On Rate of Convergence of Recursive Kernel Estimates of Probability Densities

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Abstract

Recursive estimates \( f_n^{(r)}(x) \) of the \( r \)-th derivative \( f^{(r)}(x) \)
\((r = 0, 1)\) of the univariate probability density \( f(x) \) for strictly
stationary processes \( \{X_j\} \) is considered. Asymptotic variance/covariance of
\( f_n^{(r)}(x) \) are established for stationary triangular arrays of
random variables satisfying various asymptotic independence-uncorrelatedness conditions.

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Asymptotic uncorrelatedness/Independence
1. Introduction.

Suppose \( \{X_j\} \) be a sequence of stationary random variables (r.v.'s) with common probability density function (p.d.f.) \( f(x) \). Much research in recent years is concentrated on studying properties of the kernel estimators (based on the first \( n \) observations)

\[
\hat{f}_n(x) = \frac{1}{n a_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{a_n}\right)
\]

where \( \{a_n\} \), \( n \geq 1 \), is a given sequence of positive numbers such that \( a_n \to 0 \) as \( n \to \infty \) and \( K \) is a given kernel. Roussas [16] and Rosenblatt [18] have studied the asymptotic properties of the estimators \( \hat{f}_n \) for stationary processes satisfying Doeblin's and \( C_2 \) conditions, respectively. Masry [13], Abdul-al and Siddiqui [2], Castellana and Leadbetter [5] have studied the asymptotic properties of \( \hat{f}_n \) for stationary uniform and strong mixing processes. In case of dependence it can be expected that asymptotic properties of the estimator \( f_n \) can be improved if the window width is not necessarily the same for each observation, that means the estimator

\[
f_n(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j} K\left(\frac{x - X_j}{a_j}\right)
\]

is considered. Earlier research concerning \( f_n \) in case of independent observations is done by Devroye [7], Samanta and Mugisha [20], who extended results of Yamato [22], and Davies [6]. While recently Abdul-Al [1], Masry [14], Abdul-Al and Geluk [3] have studied the properties of \( f_n \) under the uniform and strong mixing processes.

Using the first \( n \) observations from a sequence of strictly stationary
random variables, consider the estimator

\[ f_n^{(r)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j^{r+1}} k^{(r)} \left( \frac{x-X_j}{a_j} \right) \]  

(1.3)

of the \( r \)-th derivative \( f^{(r)}(x) \) \((r=0,1,...)\), where \( k^{(r)} \) is the \( r \)-th derivative of \( k \).

The aim of this paper is to study and compare the asymptotic behavior of the variance/covariance of the estimator \( f_n^{(r)} \) for strictly stationary sequence of r.v.'s satisfying various independence-uncorrelatedness conditions such as uniform and strong mixing, maximal correlation, \( G_2 \) and Doeblin's conditions.

2. Preliminaries and Notations

Let \( \mathcal{M}_a^b \) denote the \( \sigma \)-algebra of events generated by the r.v.'s \( \{X_j; a \leq j \leq b\} \), \( -\infty \leq a \leq b \leq \infty \). The concept of strong mixing is due to Rosenblatt [17]. The stationary \( \{X_j\} \) process is strong mixing if for \( k \geq 0 \)

\[ \sup_{A \in \mathcal{M}_{-\infty}^1, B \in \mathcal{M}_{1+k}^\infty} |P(AB) - P(A)P(B)| = \alpha(k) + O \quad \text{as} \quad k \to \infty \]  

(2.1)

The value \( \alpha(k) \) characterizes the mixing rate and is referred to as the strong mixing coefficient. Equation (2.1) implies that the random variables \( X_1 \) and \( X_{1+k} \) becomes asymptotically independent as the lag \( k \) tends to infinity.
A second notation of asymptotic independence is the uniform mixing due to Ibragimov [10]. The stationary process \( \{X_j\} \) is uniform mixing if for \( k \geq 0 \) and \( P(A) > 0 \)

\[
\sup_{i} \left| P(AB) - P(A) P(B) \right| = P(A) \phi(k) \downarrow 0 \quad \text{as} \quad k \to \infty. \quad (2.2)
\]

It is clear that \( \alpha(k) \leq \phi(k) \), that means a uniform mixing process is always strong mixing.

