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A.S. Ibrahim and S.A. Khatab

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

NORMAL SEQUENCES OF REFINED FUZZY SUBGROUPS

A.S. Ibrahim
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran, Saudi Arabia

and

S.A. Khatab
Department of Mathematics
Faculty of Sciences, Zagazig University
Zagazig, Egypt

Abstract:

In this paper we introduce the notions of refined fuzzy subgroups by another fuzzy subgroups, normal fuzzy subgroups (N-Fuzzy subgroups) and quasi-normal fuzzy subgroups of length 2 (QN_2-Fuzzy subgroups). We also introduce two sequences: normal sequences of refined fuzzy subgroups by some N-Fuzzy subgroup and sequences of QN_2-Fuzzy subgroups. We then use these notions together with other relevant definitions to obtain a result which gives a fuzzy-analog to Schreier's refinement theorem for normal series of groups.

1. Preliminaries:

Following the work of Zadeh [5], a mapping μ from a set S to the unit interval $[0,1]$ is called a fuzzy subset of S. Rosenfeld in [4] defined a fuzzy subset μ of a group G as a fuzzy subgroup of G if for all $x,y \in G$ the following two conditions are satisfied:

i) $\mu(xy) \geq \min(\mu(x),\mu(y))$,

ii) $\mu(x) = \mu(x^{-1})$.

Later, Das [3] proved that if e is the identity of a group G,

$t \in [0,1]$ and μ is a fuzzy subgroup of G with $\mu(e) \geq t$, then Zadeh's level subset μ_t of μ given by $\mu_t = \{x \in G | \mu(x) \geq t\}$ is a subgroup which Das called a level subgroup of μ. He also proved that the level subgroups of μ form a chain. In [1], Bhattacharya proved that for any finite chain $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_{n-1} \subseteq G_n = G$ of subgroups of G there exists a fuzzy subgroup μ of G whose level subgroups are precisely the members of the given chain. Then, Mukherjee and Bhattacharya [2] defined a fuzzy subgroup μ of G to be fuzzy normal if $\mu(xy) = \mu(yx)$ for all $x,y \in G$.

In the present work interest is concentrated on fuzzy subgroups with finite images, and hence a finite number of level subgroups.

So, as an abbreviation, the following notation:

$$\mu\colon G_{t_0} \subseteq G_{t_1} \subseteq \cdots \subseteq G_{t_{n-1}} \subseteq G_{t_n} = G,$$

will be used to mean that μ is a fuzzy subgroup of G with finite image: $Im(\mu) = \{t_0, t_1, \ldots, t_n\}$ where each $t_i \in [0,1]$ and $t_0 > t_1 > \ldots > t_n$ and whose level subgroups are $G_{t_0}, G_{t_1}, \ldots, G_{t_n} = G$.

Let $H \lhd G$ denote as usual that H is a normal subgroup of G. Then theorems 3.6 and 3.9 in [2] can be combined and restated in one theorem as follows:

<u>Theorem 1.1</u>. The fuzzy subgroup

$$\mu\colon \ G_{t_0} \subseteq G_{t_1} \subseteq \cdots \subseteq G_{t_n} = G$$

is fuzzy normal subgroup of G if and only if $G_i \lhd G$ for all $0 \leq i \leq n$.

2. Refined Fuzzy Subgroups:

<u>Definition 2.1</u>. Let $\mu: G_{t_0} \subseteq G_{t_1} \subseteq \cdots \subseteq G_{t_k} = G$ and $\nu: G_{\ell_0} \subseteq G_{\ell_1} \subseteq \cdots \subseteq G_{\ell_k} = G$ be two fuzzy subgroups of a group G. μ is said to be <u>refined</u> by ν if:

i) $\mathrm{Im}(\mu) \subseteq \mathrm{Im}(\nu)$, i.e., $\{t_0, t_1, \ldots, t_k\} \subseteq \{\ell_0, \ell_1, \ldots, \ell_n\}$ and

ii) every level subgroup of μ is also a level subgroup of ν, i.e., the chain of subgroups associated with ν is a refinement of the chain of subgroups associated with μ.

