King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 105

April 1988

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Mathematics Subject Classification: 10A20.

1. Introduction. We consider the following remarkable function on the set of positive integers

$$T(n) = \begin{cases} (3n + 1)/2 & \text{if } n = \text{odd}, \\ n/2 & \text{if } n = \text{even}. \end{cases}$$

The procedure defined by succesive iteration of T is known as the Syracuse algorithm. It is a long standing conjecture that the sequence obtained by iterating T always reaches 1 no matter which integer n we begin the sequence with. For example, the trajectory of 7 is {11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1,...}. The least positive integer k for which $T(n) = 1$ is called the total stopping time of n. E.g., the total stopping time for 7 is 11. So, the so called '3x + 1' conjecture states that every integer n > 1 has a finite total stopping time.

Yilmaz Akyildiz: I was born and raised in the eastern part of Black Sea Coast. I obtained my B.Sc.in physics in Ankara, Turkey (1970) and my Ph.D. in mathematics at the University of California, Berkeley (1976) under the supervision of Alan D. Weinstein. My present interests include applications of microcomputers to pure mathematics, theoretical computer science and watching my 13 month daughter, Selvi May, discovering something new every day.

The '3x + 1' problem, also known as the Collatz conjecture, the Syracuse problem, Hasse's conjecture and Ulam's problem, shares with other famous number-theoretic problems such as Fermat's last theorem, the attractive character of being 'so simple to state but apparently hard to solve'. During its life time of over half a century, many people, including some eminent mathematicians, worked on this conjecture. Despite the two dozen or so articles, mainly on the experimental, heuristic and probabilistic results, the behaviour of the Collatz function remains a mystery. An excellent extensive review of this problem by Lagarias appeared in this journal in 1985, [3]. There, it is stated that Paul Erdös once commented: "*Mathematics is not yet ready for such problems*". In that same paper, there is also the joke that this problem was part of a conspiracy to slow down mathematical research in the U.S.. The Collatz conjecture is also in Richard Guy's list of "*Don't try to solve these problems !*", [2]. The opinion is also held that this conjecture is unsolvable in the sense of formal logic and that the set of n which eventually returns to 1 is a non-recursive set, i.e., a set for which membership cannot be decided by a computer program that halts for all inputs, [4].

Our interest in this problem stems originally from the following two statements in published articles:

i) Lagarias in [3] writes :"*It is an interesting problem to find efficient algorithms to test the conjecture on a computer*".

Towards the end of his article Lagarias adds: "*The 3x + 1 problem is a deterministic process that simulates "random" behaviour. To the extent that the problem is structureless and "random", we have nothing to analyze and consequently cannot rigorously prove anything. Of course there remains the possibility that someone will find some hidden regularity in the 3x + 1 problem that allows some of the conjectures about it to be settled*".

ii) Crandall in [1] says : "*It is evident that numerical calculations of the stopping times of various numbers by computer methods must necessarily involve the storage of trajectory elements which are, in proportion to the starting number n, arbitrarily large; unless, of course, some theoretical method is discovered to simplify such computations*".

In the light of these questions above, we tried to see how long an odd number can keep producing odd numbers repeatedly under T, step after step. The answer to this came from the specific representation of the odd number n in the form $n = 2^i.q - 1$, where q is an odd integer. That is, the first i-1 iterates of n under T will all be odd numbers in an increasing order, and the i-th iterate will be an even number, hence it will be reduced under T at least by half. Thus, we realize that the above representation of the odd integers is the appropriate one for the Collatz problem in the sense that through this representation we can forsee many steps ahead in the iteration of the function T. This way we neither need to compute nor store many trajectory elements, hence partially answering Crandall's question. On a P.C., by using the SETL language, we also observed that our algorithm runs at least twice as fast as the Syracuse one. We expect that running our program in machine language would come close to Lagarias's wish.

Our main observation is that the function T simply switches 2 into 3 in the above representation of the odd number n. This way one skips many of the odd trajectory elements, which have to be calculated when using the Syracuse algorithm. We can still read off the exact stopping time of any number under the Collatz map T. Our procedure clearly shows that the behaviour of a Collatz sequence is intimately related to the way in which the powers of 2 are distributed among the powers of 3.