A third notation is the maximal correlation due to Kolmogorov and Rozanov [11]. The stationary process \( \{X_j\} \) satisfies the maximal correlation if for \( k \geq 0 \)

\[
\sup_{U \in L_2(\mathcal{M}_{i})} \left| \frac{\text{cov}(U, V)}{\sqrt{\text{var}(U)\text{var}(V)}} \right| = \rho(k) \downarrow 0 \quad \text{as} \quad k \to \infty \quad (2.3)
\]

where \( L_2(\mathcal{M}_a^b) \) denotes the collection of all second-order random variables measurable with respect to \( \mathcal{M}_a^b \).

It is known (see [21]) that \( 4\alpha(k) \leq \rho(k) \leq 2\phi^{1/2}(k) \). Thus a stationary process \( \{X_j\} \) satisfying the maximal correlation is uniform mixing and a uniform mixing process is strong mixing.

It will be assumed that the joint probability density \( f_{j}(x, y) \) of the r.v.'s \( X_1, X_j \) \( (j=2,3,...) \) are absolutely continuous. Define the primary measure of dependence of the sequence \( \{X_j\} \) as (see [5])

\[
\beta_n = \sup_{x,y} \sum_{j=2}^{n} \left| f_j(x,y) - f(x)f(y) \right| \quad (2.4)
\]
which is finite for each \( n \). The sequence \( \{ \beta_n \} \) is called the dependence index sequence for the process \( \{ X_j \} \). Clearly for i.i.d. sequence \( \beta_n = 0 \) for all \( n \).

Throughout this paper the functions \( k^{(r)}, r = 0, 1, \) and the constants \( a_n \) are assumed to satisfy the following conditions:

**Condition 1:**

\[
\begin{align*}
(\text{i}) & \quad \sup_x |k^{(r)}(x)| < \infty, \\
(\text{ii}) & \quad \int |k^{(r)}(x)| \, dx < \infty \quad \text{and} \\
(\text{iii}) & \quad \lim_{|x| \to \infty} |xk^{(r)}(x)| = 0.
\end{align*}
\]  

**Condition 2:**

\[
a_n \rightarrow 0 \quad \text{and} \quad na_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\] (2.6)

Define \( k_j^{(r)}(x) \) by

\[
k_j^{(r)}(x) = k^{(r)}(\frac{x}{a_j})
\]

and write the estimator \( f_n^{(r)}(x) \) as

\[
f_n^{(r)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{a_j} k_j(x - X_j).
\] (2.7)

In order to obtain the asymptotic behavior of the covariance of the estimators, the sequence \( \{ a_n \} \) has to be sufficiently smooth. It turns out that regular variation in this case is an appropriate property for the sequence \( \{ a_n \} \). The lemma below contains the basic ingredients for the proofs.
Lemma 2.1. (see Bojonic and Seneta [9]).

Suppose \( \{a_n\} \), \( n \geq 1 \), is a sequence of positive numbers, then

\[
\lim_{n \to \infty} a_{[nx]} / a_n = x^{-\alpha} (x > 0) \quad \text{for some} \quad \alpha > -1 \quad \text{if and only if}
\]

\[
\lim_{n \to \infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{1}{a_j} = \frac{1}{\alpha + 1}.
\]

A sequence \( \{a_n\} \) satisfying the assumptions of Lemma 2.1 is called a regularly varying sequence. For more properties the reader is referred to [9].

Corollary 2.2. If \( c_n > 0 \) for \( n \geq 1 \), \( c_n \sim a_n \) \( (n \to \infty) \) and the sequence \( \{a_n\} \) satisfies the assumptions of Lemma 2.1, then

\[
\lim_{n \to \infty} \frac{a_n}{n} \sum_{j=1}^{n} \frac{c_j}{c_j} = \frac{1}{\alpha + 1}.
\]

Main Result:

In this section we establish precise asymptotic expression for the covariance of the estimators \( f_{(r)} \). These results will be held under a suitable asymptotic-independence, or uncorrelatedness assumptions on the process \( \{X_j\} \).

The following lemma can be found in [20], and will be used in the sequel.