<u>Example 2.1</u>. Let Z be the additive group of integers, and consider the fuzzy subgroups μ and ν of Z given by:

$$\mu: 16Z \subseteq 4Z \subseteq Z$$

$$\nu: 32Z \subseteq 16Z \subseteq 8Z \subseteq 4Z \subseteq 2Z \subseteq Z$$

where $\mu(16Z) = t_1$, $\mu(4Z - 16Z) = t_3$, $\mu = (Z - 4Z) = t_4$

and $\nu(32Z) = t_0$, $\nu(16Z - 32Z) = t_1$, $\nu(8Z - 16Z) = t_2$, $\nu(4Z - 8Z) = t_3$, $\nu(2Z - 4Z) = t_4$, $\nu(Z - 2Z) = t_5$,

with $t_i \in [0,1]$, $0 \leq i \leq 5$ and $t_0 > t_1 > t_2 > t_3 > t_4 > t_5$.

Following definition 2.1, it is clear that μ is refined by ν.

<u>Lemma 2.1</u>. If the fuzzy subgroup μ is refined by the fuzzy subgroup ν of a group G, then $\mu \leq \nu$.

<u>Proof</u>. Let x be any element of G and let μ_t and ν_k be the smallest level subgroups of μ and ν respectively such that $x \in \mu_t \cap \nu_k$ and

$t \in Im(\mu)$, $k \in Im(\nu)$. Since μ is refined by ν, then either $\mu_t = \nu_k$ or $\nu_k \subseteq \mu_t$. Thus in either case $\mu(x) \leq \nu(x)$, i.e., $\mu \leq \nu$. Which completes the proof.

The converse of Lemma 2.1 is not always true. The following is a counter example.

Example 2.2. Consider the following fuzzy subgroups of Z:

$$\mu: 4Z \subseteq 2Z \subseteq Z \quad \text{and} \quad \nu: 9Z \subseteq 3Z \subseteq Z$$

where $\nu(4Z) = \frac{3}{4}$, $\nu(4Z - 2Z) = \frac{1}{2}$, $\nu(Z - 2Z) = \frac{1}{3}$

and $\mu(9Z) = \frac{1}{2}$, $\mu(3Z - 9Z) = \frac{1}{3}$, $\mu(Z - 3Z) = \frac{1}{9}$

It is clear that $\mu \leq \nu$, and neither μ nor ν is refined by the other.

3. Normal Sequences of Refined N-Fuzzy Subgroups:

Definition 3.1. A fuzzy subgroup μ of a group G is called normal-fuzzy subgroup (N-Fuzzy subgroup) if the level subgroups of μ form a normal chain of G, i.e.,

$$\mu: \langle e \rangle = G_{t_0} \triangleleft G_{t_1} \triangleleft G_{t_2} \triangleleft \dots \triangleleft G_{t_n} = G$$

Lemma 3.1. If $\mu: G_{t_0} \subseteq G_{t_1} \subseteq G_{t_2} \subseteq \dots \subseteq G_{t_n} = G$ is fuzzy normal subgroup of G, then μ is N-Fuzzy subgroup of G.

Proof. The proof follows immediately from theorem 1.1.

The converse of lemma 3.1 is not always true, i.e., N-Fuzzy subgroups are not necessarily fuzzy normal subgroups. The following is a counter example.

Example 3.1. Let μ be a fuzzy subgroup of the dihedral group $G = D_4 = \langle a,b: a^4 = b^2 = (ab)^2 = e\rangle$ which is given by

$$\mu: G_{t_0} \lhd G_{t_1} \lhd G_{t_2} \lhd G_{t_3} = D_4$$

where $G_{t_0} = \langle e\rangle$, $G_{t_1} = \langle a^3b\rangle$, $G_{t_2} = \{e, a^2, ab, a^3b\}$.

