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ABSTRACT: The Hankel Transform of order ν of a function if it exists, is defined by:

$$H_\nu\{f(x);\ \xi\} = \int_0^\infty x\, J_\nu(x\xi) f(x)\, dx$$

It arises in axisymmetric boundary value problems in Elasticity and Potential theory. This transform has been extended to the space of generalized functions by Zemanian and others. Sonine proved that for $f(x) = \dfrac{J_\nu(ax)J_\nu(bx)}{x^\nu}$ $H_\nu\{f(x)\xi\} = 0$ whenever, the triangular inequality $|a-b| < \xi < a+b$ is violated. We evaluate the Hankel Transform $H_\lambda\{f(x);\ c\}$ of the function $f(x) = \dfrac{J_\mu(ax)J_\nu(bx)}{x^\rho}$ and extend the result proved by Sonine.

Recently Asky, Koorwinder and Rahman have evaluated the integrals $\int_0^\infty x^{1+\nu} J_\nu(ax)J_\nu(bx)Y_\nu(cx)dx$ of the Hankel Transform type and proved that it vanishes when the triangular inequality $|a-b| < c < a+b$ is satisfied. We find the generalization of this result by evaluating the integral $\int_0^\infty x^{1-\rho} J_\mu(ax)J_\nu(bx)Y_\lambda(cx)dx$ and discuss some of its applications.

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Introduction: Integrals of products of Bessel functions have been a matter of curiosity for a long time. These integrals are not only of great interest to pure mathematicians, but they are of extreme importance in many branches of Mathematical Physics, elasticity and potential theory

We start out by considering Sonine's integral formula [14], namely

$$\int_0^\infty x^{1-\nu} J_\nu(ax)J_\nu(bx)J_\nu(cx)dx \tag{1.1}$$
$$= \begin{cases} \dfrac{2^{\nu-1}\Delta^{\nu-\frac{1}{2}}}{\sqrt{\pi}(abc)^\nu\Gamma(\frac{1}{2}+\nu)}, & \text{if } a,b,c \text{ form sides of a triangle of area } \sqrt{\Delta}, \\ 0, & \text{if } a,b,c \text{ do not form sides of a triangle.} \end{cases}$$

Recently Askey, Koornwinder and Rahman computed the following integral [3]:

$$\int_0^\infty x^{1+\nu} J_\nu(ax)J_\nu(bx)Y_\nu(cx)dx \tag{1.2}$$
$$= \begin{cases} 0, & \text{if } |a-b| < c < a+b, \\ -\dfrac{2^{-\nu-1}(-\Delta)^{-\nu-\frac{1}{2}}}{\sqrt{\pi}(abc)^{-\nu}\Gamma(\frac{1}{2}-\nu)}, & \text{if } c < |a-b|, \\ \dfrac{2^{-\nu-1}(-\Delta)^{-\nu-\frac{1}{2}}}{\sqrt{\pi}(abc)^{-\nu}\Gamma(\frac{1}{2}-\nu)}, & \text{if } c > a+b. \end{cases}$$

The formulas (1.1) and (1.2) are valid for $\operatorname{Re}\nu > -\frac{1}{2}$ and $-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$ respectively. The value of Δ in (1.1) and (1.2) is given by the same algebraic expression, namely

$$(1.3) \qquad \Delta = \frac{1}{16}[(a+b)^2 - c^2][c^2 - (a-b)^2] .$$

However, in (1.1) Δ is positive and represents the square of the area of the triangle with sides a,b,c whereas in (1.2) it is negative because a,b,c do not satisfy the triangle inequality.

Let us now consider the integral:

$$(1.4) \qquad I(a,b,c) = \int_0^\infty J_\mu(ax)J_\nu(bx)J_\lambda(cx)dx .$$

This integral is convergent for $\mathrm{Re}(\mu+\nu+\lambda) > 1$ and for arbitrary values of a,b and c, has for $c > a + b$ been represented by Bailey as a product of two hypergeometric series. Bailey's result, along with some special cases, is listed in [5]. Henrici [13] evaluated the integral in (1.4) when a,b and c satisfy the triangle inequality.