Lemma 3.1. Let \( g(x) \) be a real-valued Borel measurable function. If \( x \in C(f) \) \( [C(f) \) denote the set of continuity points of \( f(x) \)] and if
\[ \sup_{x} |g(x)| < \infty, \int |g(x)| dx < \infty \text{ and } \lim_{|x| \to \infty} |xg(x)| = 0 \]

then for every \( c > 0 \)

\[ \lim_{n \to \infty} \left\{ \frac{1}{a_n} \int_{a_n}^{\infty} |g(y)|^{1+c} f(x-y) dy \right\} = f(x) \int |g(y)|^{1+c} dy. \]

We first consider the variance/covariance of the estimators \( f_{n}^{(r)}, r = 0, 1, \) under the assumption that \( \beta_n = o\left(\frac{1}{a_n}\right) \text{ as } n \to \infty. \)

**Theorem 3.2.** Suppose the stationary sequence \( \{X_j\} \) has dependence sequence \( \{\beta_n\} \) such that \( \beta_n = o\left(\frac{1}{a_n}\right) \text{ as } n \to \infty. \)

Suppose that the following conditions hold:

(a) the sequence \( \{a_n\} \) satisfies Condition 2 and assumptions of Lemma 2.1 for some \( a > 0, \)

(b) \( K \) is a symmetric differentiable density and satisfies the following conditions

(i) Condition 1,

(ii) \( K^{(r)}(u) < c \) and \( \int K^{(r)}(u)^2 du < \infty \text{ for } r = 0, 1 \) and \( c \) is finite positive constant,

(iii) \( \int \left| K^{(r)}(u) \right| du < \infty \) and there exists \( c_1 \) such that for all \( z \in \mathbb{R}, \)

\[ t > 0 \]

We have

\[ t \left| K^{(r)}(t z) \right| \leq c_1 \left| K^{(r)}(z) \right| \text{ for } r = 0, 1. \]

Then if \( x \) and \( y \) are in \( C(f), \) we have for \( r = 0, 1, \)

\[ \lim_{n \to \infty} n a_n^{1+2r} \text{cov}\left[ f_n^{(r)}(x), f_n^{(r)}(y) \right] = \begin{cases} \frac{r}{2} f(x) \int K^{(r)}(u)^2 du & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \]

(3.1)
where

$$\theta_\tau = \frac{1}{1 + (2\tau + 1)\alpha}.$$ 

**Proof.**

$$\frac{n}{2n^{2\tau + 1}} \text{cov}(f_n^{(\tau)}(x), f_n^{(\tau)}(y)) = \frac{1}{n} \sum_{j=1}^{n} \frac{a_j^{1+2\tau}}{a_j^{2\tau}} \text{cov}(k_j^{(\tau)}(x-X_j), k_j^{(\tau)}(y-X_j))$$

$$+ \frac{a_j^{1+2\tau}}{a_j^{2\tau}} \sum_{1 \leq i, j \leq n} \frac{1}{(a_i a_j)^{\tau}} \text{cov}(k_i^{(\tau)}(x-X_i), k_j^{(\tau)}(y-X_j))$$

$$= I_1 + I_2. \quad (3.2)$$

Considering $I_1$, we have

$$\text{cov}_1 = \text{cov}[k_j^{(\tau)}(x-X_j), k_j^{(\tau)}(y-X_j)]$$

$$= \frac{1}{a_j} \int k^{(\tau)}(z)k^{(\tau)}(\frac{x-y}{a_j}) f(x-a_jz)dz$$

$$- \int k^{(\tau)}(z)f(x-a_jz)dz \int k^{(\tau)}(z)f(y-a_jz)dz. \quad (3.3)$$

In case $x = y$, we have by Lemma 3.1

$$\text{var}[k_j(x-X_j)] - \frac{1}{a_j} f(x) \int k^{(\tau)}(z)^2 dz + o\left(\frac{1}{a_j}\right) \quad (j \to \infty)$$

for $\tau = 0, 1$, at each $x$ and $y$ in $C(f)$ by Lebesgue's dominated convergence theorem.

