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0. Introduction.

The purpose of this paper is to describe the part played by measure theory in the solutions, or partial solutions, of some famous problems from geometry. To illustrate how these two apparently rather different topics can connect, let us first recall the history of the Kakeya problem.

In 1917, Kakeya (see [20]) asked the following question: find the region of smallest area A in $\mathbb{R}^2$ such that a needle (or line segment) of unit length can be turned continuously through 180° inside the region. It was conjectured that among convex sets, an equilateral triangle of height one should be the smallest area; and, indeed, this was confirmed by Pàl [32]. Without the restriction of convexity, it was thought that the solution would be a three cusped hypercycloid. To everyone's surprise Besicovitch proved this false, in a dramatic way. Earlier, in 1917, he had constructed a set in $\mathbb{R}^2$, of Lebesgue measure zero, which contains a translate of each straight line. (He had used this to answer a question in Riemann integration.) Now, in 1928 [3], Besicovitch showed that an arbitrarily small area could be added to his set in such a way as to allow the needle to turn.

This construction of Besicovitch began a long series of papers in which planar sets of measure zero were found which contain translates, or congruent copies, of polygonal arcs, circles of different radii, and so on. A whole new section of measure theory developed, some of it complicated and sophisticated, such as Marstrand's result [28]. (The latter implies that there is no set in $\mathbb{R}^2$ of Lebesgue measure zero containing a translate of each rectifiable curve.) moreover, the original construction found an important use in functional analysis in C. Fefferman's 1971 solution of the multiplier problem for the ball (see [8]).

The story of the Kakeya problem serves nicely to demonstrate some points I wish to emphasize in this article. Firstly, many problems from elementary geometry are, like some in number theory, not only easy to state and to understand, but beautiful in a way only simple things can be. Their solutions, however, may be far from simple, and might involve ideas or constructions from measure theory. It will therefore pay those who work in measure theory to ponder, from time to time, over geometrical problems. Next, measure theory itself may be enriched by the methods employed in solving these problems. And finally, there is always the possibility that these techniques may have applications in some other part of mathematics.

Four geometrical problems will be considered here. The first is to a large extent solved, though some questions remain. The others are all more or less completely unsolved. They have achieved a certain notoriety among geometers, and are often included in papers or books on unsolved problems. Since they are popularly considered important, I wish to include here six related conjectures, the study of which may lead to progress. In any case, the problems have already yielded some interesting measure theory, and to explain this we take them one by one.

1. Hammer's X-ray problem.

Suppose K is a convex body in $\mathbb{R}^n$, and s is a direction. Let H_s be the hyperplane orthogonal to s and containing the origin. For each $x \in H_s$, let $f_s(x)$ be the length of the chord of K that lies on the line through x which is parallel to s. This is called the chord function of K in the direction s. The information it gives is, ideally, that which would be given by an X-ray picture of K, assuming K to be of uniform density.

In 1963, P.C. Hammer [18] asked some questions, one variant of which is the following: *find sets of directions such that the chord functions of a convex body* K *in* $\mathbb{R}$, *taken in those directions, determine* K *uniquely*.

This, or modified versions of it, became known to geometers as the X-ray problem. Of course, if one regards K as a solid, there would be little point in taking X-rays to determine its shape. But Hammer envisaged instead a convex hole in a uniform solid; and I am told that this is not too far-fetched, since bubbles formed in metal pipes cause weakness, and their position, if not their shape, needs somehow to be discovered.

It turned out that Hammer had almost been anticipated. The same year, a paper by O. Giering appeared (see [17]; also [11], [39]) in which it is shown that if K in $\mathbb{R}^2$ is known in advance, three directions may be chosen so that the Steiner symmetrals of K in those directions distinguish K from all other convex bodies. (In the notation above, the Steiner symmetral of K in the direction s is formed by sliding each chord of K parallel to s so that it is bisected by H_s; it is easy to see that in the context of Hammer's question, chord functions and Steiner symmetrals are the same.) It is an important feature of the X-ray problem, however, that K is not fixed in advance.

Some simple examples show that no matter how many directions are used, they must be carefully chosen. For, let H be a regular n-gon, and let K be H rotated by π/n. Suppose s is any direction parallel to an edge of the convex hull of H and K. Then H and K have the same chord functions in the direction s. Furthermore, the problem is affine invariant, so affine images provide further examples. In [16], Peter McMullen and I showed that the 'bad' sets of directions given by these are the only ones.

<u>THEOREM</u> 1. Chord functions in a set S of directions will distinguish between different convex bodies in $\mathbb{R}^2$ if and only if S is not an affine image of a subset of the directions of edges of a regular polygon.

Measure theory comes into the proof, albeit in a superficial way, via the Cavalieri principle. Let S be a set of directions, and suppose H and K are two different convex bodies in $\mathbb{R}$ which have the same chord functions in each direction in S. Provided S contains at least two directions, as it must, H and K have the same centroid and so the symmetric difference $H \Delta K$ contains a non-empty open component A. If $s \in S$, there must correspond to A another component sA such that each line parallel to s meets A and sA in chords of equal length (see Fig. 1). The Cavalieri principle says that $\lambda_2(sA) = \lambda_2(A)$ (we use λ_n to denote n-dimensional Lebesgue measure). Furthermore, A and sA are disjoint. Now beginning with $A = A_1$, we apply this argument to all directions in S, to obtain disjoint components $A_2, A_3, \ldots,$ of $H \Delta K$. Since each has the same measure, the process must close up. Now let c_k be the centroid of A_k. The finite set of points c_k form the vertices of a convex polygon with a special property: the vertices match up in pairs, in each direction in S. Finally, a geometrical trick can be used to show that the only polygons which have this property are affinely regular, proving the theorem.

