the estimators $f_n(x)$, $r = 0, 1$ for p.d.f. $f(x)$. The results obtained are valid under asymptotic uncorrelatedness/independence on the process X_j.

2. The asymptotic uncorrelatedness condition imposed on $\{X_j\}$ in Theorem 3.2 is weaker than the conditions imposed on the process $\{X_i\}$ in other theorems. However, this freedom leads us to impose relatively stronger condition(s) on the kernel in Theorem 3.2 than that imposed on the kernels in the other theorems.

3. G_2 and Doeblin's conditions are the weakest among the various mixing conditions imposed on the process since it involves only that the dependence index β_n is $o(\frac{1}{a_n})$.

4. It is interesting to note that the asymptotic variance/covariance expressions (3.1) obtained here under the variance asymptotic uncorrelatedness/independence conditions coincides with classical results for independent observations.

5. It is shown that the asymptotic variance of $f_n(x)$ is smaller than the asymptotic variance of $\hat{f}_n(x)$.

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