Clearly μ is N-Fuzzy subgroup of D_4, which is not fuzzy normal subgroup since G_{t_1} is not a normal subgroup of D_4.

Now, given the N-Fuzzy subgroup μ of a group G:

$$\mu: \langle e\rangle = G_{t_0} \lhd G_{t_1} \lhd \; . \; . \; . \lhd G_{t_{n-1}} \lhd G_{t_n} = G$$

we are going to construct a sequence $\{\mu_i\}_{1 \le i \le n}$ of fuzzy subgroups of G for which each member is refined by μ in a special way as follows:

$$\begin{array}{ll}
\mu = \mu_n : & G_{t_0} \lhd G_{t_1} \lhd G_{t_2} \lhd \cdots\cdots \lhd G_{t_{n-1}} \lhd G_{t_n} = G \\
\mu_{n-1} : & G_{t_1} \lhd G_{t_2} \lhd \cdots\cdots \lhd G_{t_{n-1}} \lhd G_{t_n} = G \\
\vdots & \vdots \\
\mu_i : & G_{t_{n-i}} \lhd G_{t_{n-i+1}} \lhd \cdots \lhd G_{t_{n-1}} \lhd G_{t_n} = G \\
\vdots & \vdots \\
\mu_1 : & G_{t_{n-1}} \lhd G_{t_n} = G
\end{array}$$

Definition 3.2. The sequence $\{\mu_i\}_{1\leq i\leq n}$ constructed above is called the normal sequence of refined fuzzy subgroups by the N-Fuzzy subgroup μ of the group G.

As an immediate result of lemma 2.1, we have

Lemma 3.2. If $\{\mu_i\}_{1\leq i\leq n}$ is a normal sequence of refined fuzzy subgroups by some N-Fuzzy subgroup μ of G, then $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_n$.

Example 3.2. The members of the normal sequence $\{\mu_i\}_{1\leq i\leq 3}$ of the N-Fuzzy subgroup μ of the dihedral group D_4 given in example 3.1 are:

$$\mu_3\colon \langle e\rangle = G_{t_0} \lhd \langle a^3b\rangle = G_{t_1} \lhd \{e, a^2, ab, a^3b\} = G_{t_2} \lhd G_{t_3} = D_4$$

$$\mu_2\colon \langle a^3b\rangle = G_{t_1} \lhd \{e, a^2, ab, a^3b\} = G_{t_2} \lhd G_{t_3} = D_4$$

$$\mu_1\colon \{e, a^2, ab, a^3b\} = G_{t_2} \lhd G_{t_3} = D_4$$

4. Sequences of QN_2-Fuzzy Subgroups:

Definition 4.1. A fuzzy subgroup μ of a group G given by:

$$\mu\colon G_{t_0} \lhd G_{t_1} \subseteq G_{t_2} = G$$

is called a quasi-normal fuzzy subgroup of G of length 2 (QN_2-Fuzzy subgroup).

Notice that G_{t_1} is not necessarily normal in G_{t_2} or not necessarily distinct from G_{t_2}.

Since every level subgroup of a fuzzy normal subgroup μ of a group G

is a normal subgroup of G, and every level subgroup of an N-Fuzzy subgroup μ is normal in its immediate successor level subgroup of μ, then, we have immediately the following lemma

<u>Lemma 4.1</u>. Every fuzzy normal or N-Fuzzy subgroup $\mu: G_{t_0} \lhd G_{t_1} \lhd G_{t_2} = G$ is a QN_2-Fuzzy subgroup.

The following is an example of QN_2-Fuzzy subgroup which is neither fuzzy normal nor N-Fuzzy subgroup.