The purpose of this paper is to seek generalizations the results in (1.1), (1.2) and (1.4). In section 2 we compute the following two integrals:

$$(1.5) \quad L(a,b,c) = \int_0^\infty x^{1-\rho}J_\mu(ax)J_\nu(bx)J_\lambda(cx)dx \;;\; \mathrm{Re}(\mu+\nu+\lambda-\rho) > -2 ,$$

and

$$(1.6) \quad M(a,b,c) = \int_0^\infty x^{1-\rho}J_\mu(ax)J_\nu(bx)Y_\lambda(cx)dx \;;\; \mathrm{Re}(\mu+\nu-\lambda-\rho) > -2 .$$

In section 3 we give some special cases of (1.5) and (1.6). As an application to our results, we show in section 4 the following two important relations.

$$\int_0^\infty t^{1-\mu} J_\mu(t \sin\phi \sin\psi) J_\nu(t \cos\phi \cos\psi) J_\nu(t \cos\theta)\,dt$$

(1.7)
$$= \frac{2^{\nu+2}}{\Gamma(\mu+1)} (\sin\phi \sin\psi \sin^2\theta)^\mu (\cos\phi \cos\psi \cos\theta)^\nu$$

$$\cdot \sum_{n=0}^{\infty} h_n^{(\mu,\nu)} P_n^{(\mu,\nu)}(\cos 2\phi) P_n^{(\mu,\nu)}(\cos 2\psi) P_n^{(\mu,\nu)}(\cos 2\theta) \frac{n!}{(\mu+1)_n},$$

$$\mathrm{Re}\,\mu > -\frac{1}{2},\ \mathrm{Re}\,\nu > -1,\ 0 < \theta, \phi, \psi < \frac{\pi}{2},$$

$$\int_0^\infty t^{1+\mu} J_\mu(t \cos\phi \cos\psi) J_\nu(t \sin\phi \sin\psi) Y_\nu(t \cos\theta)\,dt$$

(1.8)
$$= 2^{2\mu+2\nu+3} \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)} (\cos\phi \cos\psi)^\mu (\sin\phi \sin\psi \cos\theta)^\nu$$

$$\cdot \sum_{n=0}^{\infty} h_n^{(\mu,\nu)} P_n^{(\mu,\nu)}(\cos 2\phi) P_n^{(\mu,\nu)}(\cos 2\psi) Q_n^{(\mu,\nu)}(\cos 2\theta) \frac{(\mu+\nu+1)_n}{(\mu+1)_n},$$

$$\mathrm{Re}\,\mu > -\frac{1}{2},\ -1 < \mathrm{Re}\,\nu < \frac{1}{2},\ 0 < \phi, \psi < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2},$$

where

(1.9)
$$h_n^{(\mu,\nu)} = \left[\int_{-1}^{1} (1-x)^\mu (1+x)^\nu [P_n^{(\mu,\nu)}(x)]^2 dx \right]^{-1}, \text{ and}$$

$P_n^{(\mu,\nu)}(x)$ and $Q_n^{(\mu,\nu)}(x)$ are Jacobi polynomial and Jacobi functions of the second kind, respectively.

Without loss of generality, we may assume that $c > b > a$ since this can always be achieved by permutation of indices. We first consider the following function:

(2.1)
$$g(z) = \int_0^\infty t^{1-\rho} J_\mu(at) J_\nu(bt) K_\lambda(zt)\,dt$$

where a,b are fixed and z is a arbitrary complex number. To ensure convergence we must now assume temporarily that $\mathrm{Re}(\mu+\nu\pm\lambda-\rho) > -2$. For convenience, we substitute

(2.2) $$\alpha = a^2 \ , \ \beta = b^2 \ , \ \gamma = c^2 \ , \ \xi = z^2 \ .$$

Since,

$$J_\mu(at)J_\nu(bt) = \frac{2^{-\mu-\nu}(at)^\mu(bt)^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\beta-\alpha)^n t^{2n}}{4^n(\mu+1)_n(\nu+1)_n} P_n^{(\mu,\nu)}\left(\frac{\beta+\alpha}{\beta-\alpha}\right)$$

[9,p.11.eq.(4)]

and

(2.4) $$\int_0^\infty K_\lambda(zt)t^{k-1}dt = \frac{1}{4}\left(\frac{z}{2}\right)^{-k}\Gamma\left(\frac{k+\lambda}{2}\right)\Gamma\left(\frac{k-\lambda}{2}\right) \ , \ (\mathrm{Re}(k-\lambda) > 0)$$