Application of Lemma 2.1, Corollary 2.2, and Toeplitz Lemma, [12, p. 253] now gives

$$I_1 = \theta_\tau f(x) \int k^{(\tau)}(z)^2 dz + o(1) \quad \text{as } n \to \infty \quad \text{at each } x \in C(f).$$

In case $x \neq y$, the first term on the right hand side of (3.3) can be estimated as follows. For $j$ is sufficiently large we have
\[
\left| \frac{1}{n_j} \int \chi^{(r)}(z) K^{(r)}(x - y - \frac{a_j}{z}) f(x - a_j z) dz \right|
\leq c_1 \left| \int \chi^{(r)}(z) K^{(r)}(x - y + a_j z) F(x - a_j z) dz \right|
- c_1 f(x) |K^{(r)}(x - y)| \int |K^{(r)}(z)| dz \quad (j \to \infty)
\]

hence the first term on the right hand side of (3.3) is bounded as \( j \to \infty \). The second term can be estimated by similar arguments using the Lebesgue's dominated convergence theorem. As a consequence \( \text{cov}_1 \) is bounded as \( j \to \infty \) in case \( x \neq y \). Using (3.3), Lemma 2.1 and Toeplitz Lemma, we find that \( I_1 = o(a_n) \) as \( n \to \infty \) in \( x \neq y \). Finally we estimate \( I_2 \).

Since the constants \( a_n \) are nondecreasing, we have

\[
|I_2| \leq \frac{a_n}{n} \sum_{j \neq i} \sum_{1 \leq i, j \leq n} |\text{cov}(K^{(r)}_i (x - x_j), K^{(r)}_j (y - x_j))|
\leq \frac{a_n}{n} \sum_{j \neq i} \int \int |K^{(r)}_i (z_1) K^{(r)}_j (z_2)| f_{ij}(x - z_1, y - z_2) dz_1 dz_2
- f(x - z_1) f(y - z_2) dz_1 dz_2.
\] (3.4)

By stationarity, we have

\[
\frac{1}{n} \sum_{1 < i < j \leq n} |f_{ij}(x - z_1, y - z_2) - f(x - z_1) f(y - z_2)|
= \sum_{j=1}^{n} \frac{n - j + 1}{n} |f_{1j}(x - z_1, y - z_2) - f(x - z_1) f(y - z_2)|
= o\left(\frac{1}{n}\right).
\]

Moreover, a similar equality holds in case \( i > j \).
Combination with (2.4) and (3.4) gives

\[ |I_2| \leq 2c_2 a_n \mathbb{1}_{a_n \rightarrow n} \int |K(r)(z_1)K(r)(z_2)|dz_1dz_2 \]

\[ = (1) \quad \text{as} \quad n \rightarrow \infty. \]

**Remark 1.** Note that if \( a \in (0,1) \), then \( a_n \rightarrow 0 \) and \( na_n \rightarrow \infty \) \( (n \rightarrow \infty) \).

Hence in this case we have \( \text{var} f_n^{(r)}(x) \rightarrow 0 \) \( (n \rightarrow \infty) \) at all \( x \in C(f) \).

Second, we consider the variance/covariance of \( f_n^{(r)} \) under the uniform and strong mixing condition on the process \( \{X_j\} \).

**Theorem 3.3.** Assume Conditions 1 and 2 and assumptions of Lemma 2.1 on the function \( K \) and the sequence \( \{a_n\} \) are satisfied.

(a) If \( \{X_j\} \) is a stationary sequence of r.v.'s satisfying the uniform mixing condition (2.2) such that \( \beta_n = o\left(\frac{1}{a_n}\right) \) and \( \sum_{j=1}^{\infty} [\phi(j)]^{1/2} < \infty \), then, for \( r = 0, 1 \), (3.1) continues to hold.

(b) If \( \{X_j\} \) is a stationary sequence of r.v.'s satisfying the strong mixing condition (2.1) such that \( \beta_n = o\left(\frac{1}{a_n}\right) \) and \( \sum_{j=1}^{\infty} \left[\frac{a(j)}{a_j}\right]^q < \infty \) for some \( q \in (0, 1) \), then, for \( r = 0, 1 \), (3.1) continues to hold.

**Proof.**

(a) It is enough to show that \( I_2 \) in (3.2) converges to zero as \( n \) tends to \( \infty \).