It has been noted by other researchers [3] that coalescence of trajectories occurs for certain nearby numbers. This paper will demonstrate that such coalescence does also occur for numbers which are far apart. This we shall show by proving that the total stopping time of $2^{2k-1} - 1$ is one less than the total stopping time for $2^{2k} - 1$, where k is a positive integer > 1.

2. The Algorithm: "Switching 2 to 3". It is clear that the '3x + 1' problem is really about odd integers. Therefore, we shall work with the following variant of the Collatz function defined on positive odd integers $C(n) = (3n + 1)/2^{e(n)}$, where

$e(n)$ is the largest power of 2 that divides $3n + 1$, i.e., $C(n)$ is again an odd integer. Then the '3x + 1' conjecture is equivalent to the assertion that every positive odd integer is eventually taken to 1 by iteration of C. Our main observation is the following: For any positive integer n let i be the highest power of 2 dividing n+1, i.e., $n = 2^i.q - 1$, where q is an odd positive integer. Now consider the even integer $3^i.q - 1$ and let j be the highest power of 2 dividing this even number, i.e., $3^i.q - 1 = 2^j.m$, where m is an odd positive integer. Then we have $C^i(n) = m$. This procedure can now be repeatedly applied to the new odd integer m. The total stopping time with respect to the iterations of the Collatz map T can be obtained from the sum of all the values of i and j in the above procedure. Crandall [1] works with the height function h(n), which is defined as the cardinality of the trajectory of n under the map C. According to our procedure above, h(n) is obtained by the sum of all the values of i. One sees that this algorithm skips many trajectory elements. Therefore one needs to store fewer numbers in order to study the growth properties of a Collatz sequence. For example, Crandall gives the trajectory of 27 as

{41,31,47,71,107,161,121,91,137,103,155,233,175,263,395,593, 445,167,251,377,283,425,319,479,719,1079,1619,2429,911,1367,2051, 3077,577,433,325,61,23,35,53,5,1}, with 3077 being the largest element, whereas our algorithm lists only

{31,121,91,103,175,445,167,283,319,911,577,433,325,61,23,5,1}, with the maximum element 911.

For the proof of the above algorithm we simply observe that

$$n=2^{i}.q-1 \xrightarrow{C} 2^{i-1}.3q-1 \xrightarrow{C} 2^{i-2}.3^{2}.q-1 \; ..... \xrightarrow{C} 2.3^{i-1}.q-1 \xrightarrow{C} (3^{i}.q-1)/2^{j}=m \quad (=\text{odd}).$$

(Note that the Syracuse algorithm would count $i+j$ as the number of T iterations from n to m, while we count i as the number of C iterations. This difference stems from the fact that we concentrate on odd integers only, which is enough).

3. Coalescence of Trajectories. Very little is known about the stopping time function. It has been experimentally observed that over short ranges of n the stopping time function tends to assume only a few values. This is caused by coalescence of trajectories of certain nearby numbers after a few steps, [3, p.12].

To conclude, we now show, by using our algorithm, that coalescence is also possible for numbers which are not close to each other.This we do by proving that the stopping times under C of $2^{2k-1}-1$ and $2^{2k}-1$ are the same. We first observe that $(3^{2k-1}-1)/2$ and $((3^{2k-1}-1)/2+1)/2$ are both odd numbers, (proof by contradiction). Now, our algorithm gives

$$2^{2k-1}-1 \xrightarrow{C^{2k-1}} (3^{2k-1}-1)/2=2q-1 \xrightarrow{C} (3q-1)/2^{e}=(3^{2k}-1)/2^{e+2},$$

which is nothing but the first element in the trajectory of $2^{2k}-1$. Clearly, $C^{2k}(2^{2k-1}-1) = C^{2k}(2^{2k}-1)$, and this completes the proof. (Note that the total stopping time for these two numbers relative to the Collatz map T differs by one).

Acknowledgements. Special thanks are due to Dr. Mohammad I. Al-Suwaiyel for the computer aspects of this work and to Dr. Hashim A. Yamani who (without knowing) initiated this research. I am also grateful to King Fahd University of Petroleum and Minerals where this research has been done.

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