Therefore, by substituting these expressions in (2.1) and by using [7,p.255] we get:

(2.5) $$g(z) = \frac{\Gamma\left(\frac{\mu+\nu+\lambda-\rho+2}{2}\right)\Gamma\left(\frac{\mu+\nu-\rho-\lambda+2}{2}\right)}{2^\rho\Gamma(\mu+1)\Gamma(\nu+1)}\left(\frac{\alpha^\mu\beta^\nu}{\xi^{\mu+\nu-\rho+2}}\right)^{\frac{1}{2}}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-\rho+2}{2}\right)_n\left(\frac{\mu+\nu-\rho-\lambda+2}{2}\right)_n}{(\mu+1)_n(\nu+1)_n}\left(\frac{\beta-\alpha}{\xi}\right)^n P_n^{(\mu,\nu)}\left(\frac{\beta+\alpha}{\beta-\alpha}\right) \ .$$

We also have:

(2.6) $$P_n^{(\mu,\nu)}(y) = \frac{(1+\nu)_n}{n!}\left(\frac{y-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n \ , \ -n-\mu \\ \nu+1\end{matrix} \ ; \ \frac{y+1}{y-1}\right]$$ [17,19]

we obtain

(2.7) $$g(z) = C(\alpha,\beta,\xi)F_4\left(\frac{\mu+\nu+\lambda-\rho+2}{2}, \frac{\mu+\nu-\lambda-\rho+2}{2} \ ; \ \mu+1 \ , \ \nu+1 \ ; \ \frac{\alpha}{\xi} \ , \ \frac{\beta}{\xi}\right)$$

where

(2.8) $$C(\alpha,\beta,\xi) = \frac{\Gamma\left(\frac{\mu+\nu+\lambda-\rho+2}{2}\right)\Gamma\left(\frac{\mu+\nu-\rho-\lambda+2}{2}\right)}{2^\rho\Gamma(\mu+1)\Gamma(\nu+1)}\left(\frac{\alpha^\mu\beta^\nu}{\xi^{\mu+\nu-\rho+2}}\right)^{\frac{1}{2}},$$

and F_4 is an Appell function [4, 18] defined by

(2.9) $$F_4(\alpha_1,\alpha_2\ ;\ \beta_1,\beta_2\ ;\ u,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{m+n}(\alpha_2)_{m+n}}{m!n!(\beta_1)_m(\beta_2)_n}u^m v^n .$$

By using the relations

$$J_\lambda(tc) = \frac{i}{\pi}\lim_{\varepsilon\to 0}\{e^{i\lambda\pi/2}K_\lambda(itc+i\varepsilon) - e^{-i\lambda\pi/2}K_\lambda(-itc-i\varepsilon)\}$$

$$Y_\lambda(tc) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\{e^{i\lambda\pi/2}K_\lambda(itc+i\varepsilon) + e^{-i\lambda\pi/2}K_\lambda(-itc-i\varepsilon)\}$$

[1],[7],[15] & [16].

we obtain

(2.10) $$L(a,b,c) = \frac{iC(\alpha,\beta,\gamma)}{\pi}\left[e^{\frac{i\pi}{2}(\mu+\nu-\lambda-\rho+2)}A(\alpha,\beta,\gamma+io) - e^{-\frac{i\pi}{2}(\mu+\nu-\lambda-\rho+2)}A(\alpha,\beta,\gamma-io)\right] ,$$

and

(2.11) $$M(a,b,c) = \frac{C(\alpha,\beta,\gamma)}{\pi}\left[e^{\frac{i\pi}{2}(\mu+\nu-\lambda-\rho+2)}A(\alpha,\beta,\gamma+io) + e^{-\frac{i\pi}{2}(\mu+\nu-\lambda-\rho+2)}A(\alpha,\beta,\gamma-io)\right] ,$$

where, for abbreviation, we write

(2.12) $$A(\alpha,\beta,\gamma) = F_4\left(\frac{\mu+\nu+\lambda-\rho+2}{2},\frac{\mu+\nu-\lambda-\rho+2}{2}\ ;\ \mu+1\ ,\ \nu+1\ ;\ \frac{\alpha}{\gamma}\ ,\ \frac{\beta}{\gamma}\right) .$$

