King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 101

January 1988

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COMPLETELY QUASI-PROJECTIVE MONOIDS

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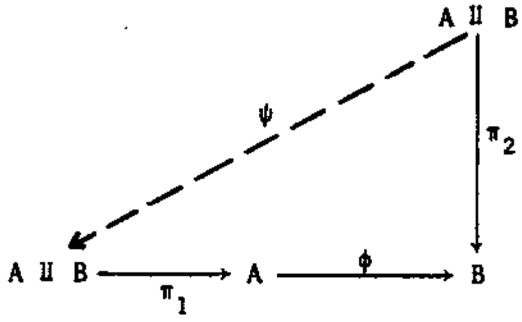
Let S be a monoid with a two-sided zero, 0. A unitary right S-system M over S is a nonempty set M with a multiplication $M \times S \to M$ satisfying $m1 = m$ and $m(st) = (ms)t$ for all $m \in M$ and $s, t \in S$. M is centered if it has a fixed element θ such that $\theta s = \theta$ and $m0 = \theta$ for all $s \in S$ and $m \in M$. A subsystem of M_S, denoted by $N_S \subseteq M_S$, is a subset N of M such that $ns \in N$ for all $n \in N$ and $s \in S$. A one element subsystem of M is called a zero of M. A mapping $f: A_S \to B_S$ is an S-homomorphism if for any $a \in A$ and $s \in S$, $f(as) = f(a)s$. Let $\{M_i: i \in I\}$ be a family of right S-systems and let every M_i contain a fixed one element subsystem θ_i, then the product $\prod_{i \in I} M_i$ and the coproduct $\coprod_{i \in I} M_i$ exist in the category of S-systems and are isomorphic, respectively, to the cartesian product and the disjoint union of the sets M_i with their respective zeros identified, and with the suitable action of S. By the <u>direct sum</u>, $\bigoplus_{i \in I} M_i$ of $\{M_i: i \in I\}$ we mean the subset of $\prod_{i \in I} M_i$, consisting of $(m_i)_{i \in I} \in \prod M_i$ for which $\{i: m_i \neq \theta_i\}$ is finite. Then $\bigoplus_{i \in I} M_i$ is a right S-system under componentwise multiplication. A right S-system P is projective if, given an S-epimorphism $g: A \to B$ and an S-homomorphism $h: P \to B$, there exists an S-homomorphism $k: P \to A$ such that $gk = h$ ([4]). A monoid S is called <u>completely projective</u> if each right S-system is projective. Completely projective monoids have been investigated by Skornjakov [8], Kilp [3], and Isbell [2]. Luedeman [5] called a monoid S completely cyclic projective if each cyclic right S-system is projective. He proved that if S is a monoid with 0, then all centered cyclic unitary right S-systems are projective if and only if $S = \{1, 0\}$.

Generalizing the definition of a projective S-system, we call an S-system M quasi-projective if, for a given S-epimorphism $\mu: M \to A$ and an S-homomorphism $f: M \to A$, there exists an S-homomorphism $g: M \to M$ such that $\mu g = f$. Clearly, every projective S-system is quasi-projective. But the converse is false. The dual notion of quasi-injective S-systems has been investigated by Satyanarayana [7], Lopez and Luedeman [6], and Ahsan [1]. We call a monoid S completely quasi-projective if each right S-system is quasi-projective. In this paper we show that if S is such a monoid then $S = \{1, 0\}$. From this it follows that completely quasi-projective monoids need not be completely projective. In what follows, it is assumed that S is a monoid having a two-sided zero, and all S-systems are right, unitary and centered. Consequently, all S-homomorphisms map fixed points to fixed points. We now prove our results.

Lemma 1. Let $\phi: A_S \to B_S$ be an S-epimorphism. If $A_S \amalg B_S$ is quasi-projective, then B is a retract of A.

Proof. There exists a homomorphism $\psi: A_S \amalg B_S \longrightarrow A_S \amalg B_S$ such that the diagram:



where $\pi_1|_A = id_A$, $\pi_1(B) = \theta_A$,

$\pi_2(A) = \theta_B$, $\pi_2|_B = id_B$

is commutative. If $i: B_S \to A_S \amalg B_S$ is the canonical injection, then we get

$1_B = \pi_2 i = \phi\pi_1\psi i$. Hence B is a retract of A.