<u>Example 4.1</u>. Let μ be the fuzzy subgroup of the dihedral group D_4 given by:

$$\mu: \langle e\rangle = G_{t_0} \lhd \langle ab\rangle = G_{t_1} \leq G_{t_2} = D_4.$$

Clearly μ is QN_2-Fuzzy which is neither fuzzy normal nor N-Fuzzy.

Now, we construct another sequence $\{\mu_{i-1,i}\}_{1\leq i\leq n}$ of fuzzy subgroups of a group G, where this time each member of the sequence is a QN_2-Fuzzy subgroup which is refined by some N-Fuzzy subgroup μ of G, as follows:

Let $\mu = \mu_n: \langle e\rangle = G_{t_0} \lhd G_{t_1} \lhd \ldots \lhd G_{t_{n-1}} \lhd G_{t_n} = G.$

Define $\mu_{0,1}: G_{t_{n-1}} \lhd G_{t_n} \leq G_{t_n} = G$

and $\mu_{i-1,i}: G_{t_{n-i}} \lhd G_{t_{n-i+1}} \leq G_{t_n} = G, \quad 1 \leq i \leq n.$

<u>Definition 4.2</u>. The sequence $\{\mu_{i-1,i}\}_{1\leq i\leq n}$ constructed above is called <u>the sequence of QN_2-Fuzzy subgroups</u> of the N-Fuzzy subgroup μ of the group G.

<u>Example 4.2</u>. The members of the sequence of QN_2-Fuzzy subgroups of length 2 of the N-Fuzzy subgroup μ of example 3.1 are:

$$\mu_{0,1}: \; \{e, a^2, ab, a^3b\} = G_{t_2} \subseteq G_{t_3} = D_4 \subseteq G_{t_3} = D_4$$

$$\mu_{1,2}: \; \langle a^3b \rangle = G_{t_1} \subseteq G_{t_2} \subseteq G_{t_3} = D_4$$

$$\mu_{2,3}: \; \langle e \rangle = G_{t_0} \subseteq G_{t_1} \subseteq G_{t_3} = D_4$$

Definition 4.3. Two QN_2-Fuzzy subgroups of a group G

$$\mu: G_{t_0} \lhd G_{t_1} \subseteq G_{t_2} = G \quad \text{and} \quad \nu: G_{\ell_0} \lhd G_{\ell_1} \subseteq G_{\ell_2} = G$$

are said to be *similar* if the factor groups G_{t_1}/G_{t_0} and G_{ℓ_1}/G_{ℓ_0} are isomorphic.

Example 4.3. Let $G = \langle a: a^8 = e \rangle$. The QN_2-Fuzzy subgroups μ and ν of G given by:

$$\mu: \langle e \rangle = G_{t_0} \lhd \langle a^4 \rangle = G_{t_1} \subseteq \langle a \rangle = G_{t_2} = G$$

and $$\nu: \langle a^4 \rangle = G_{\ell_0} \lhd \langle a^2 \rangle = G_{\ell_1} \subseteq \langle a \rangle = G_{\ell_2} = G$$

are similar.

Definition 4.4. Two normal sequences $\{\mu_i\}_{1 \le i \le m}$ and $\{\nu_j\}_{1 \le j \le n}$ of refined fuzzy subgroups by the N-Fuzzy subgroups μ and ν respectively of a group G are said to be *similar sequences* if

i) $m = n$, and

ii) the sequences $\{\mu_{i-1,i}\}_{1 \le i \le m}$ and $\{\nu_{j-1,i}\}_{1 \le j \le n}$ of the QN_2-Fuzzy subgroups of μ and ν respectively can be put in one-to-one correspondence such that corresponding QN_2-Fuzzy subgroups are similar.