The Appell function on the right of (2.11) is absolutely convergent if

$$\left|\frac{\alpha}{\gamma}\right|^{\frac{1}{2}} + \left|\frac{\beta}{\gamma}\right|^{\frac{1}{2}} < 1 .$$

In terms of a,b,c this condition means

(2.13) $$a + b < c .$$

But, if this inequality is satisfied then $A(\alpha,\beta,\gamma)$ in (2.12) as a function γ is single-valued on the real axis and so

(2.14) $L(a,b,c) = \frac{2}{\pi}C(\alpha,\beta,\gamma)A(\alpha,\beta,\gamma)\sin(\mu+\nu-\lambda-\rho+2)\frac{\pi}{2}$, if $c > a + b$,

(2.15) $M(a,b,c) = \frac{2}{\pi}C(\alpha,\beta,\gamma)A(\alpha,\beta,\gamma)\cos(\mu+\nu-\lambda-\rho+2)\frac{\pi}{2}$, if $c > a + b$.

To consider $L(a,b,c)$ and $M(a,b,c)$ in the triangular region namely,

(2.16) $$|a - b| < c < a + b$$

we substitute

$$\frac{\alpha}{\gamma} = x(1 - y) \ , \ \frac{\beta}{\gamma} = y(1 - x) \ .$$

Solving for x and y we obtain

(2.17) $$x = \frac{\gamma + \beta - \alpha - 4\sqrt{-\Delta}}{2\gamma} \ , \ y = \frac{\gamma + \alpha - \beta - 4\sqrt{-\Delta}}{2\gamma} \ .$$

where Δ is the same as defined in (1.3). Here signs of the square root are choosen such that the roots are asymptotically equal to $+\gamma$ for large values of γ , so that

$$x = -\frac{\alpha}{\gamma} + O(\gamma^{-2}) \ , \ y = -\frac{\beta}{\gamma} + O(\gamma^{-2}) \ .$$

As we have mentioned earlier Δ is negative for $c > a + b$ whereas it is positive when triangular inequality (2.16) holds. Therefore, when the inequality (2.16) is satisfied, we can write

(2.18) $$x = \frac{\gamma + \beta - \alpha + 4i\sqrt{\Delta}}{2\gamma} \ , \ y = \frac{\gamma + \alpha - \beta + 4i\sqrt{\Delta}}{2\gamma}$$

in the upper half-plane and

(2.19) $$x = \frac{\gamma + \beta - \alpha - 4i\sqrt{\Delta}}{2\gamma} \ , \ y = \frac{\gamma + \alpha - \beta - 4i\sqrt{\Delta}}{2\gamma}$$

in the lower half-plane.

Defining two positive angles ϕ and ψ by

(2.20) $$\phi = \tan^{-1} \frac{4\sqrt{\Delta}}{\gamma + \beta - \alpha}, \ \psi = \tan^{-1} \frac{4\sqrt{\Delta}}{\gamma + \alpha - \beta}$$

we thus have

(2.21) $$x = \sqrt{\frac{\alpha}{\beta}}\, e^{+i\phi}, \ y = \sqrt{\frac{\beta}{\gamma}}\, e^{+i\psi}$$

on the upper half-plane.

By the lower assumption $c > b > a$ we have $|x| < 1$ and $|y| < 1$.

Using the Burchnall and Chaundy's expansion formula:

$$F_4[a,b\ ;\ c,c'\ ;\ x(1-y),y(1-x)]$$
$$= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(1+a+b-c-c')_n}{n!(c)_n(c')_n} x^n y^n$$
$$\cdot\ {}_2F_1\begin{bmatrix} a+n\ ,\ b+n \\ c+n \end{bmatrix};x\Bigg]\ {}_2F_1\begin{bmatrix} a+n\ ,\ b+n \\ c+n \end{bmatrix};y\Bigg] \qquad [18,\ p.223]$$