First, we shall get a bound on \( I_2 \) under the assumptions that the sequence \( \{X_j\} \) satisfies the uniform mixing with coefficient \( \phi(u) \) such that (see Abdul-Al [1])

\[ \sum_{j=1}^{\infty} [\phi(j)]^{1/2} < \infty. \]
We have by fact that \( \{a_n\} \) is nonincreasing sequence, Lemma 2.1 in [1] and Lemma 3.1 that

\[
|I_2| \leq \frac{2a_n}{n} \sum_{i>j} |\text{cov}[K_i^{(r)}(x-X_i), K_j^{(r)}(x-X_j)]|
\]

\[
\leq \frac{4a_n}{n} \sum_{i>j} \left[ \left( \phi(i-j) \right)^{1/2} E^{1/2} |K_i^{(r)}(x-X_i)|^2 E^{1/2} |K_j^{(r)}(x-X_j)|^2 \right]
\]

\[
= \frac{4a_n}{n^{1/2} a_i a_j} \sum_{i>j} \left[ \phi(i-j) \right]^{1/2} f(x) \int k^{(r)}(z)^2 dz
\]

\[
\leq (4f(x) \int k^{(r)}(z)^2 dz) \left( \sum_{j=1}^{\infty} \left[ \phi(j) \right]^{1/2} \right) < \infty. \tag{3.6}
\]

By the fact that \( \{X_j\} \) is a stationary sequence of r.v.'s, \( a_n b_n = o(1) \) and \( \{a_n\} \) is a nonincreasing sequence, we get

\[
|I_2| \leq \frac{2a_n}{n} \sum_{i>j} | \text{cov}[K_i^{(r)}(x-X_i), K_j^{(r)}(x-X_j)] |
\]

\[
= \frac{2a_n}{n} \sum_{i>j} \int \int |K_i^{(r)}(x-u)||K_j^{(r)}(x-w)||f_{i,j}(u,w) - f(u)f(w)| du dw
\]

\[
\leq \frac{2a_n}{n} \sum_{i>j} \sup_{u,w} |f_{i,j}(u,w) - f(u)f(w)| \int k^{(r)}(z)^2 dz
\]

\[
= \frac{2a_n}{n} \sup_{u,w} \sum_{j=2}^{\infty} (u-j+1) |f_{i,j}(u,w) - f(u)f(w)| \int k^{(r)}(z)^2 dz
\]

\[
= o(1) \text{ as } n \to \infty. \tag{3.7}
\]

(b) Proceed exactly as in Part (a) except we use a lemma due to Doe [8] to find the bound on \( I_2 \) as in (3.6).
\[ |I_2| \leq \frac{2a}{n} \sum_{i>j} \left| \text{cov}[K_1^{(r)}(x-X_i), K_j^{(r)}(x-X_j)] \right| \]
\[
\leq \frac{20a}{n} \sum_{i>j} [a(i-j)]^{1-\gamma} (E|K_1^{(r)}(x-X_i)|^{2+\delta}E|K_j(x-X_j)|^{2+\delta})^{\frac{1}{2+\delta}}
\]

(3.8)

where \( \gamma = \frac{2}{2+\delta} \) for some \( \delta > 0 \).

Note that

\[ E|K_1^{(r)}(x-X_i)|^{2+\delta} \sim \frac{1}{a_{i+\delta}} f(x) \int |K^{(r)}(z)|^{2+\delta} dz \text{ as } i \to \infty \]

(3.7), and (3.8) imply that

\[
|I_2| \leq \frac{M}{n} \sum_{i>j} \frac{(a_i a_j)}{a_i a_j} [a(i-j)]^{1-\gamma}
\]
\[
= \frac{M}{n} \sum_{i>j} \left( \frac{1}{a_i a_j} \right)^{(2+\delta)} [a(i-j)]^{1-\gamma}
\]
\[
\leq \frac{M}{n} \sum_{i>j} \frac{\delta}{a_i a_j} [a(i-j)]^{1-\gamma}
\]
\[
\leq \sum_{k=1}^{\infty} \left( \frac{a(k)}{a_k} \right)^{1-\gamma} ,
\]

(3.9)

where \( M \) is finite constant.