<u>Example 4.4</u>. Consider the N-Fuzzy subgroups μ and ν of $G = \langle a : a^6 = e\rangle$ given by:

$$\mu: \langle e\rangle = G_{t_0} \lhd \langle a^2\rangle = G_{t_1} \lhd G_{t_2} = G$$

and

$$\nu: \langle e\rangle = G_{\ell_0} \lhd \langle a^3\rangle = G_{\ell_1} \lhd G_{\ell_2} = G$$

The elements of the normal sequence $\{\mu_i\}_{1\le i\le 2}$ and $\{\nu_j\}_{1\le j\le 2}$ refined by μ and ν respectively are:

$$\mu = \mu_2: G_{t_0} \lhd G_{t_1} \lhd G_{t_2} = G$$

$$\mu_1: \quad G_{t_1} \lhd G_{t_2} = G$$

and

$$\nu = \nu_2: G_{\ell_0} \lhd G_{\ell_1} \lhd G_{\ell_2} = G$$

$$\nu_1: \quad G_{\ell_1} \lhd G_{\ell_2} = G$$

while the elements of the sequences $\{\mu_{i-1,i}\}_{1\le i\le 2}$ and $\{\nu_{j-1,i}\}_{1\le j\le 2}$ of the QN -Fuzzy subgroups of μ and ν respectively are:

$$\mu_{0,1}: G_{t_1} \lhd G_{t_2} \subseteq G_{t_2} = G$$

$$\mu_{1,2}: G_{t_0} \lhd G_{t_1} \subseteq G_{t_2} = G$$

and

$$\nu_{0,1}: G_{\ell_1} \lhd G_{\ell_2} \subseteq G_{\ell_2} = G$$

$$\nu_{1,2}: G_{\ell_0} \lhd G_{\ell_1} \subseteq G_{\ell_2} = G$$

It is clear that $\mu_{0,1}$ is similar to $\nu_{1,2}$ since G_{t_2}/G_{t_1} is isomorphic to G_{ℓ_1}/G_{ℓ_0} and $\mu_{1,2}$ is similar to $\nu_{0,1}$ since G_{t_1}/G_{t_0} is isomorphic to G_{ℓ_2}/G_{ℓ_1}. Thus the sequences $\{\mu_i\}_{1\le i\le 2}$ and $\{\nu_i\}_{1\le i\le 2}$ are similar.

5. Fuzzy-Analog to Schreier's Refinement Theorem:

In this section we state and prove a fuzzy-analog to Schreier's refinement theorem for normal series of groups.

<u>Theorem 5.1</u>. Any two N-Fuzzy subgroups μ and ν of a group G:

$$\mu: \langle e\rangle = G_{t_0} \lhd G_{t_1} \lhd \dots \lhd G_{t_n} = G \qquad (*)$$

and

$$\nu: \langle e\rangle = G_{\ell_0} \lhd G_{\ell_1} \lhd \dots \lhd G_{\ell_m} = G \qquad (**)\ ,$$

can be refined by two N-Fuzzy subgroups $\bar{\mu}$ and $\bar{\nu}$ of G respectively such that

i) the chains of $\bar{\mu}$ and $\bar{\nu}$ are of the same length; and

ii) the sequences of QN_2-Fuzzy subgroups $\{\bar{\mu}_{i-1,i}\}_{1\leq i\leq k}$ and $\{\bar{\nu}_{i-1,i}\}_{1\leq i\leq k}$ of $\bar{\mu}$ and $\bar{\nu}$ respectively are similar.

<u>Proof</u>. From the classical Schreier's refinement theorem, the normal series of G in $*$ and $**$ have isomorphic refinements, say, $*'$ and $**'$, respectively. Now, consider the N-Fuzzy subgroups $\bar{\mu}$ and $\bar{\nu}$ given by

$$\bar{\mu}: \ *'$$

and

$$\bar{\nu}: \ **'$$

such that μ is refined by $\bar{\mu}$ and ν is refined by $\bar{\nu}$.

Clearly $\bar{\mu}$ and $\bar{\nu}$ satisfy the conditions in the theorem. This completes the proof.

In subsequent paper(s) it is hoped to develop further the theory discussed in this paper so that it may be used to obtain fuzzy-analogs of the classical notions of composition series and solvable groups.

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