we obtain

(2.23) $$L(a,b,c) = \begin{cases} \frac{2}{\pi}C(\alpha,\beta,\gamma)\sin(\mu+\nu-\lambda-\rho+2)\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{\mu+\nu+\lambda-\rho+2}{2})_n(\frac{\mu+\nu-\lambda-\rho+2}{2})_n(1-\rho)_n}{n!(\mu+1)_n(\nu+1)_n} x^n y^n \\ \qquad \cdot\ F_1(n;x)F_2(n;y)\ ,\ \text{if}\ \ c > a+b\ , \\ \frac{2}{\pi}C(\alpha,\beta,\gamma)\,\mathrm{im}\{e^{i\frac{\pi}{2}(\mu+\nu-\lambda-\rho+2)} \sum_{n=0}^{\infty} \frac{(\frac{\mu+\nu+\lambda-\rho+2}{2})_n(\frac{\mu+\nu-\lambda-\rho+2}{2})_n(1-\rho)_n}{n!(\mu+1)_n(\nu+1)_n} x^n y^n \\ \qquad F_1(n;x)F_2(n;y)\}\ ,\ \text{if}\ \ |a-b| < c < a+b\ , \end{cases}$$

and

$$(2.24)\quad M(a,b,c)=\begin{cases}\frac{2}{\pi}C(\alpha,\beta,\gamma)\cos(\mu+\nu-\lambda-\rho+2)\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{\mu+\nu+\lambda-\rho+2}{2})_n(\frac{\mu+\nu-\lambda-\rho+2}{2})_n(1-\rho)_n}{n!(\mu+1)_n(\nu+1)_n}x^ny^n \\ \qquad\cdot F_1(n;x)F_2(n;y)\ ,\ \text{if}\quad c > a+b\ , \\ \frac{2}{\pi}C(\alpha,\beta,\gamma)\mathrm{Re}\{e^{i\frac{\pi}{2}(\mu+\nu-\lambda-\rho+2)}\sum_{n=0}^{\infty}\frac{(\frac{\mu+\nu+\lambda-\rho+2}{2})_n(\frac{\mu+\nu-\lambda-\rho+2}{2})_n(1-\rho)_n}{n!(\mu+1)_n(\nu+1)_n}x^ny^n \\ \qquad\cdot F_1(n;x)F_2(n;y)\}\ ,\ \text{if}\quad |a-b| < c < a+b\ ,\end{cases}$$

where

$$(2.25)\qquad F_1(n;x) = {}_2F_1\left[\begin{matrix}\frac{\mu+\nu+\lambda-\rho+2}{2}+n\ ,\ \frac{\mu+\nu-\lambda-\rho+2}{2}+n \\ \mu+1+n\end{matrix}\ ,\ x\right]$$

and

$$(2.26)\qquad F_2(n;y) = {}_2F_1\left[\begin{matrix}\frac{\mu+\nu+\lambda-\rho+2}{2}+n\ ,\ \frac{\mu+\nu-\lambda-\rho+2}{2}+n \\ \nu+1+n\end{matrix}\ ,\ x\right] .$$

3. Special Cases

I. The substitution $\rho = 1$ in (2.23) immediately lead to Bailey-Henrici formula [13, p. 154].

II. If we let $\lambda = \rho = \mu = \nu$ in (2.23) we obtain the extension of the Sonine formulae:

$$\int_0^{\infty} t^{1-\nu} J_\nu(at)J_\nu(bt)J_\nu(ct)dt$$

$$(3.1)\quad = \begin{cases}0 & \text{if}\quad c > a+b\ , \\ \frac{\Gamma(\nu+1)}{2^{\nu-1}\pi c^2}(\frac{ab}{c})^\nu \mathrm{Im}\left\{\sum_{n=0}^{\infty}\frac{(1)_n(1-\nu)_n}{n!(1+\nu)_n}x^ny^n \cdot {}_1F_0\left[\begin{matrix}n+1 \\ -\end{matrix};x\right]{}_1F_0\left[\begin{matrix}n+1 \\ -\end{matrix};y\right]\right\} \\ \qquad \text{if}\quad |a-b| < c < a+b\ .\end{cases}$$

$$= \begin{cases} 0 & \text{if } \quad c > a + b\,, \\ \dfrac{\Gamma(\nu+1)}{2^{\nu-1}\pi c^2}\left(\dfrac{ab}{c}\right)^{\nu} \operatorname{Im}\left\{(1-x)^{-1}(1-y)^{-1}\,{}_2F_1\left[\begin{matrix} 1\,,\ 1-\nu \\ 1+\nu \end{matrix}\,;\ \dfrac{xy}{(1-x)(1-y)}\right]\right\}, \\ & \text{if } \quad |a-b| < c < a+b\,. \end{cases}$$