Remark 1. Replacing $A \amalg B$ by $A \oplus B$ and π_1, π_2, i respectively, by canonical projections and injection, it turns out that if there exists an epimorphism $A \to B$ and $A \oplus B$ is quasi-projective, then B is a retract of A.

Theorem 1. The following conditions are equivalent for a monoid S with 0:

(1) Each (finite) coproduct of quasi-projective S-systems is quasi-projective.

(2) Each quasi-projective S-system is projective.

Proof. (1) $\Rightarrow$ (2): Let M be a quasi-projective S-system. Then there exists a free S-system F and an epimorphism $\phi: F \to M$. Since F is projective, $F \amalg M$ is quasi-projective by the hypothesis. Hence M is a retract of F by the above lemma, and so M is projective.

(2) $\Rightarrow$ (1): Follows from the fact that the coproduct of each family of projective S-systems is projective ([4], Proposition 3.3).

Remark 2. If we assume that each (finite) direct sum of quasi-projective S-systems is quasi-projective, then as remarked above, it turns out that each quasi-projective S-system is projective.

Theorem 2. For a monoid S with zero, the following are equivalent:

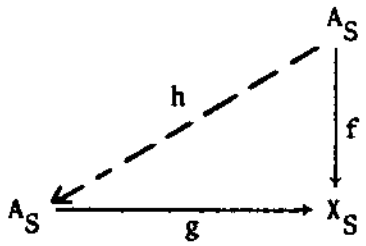
(1) S is completely quasi-projective.

(2) $S = \{1, 0\}$.

Proof. (1) $\Rightarrow$ (2): Let M be a cyclic S-system. Then there exists an epimorphism $\phi: S \to M$. On the other hand, $S \oplus M$ is quasi-projective as an

S-system, since S is completely quasi-projective. Hence by Remark 1, M is a retract of S, showing that M is projective. Thus S is completely cyclic projective. Hence by Luedeman ([4], Theorem on p. 51) S = {1, 0}.

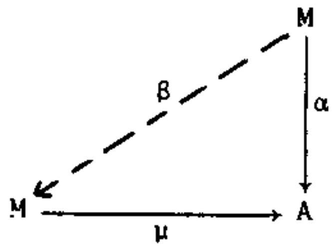
(2) ⇒ (1): Assume that S = {1, 0} and let A_S be an arbitrary unitary right centered S-system. Consider the diagram



in which g is an S-epimorphism and f an S-homomorphism. Let $a \in A$. Then $f(a) \in X_S$. Hence there exists an element $b \in A$ such that $g(b) = f(a)$. Define $h: A \to A$ by $h(a) = b$, where b is an arbitrary element of A such that $g(b) = f(a)$ and b is zero if $a = 0$. It is easy to see that h is well-defined and $gh = f$. Hence A_S is quasi-projective.

Lemma 2. Over a commutative monoid S, each cyclic S-system is quasi-projective.

Proof. Let M be a cyclic S-system generated by g. Consider the diagram:



in which μ is an S-epimorphism and α an S-homomorphism. Let $\alpha(g) = a \in A$. So, $\alpha(gs) = \alpha(g)s = as$, $s \in S$. Since μ is an epimorphism, there is $t \in M$ such that $\mu(t) = a$. Since g generates M, we can write $t = gx$ for some

$x \in S$. Define $\beta: M \to M$ by $\beta(gs) = ts (s \in S)$. If $gs = \theta$ then $gsx = \theta x = \theta$. Then since S is commutative $\theta = gsx = gxs = ts$. Furthermore, if $gs = gs_1$, then $ts = ts_1$. Thus β is independent of s. Easily, $\mu\beta = \alpha$. Hence M is quasi-projective.

Theorem 3. Let S be a commutative monoid with 0. Then the following are equivalent:

(1) S is completely quasi-projective.

(2) Each (finite) coproduct of quasi-projective S-systems is quasi-projective.

(3) $S = \{1, 0\}$.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let M be a cyclic S-system. Then M is quasi-projective by the above lemma. Hence, as argued in Theorem 1, M is projective. Consequently, S is completely cyclic projective, and hence it follows from Luedeman's result ([5], Theorem on p. 51) that $S = \{1, 0\}$.

(3) $\Rightarrow$ (1): Follows from Theorem 2.

In conclusion, we wish to express our indebtedness to Professor Peeter Normak for his comments and suggestions for the improvement of this paper.

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