**Remark 2.** If we define the triangular array \( \alpha_n(j) \) of mixing coefficient as in [ ] by

\[
\alpha_n(k) = \begin{cases} 
\max_{1 \leq i < n-k} \sup_{A \in M_1^i} |P(AB)-P(A)P(B)|, & k=1, \ldots, n-1 \\
0, & k \geq n
\end{cases}
\]

Then
\[ \alpha_n(k) = \sup_n \alpha_n(k), \quad k \geq 1. \]

Also replace the condition \[ \sum_{j=1}^{\infty} \left( \frac{\alpha(j)}{a_j} \right)^q < \infty \] for some \( q \in (0, 1) \) in the above theorem by \[ \sum_{j=1}^{\infty} \left[ a(j) \right]^q < \infty. \] Let \( \{C_n\} \) be a sequence of real numbers in \([1, \infty)\) such that \( C_n \to \infty \) and \( C_n a_n \to 0 \) as \( n \to \infty \). It can be proved that for some \( q \in (0, 1) \)

\[ \frac{1}{a_n^q} \sum \left[ \alpha_n(k) \right]^q \to 0 \quad \text{as} \quad n \to \infty \]

and \[ I_2 \to 0 \quad \text{as} \quad n \to \infty. \] (see [14] for the proof.

The results of Theorem 3.3 hold in particular for asymptotically uncorrelated processes, provided \[ \sum_{j=1}^{\infty} [\rho(j)]^q < \infty \] for some \( 0 < q < 1 \). However, for this class of processes of Theorem 3.3 can be established under the weaker condition \[ \sum_{k=1}^{\infty} \rho(k) < \infty. \]

**Theorem 3.4.** Let \( \{X_j\} \) be a stationary sequence of r.v.'s satisfying the asymptotic uncorrelatedness condition such that \( \beta_n = o(\frac{1}{a_n}) \) and \[ \sum_{j=1}^{\infty} \rho(j) < \infty, \] the kernel \( k \) and its first derivative satisfy Condition 1 and the sequence \( \{a_n\} \) satisfy the assumptions of Lemma 2.1 and Condition 2; then, for \( r = 0, 1 \), (3.6) continues to hold.

**Proof.** Proceed exactly as the proof in part (a) of Theorem 3.3 except we use inequality (2.3).

If \( \{X_j\} \) is a discrete time stationary Markov process satisfying Doeblin's condition \((D_0)\), then it can be shown that \( \{X_j\} \) is uniform mixing with \( \phi(j) = a^j \rho \) for \( a \geq 1 \) and \( \rho \in (0, 1) \). Hence, the class of uniform and strong mixing processes contains the class of Markov processes satisfying the Doeblin's condition \((D_0)\). This means that Theorem 3.3 is true.
if the process \( \{X_j\} \) satisfies the Doeblin's Condition \((D_0)\).

Let \( \{X_j\} \) be a stationary process and define the transition operator \( H_n \) by

\[
(H_n f)(x) = E[f(X_{n+1})|X_1 = x]
\]

for all \( x \), where \( f \) is a bounded measurable function on \( \mathbb{R} \). Then define

\[
|H_n|_2 = \sup_{\{f: E[f(X)] = 0\}} E^{1/2}(H_n f)^2 / E^{1/2}(f^2).
\]

The transition operator \( H_n \) is said to satisfy the Condition \( G_2(m, \alpha) \) if there exists a positive integer \( m \) such that \( |H_m|_2 < \alpha \) with \( \alpha \in (0, 1) \).

If \( \{X_j\} \) is a stationary Markov process, then

\[
H_{m+n} = H_m H_n = H_n H_m
\]

and for every \( n \geq 0 \),

\[
|H_n|_2 < \beta^n / \alpha
\]

(3.11)

with \( \beta = \alpha^{1/m} \) in \((0, 1)\). (see [15, p. 322]).

**Theorem 3.5.** Let \( \{X_j\} \) be a stationary Markov process satisfying \( G_2(\alpha, m) \).

Let \( K \) and the sequence \( \{a_n\} \) satisfy Conditions 1 and 2, respectively.

Then for \( r = 0, 1 \), (3.6) continues to hold.

**Proof.** It is sufficient to prove that \( I_2 \) in (3.2) converges to 0 as \( n \to \infty \).

\[
|I_2| \leq \frac{2a_n}{n} \sum_{i > j} |\text{cov}[k_i^{(r)}(x-X_i), k_j^{(r)}(x-X_j)]|
\]

\[
\leq \frac{2a_n}{n} \sum_{i > j} \frac{\beta^{(i-j)}}{\alpha} E^{1/2} |k_i^{(r)}(x-X_i)|^2 E^{1/2} |k_j^{(r)}(x-X_j)|^2
\]

\[
\leq \frac{2}{\alpha} M_1 \sum_{j=1}^{\infty} \beta^j \leq 2M_1/(1-\beta) < \infty,
\]
because $\beta \in (0, 1)$ and $M_1$ is a finite constant.