To find the imaginary part of the quantity inside the parenthesis of the last equation we use (2.19) to get in the upper half-plane $(1 - x)^{-1}(1 - y)^{-1} = \frac{c^2}{ab}e^{i(\psi+\phi)}$, $\frac{xy}{(1 - x)(1 - y)} = e^{2i(\phi+\psi)}$ and so, in the upper half-plane [4, 18]

$$(1-x)^{-1}(1-y)^{-1}\,{}_2F_1\left[\begin{matrix} 1\,,\ 1-\nu \\ 1+\nu \end{matrix}\,;\ \frac{xy}{(1-x)(1-y)}\right]$$
$$= \frac{\nu c^2 e^{i(\phi+\psi)}}{ab}\int_0^1 [(1-u)(1-u\,e^{2i(\phi+\psi)})]^{\nu-1}du$$
$$= \frac{\nu c^2 e^{i(\phi+\psi)}}{ab}\int_0^{e^{i(\phi+\psi)}} [(1-e^{-i(\phi-\psi)}u)(1-e^{i(\phi+\psi)}u)]^{\nu-1}du\,.$$

The contribution from the lower half-plane is complex conjugate of this. Therefore

$$\operatorname{Im}\ (1-x)^{-1}(1-y)^{-1}\,{}_2F_1\left[\begin{matrix} 1\,,\ 1-\nu \\ 1+\nu \end{matrix}\,;\ \frac{xy}{(1-x)(1-y)}\right]$$
$$= \frac{\nu e^2}{2iab}\int_{0\,e^{-i(\psi+\psi)}}^{e^{i(\phi+\psi)}} [(1-e^{-i(\phi+\psi)}u)(1-e^{i(\phi+\psi)}u)]^{\nu-1}du$$
$$= \frac{\nu e^2}{2iab}\, i[2\sin(\phi+\psi)]^{2\nu-1}\,\frac{\Gamma(\nu)\Gamma(\nu)}{\Gamma(2\nu)}\,.$$

Finally using

(3.2) $$\sin(\phi + \psi) = \frac{2\sqrt{\Delta}}{ab}\,, \text{ (computing from (2.19), (2.21))}$$

and $$\Gamma(\tfrac{1}{2})\Gamma(2\nu) = 2^{2\nu-1}\Gamma(\nu)\Gamma(\nu + \tfrac{1}{2})$$

we obtain the Sonine's formula (1.1) for $|a - b| < c < a + b$.

III. If we put $-\rho = \lambda = \mu = \nu$ in (2.24) we obtain

$$\int_0^\infty x^{1+\nu} J_\nu(ax) J_\nu(bx) Y_\nu(cx)dx$$

$$= \begin{cases} \frac{2^{\nu+1}}{\pi} \frac{\Gamma(2\nu + 1)}{\Gamma(\nu + 1)} \frac{(ab)^\nu}{c^{3\nu+1}} \cos(\nu + 1)\pi \ (1-x-y)^{-(1+2\nu)} , & \text{if} \quad c > a + b \\ \frac{2^{\nu+1}}{\pi} \frac{\Gamma(2\nu + 1)}{\Gamma(\nu + 1)} \frac{(ab)^\nu}{c^{3\nu+1}} \mathrm{Re}\{e^{i(\nu+1)\pi}(1-x-y)^{-(1+2\nu)}\} & \text{if} \quad |a - b| < c < a + b \end{cases}$$

Using (2.17), (2.19) and the relations:

(3.3) $$\sqrt{\pi}\, \Gamma(2\nu + 1) = 2^{2\nu}\Gamma(\nu + \tfrac{1}{2})\Gamma(\nu + 1) ,$$

(3.4) $$\pi(\tfrac{1}{2} + \nu)\Gamma(\tfrac{1}{2} - \nu) = \frac{x}{\cos x\nu} ,$$

(3.5) and $$\mathrm{Re} \quad e^{i(\nu+1)\pi} \left(\frac{4i\sqrt{\bar{\Delta}}}{c^2}\right)^{-2\nu-1} = \mathrm{Re} \quad i\left(\frac{4i\sqrt{\bar{\Delta}}}{c^2}\right)^{-2\nu-1} = 0$$

we get Askey, Koornwinder and Rahman's formula (1.2) for $c > a + b$ and $|a - b| < c < a + b$.