From (3.7) and (3.12), we have

$$I_2 \to 0 \text{ as } n \to \infty. \quad \Box$$

**Theorem 3.6.** Suppose the sequence $\{a_n\}$ satisfies Condition 2 and the assumptions of Lemma 2.1 with $\alpha \in (0, 1/2)$ and $K$ is a symmetric density with $\int |z|^3 K(z) \, dz < \infty$. Assume that $X_1, X_2, \ldots$ are identically distributed with density $f$. If $f_n$ is defined as in (1.2), $f$ is 3 times differentiable in a neighborhood of $x$ and $f''(x) \neq 0$, then

$$Ef_n(x) - f(x) - \frac{1}{2(1-2\alpha)} f''(x) a_n^2 \int z^2 K(z) \, dz \quad (n \to \infty). \quad (3.12)$$

**Proof.** Since $K$ is symmetric, we have, using Taylor's theorem

$$Ef_n(x) - f(x) = \frac{1}{n} \sum_{j=1}^{n} K(z) f(x - a_j z) - f(x) \, dz$$

$$= \frac{1}{n} \sum_{j=1}^{n} \int K(z) \left(-a_j^2 z f'(x) + \frac{a_j^2 z^2}{2} f''(x) + O(a_j^3 z^3) \right) \, dz$$

$$= \frac{1}{2} f''(x) \left( \int z^2 K(z) \, dz \right) \frac{1}{n} \sum_{j=1}^{n} a_j^2 (1 + o(1)) \quad (n \to \infty)$$

Application of Lemma 2.1 finally gives (3.12).

**Remark 3.** Under the assumptions of Theorem 3.2 - Theorem 3.5 we find that the mean square error is equal to

$$E[f_n(x) - f(x)]^2 = \frac{1}{1+\alpha} f(x) \int k^2(z) \, dz \frac{1}{n a_n}$$

$$+ \frac{1}{4(1-2)^2} f''(x)^2 \left( \int z^2 K(z) \, dz \right)^2 a_n^4 + o\left(\frac{1}{n a_n}\right) + o\left(a_n^4\right) \quad (n \to \infty).$$

It is easily seen that in case $a_n = cn^{-\alpha}$, the optimal choice for $\alpha$
is $1/5$. In that case we have

$$E(f_n(x) - f(x))^2 = \frac{5}{6} f(x) (\int K^2(z) dz) c^{-1} + \frac{25}{36} f''(x) (\int z^2 K(z) dz)^2 c^4) n^{-4/5}.$$ 

The optimal choice of $c$ then finally depends upon the kernel $K$. Depending upon the kernel and the values of $f(x)$ and $f''(x)$ the result may be better or worse than the usual estimate.

Conclusions:

1. We have established in Theorems 3.2 - 3.6 the asymptotic expressions for the variance/covariance bias and the quadratic mean consistency $r$, of the estimators $f_n(x)$, $r = 0, 1$ for p.d.f. $f(x)$. The results obtained are valid under asymptotic uncorrelatedness/independence on the process $X_j$.

2. The asymptotic uncorrelatedness condition imposed on $\{X_j\}$ in Theorem 3.2 is weaker than the conditions imposed on the process $\{X_j\}$ in other theorems. However, this freedom leads us to impose relatively stronger condition(s) on the kernel in Theorem 3.2 than that imposed on the kernels in the other theorems.

3. $G_2$ and Doeblin's conditions are the weakest among the various mixing conditions imposed on the process since it involves only that the dependence index $g_n$ is $o(\frac{1}{n})$.

4. It is interesting to note that the asymptotic variance/covariance expressions (3.1) obtained here under the variance asymptotic uncorrelatedness/independence conditions coincides with classical results for independent observations.
5. It is shown that the asymptotic variance of \( f_n(x) \) is smaller than the asymptotic variance of \( \hat{f}_n(x) \).

References


[18] M. Rosenblatt, Density estimates of Markov sequences, Nonparametric Techniques in Statistical Inference,