IV. If we substitute $\rho = 0$ and $\lambda = \mu - \nu$ in (2.22) and (2.24) then after some simplification we get:

$$\int_0^\infty x\, J_\mu(ax) J_\nu(bx) J_{\mu-\nu}(cx)dx$$

(3.6) $$= \begin{cases} \frac{8\sqrt{-\Delta}}{\pi\, c^{\mu+\nu+4}} \left(\frac{a}{1 - y}\right)^\mu \left(\frac{b}{1 - x}\right)^\nu \sin(1+\nu)\pi , & \text{if} \quad c > a + b , \\ \frac{8\sqrt{\Delta}}{\pi\, c^4} \cos(\phi\mu + \psi\nu + \pi\nu) , & \text{if} \quad |a - b| < c < a + b , \end{cases}$$

and

$$\int_0^\infty x\, J_\mu(ax) J_\nu(bx) Y_{\mu-\nu}(cx)dx$$

(3.7)
$$= \begin{cases} \dfrac{8\sqrt{-\Delta}}{\pi\, c^{\mu+\nu+4}}\left(\dfrac{a}{1-y}\right)^\mu \left(\dfrac{b}{1-x}\right)^\nu \cos(1+\nu)\pi \ , & \text{if } \ c > a+b \ , \\ \dfrac{8\sqrt{\Delta}}{\pi\, c^4} \sin(\phi\mu + \psi\nu + \pi\nu) \ , & \text{if } \ |a-b| < c < a+b \ . \end{cases}$$

4. An application: Let us set $\rho = \mu$, $\lambda = \nu$ and $a = \sin\phi \sin\psi$, $b = \cos\phi \cos\psi$, $c = \cos\theta$ $(0 < \phi \ , \psi \ , \theta < \frac{\pi}{2})$ in (2.10). Then, after some algebraic manipulation, we get:

$$\int_0^\infty t^{1-\mu} J_\mu(t \sin\phi \sin\psi) J_\nu(t \cos\phi \cos\psi) J_\nu(t \cos\theta)dt$$

(4.1)
$$= \frac{2^{\nu+2}}{\Gamma(\mu+1)} (\sin\phi \sin\psi \sin^2\theta)^\mu (\cos\phi \cos\psi \cos\theta)^\nu$$

$$\cdot \sum_{n=0}^\infty h_n^{(\mu,\nu)} P_n^{(\mu,\nu)}(\cos 2\phi) P_n^{(\mu,\nu)}(\cos 2\psi) P_n^{(\mu,\nu)}(\cos 2\theta) \frac{n!}{(\mu+1)_n} \ ,$$

$$\operatorname{Re} \mu > -\frac{1}{2} \ , \quad \operatorname{Re} \nu > -1,$$

which is an important relation proved by Gasper in [12, p.116].

Similary if we set $\rho = -\mu$, $\lambda = \nu$ and $a = \cos\phi \sin\psi$, $b = \sin\phi \sin\psi$, $c = \cos\theta$ $(0 < \phi \ , \psi \ , \theta < \frac{\pi}{2})$ in (2.11), we get:

$$\int_0^\infty t^{1+\mu} J_\mu(t \cos\phi \cos\psi) J_\nu(t \sin\phi \sin\psi) Y_\nu(t \cos\theta)dt$$

(4.2)
$$= 2^{2\mu+2\nu+3} \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)} (\cos\phi \cos\psi)^\mu (\sin\phi \sin\psi \cos\theta)^\mu$$

$$\cdot \sum_{n=0}^\infty h_n^{(\mu,\nu)} P_n^{(\mu,\nu)}(\cos 2\phi) P_n^{(\mu,\nu)}(\cos 2\psi) Q_n^{(\mu,\nu)}(\cos 2\theta) \frac{(\mu+\nu+1)_n}{(\mu+1)_n} \ ,$$

$$\operatorname{Re} \mu > -\frac{1}{2} \ , \ -1 < \operatorname{Re} \nu < \frac{1}{2} \ .,$$

which is another important relation proved earlier in [12], [15] and [16].

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