For uniqueness, then, 3 directions are not enough, since each set of 3 directions is an affine image of the directions of edges of an equilateral triangle. But certain sets of 4 directions will do; specifically, one can take a set of 4 directions, the cross-ratio of whose slopes is a transcendental number.

At this stage the reader may wonder why Hammer's question was set in $\mathbb{R}^2$. The reason is that by taking chord functions in a suitable coplanar set of directions, we can determine each 2-dimensional slice of a convex body in $\mathbb{R}^n$, and so determine the whole body itself. The result above then extends automatically to $\mathbb{R}^3$, for example. This is rather satisfying, and (as we shall explain later) no counterpart of the theorem exists outside convexity. Nevertheless, there is a drawback. A glance at the examples above shows that, for each k, the 'bad' sets of k directions are dense in the class of all sets of k directions. Therefore, it would be impossible, in practice, to set up X-ray machines to guarantee uniqueness.

It makes sense, because of this, to ask which sets of directions in $\mathbb{R}^3$, no 3 of which are coplanar, are such that chord functions of a convex body in $\mathbb{R}^3$ determine it uniquely; indeed, I asked myself this in 1979. We can get some examples of sets of directions to avoid as follows. Labelling the vertices of a cube alternately black and white, we get two tetrahedra — one black, one white — which have the same chord functions in the directions of the edges of the cube. Elaborating on this idea, with J. Wills and A. Volčič, we found an example of a 'bad' set of 6 directions, no 3 of which are coplanar. Indeed, it can be shown that labelling the vertices of the regular polyhedron (4, 6, 10) (see [9], p. 110) alternately as above, one obtains two copies of the polyhedron (3, 3, 3, 3, 4), which have equal chord functions in such a set of directions. The difficulty in improving on this leads to the following conjecture.

<u>CONJECTURE</u> 1. There is a k such that in $\mathbb{R}^3$, any set of k directions, no 3 of which are coplanar, will give uniqueness. Perhaps we can take $k = 7$.

Hammer also imagined X-rays emanating from a point source, rather than from infinity (the parallel X-ray case considered above). We prefer to speak of a chord function of a convex body K in $\mathbb{R}^2$ at a point p. This function specifies the length of each chord which lies on a ray with endpoint at p.

(This differs slightly from the definition in the literature, where doubly infinite lines through p, rather than rays, are used; of course, our chord functions give rather more information, distinguishing for example between two discs of equal radii placed symmetrically with respect to p.) The question is now: which sets of points are such that chord functions at those points determine shape?

Let us try to repeat the arguments used in Theorem 1. If H and K are two different convex bodies with equal chord functions at at least two points, $H \Delta K$ again contains a non-empty open component A. Also, there is a corresponding and disjoint component pA such that A and pA have equal chord functions at p. But now the Cavalieri principle does not apply, and we can only deduce that $\lambda_2(pA) > \lambda_2(A)$, assuming pA to be further from p than A. For this reason, progress was slow, and the breakthrough came only in 1986, with A.Volčič's paper [38], providing the next powerful theorem.

THEOREM 2. Chord functions at 3 non-collinear points determine a convex body uniquely.

It was Volčič's insight that a different measure can be found to do the job of Lebesgue measure for the parallel case. With respect to a fixed set of cartesian axes, let

$$\mu(E) = \iint_E \frac{dx\,dy}{|y|}$$

for bounded λ_2-measurable sets E. Suppose p is any point on the x-axis, and let A and pA be two sets in the upper half-plane which are not necessarily convex but which have well-defined and equal chord functions at p (see Fig. 2). Then it can be shown that $\mu(pA) = \mu(A)$, and this repairs the loss of the Cavalieri principle. It seems only fair to call the phenomenon the

Volčič principle.

Volčič's measure μ needs a little extra care in its use, however, since a convex body met in its interior by the x-axis will have infinite μ-measure. A useful fact is that a triangle contained in the upper half-plane and with one vertex on the x-axis has finite μ-measure.

Let us see how Volcic's measure may be used to obtain a uniqueness result. Consider two points, p_1 and p_2, and suppose the line through them is the x-axis. Let H and K be two different convex bodies with equal chord functions at p_1 and p_2. We need to assume also something about the position of H and K, namely, that they are both met in their interiors by the line segment $[p_1, p_2]$. We can find a non-empty open component A_1 of $H \Delta K$, and we shall assume its closure lies in the upper open half-plane. (If this is not the case, an easy argument leads to a contradiction.) The chord functions of H and K at p_1 tell us that there is a non-empty component A_2 of $H \Delta K$, which we may suppose is nearer to p_1 than A_1, which by the Volčič principle has the same μ-measure as A_1. Now chord functions at p_2 give another component A_3, again with $\mu(A_3) = \mu(A_2) = \mu(A_1)$. Continuing, we obtain a sequence $\{A_n\}$ of disjoint components of equal μ-measure. By convexity, the subsequence $\{A_{2n}\}$ all lie in a triangle in the upper half-plane with one vertex on the x-axis, and since this has finite μ-measure, we have a contradiction.

The 3-point result of Theorem 2 is proved by the technique above, together with others. It leaves open the possibility that 2 points would be enough. In view of the argument above, we need to decide whether there can be two different convex bodies which do not meet the line joining two points p_1 and p_2, with equal chord functions at p_1 and p_2, as in Fig. 3. Without any great conviction we state the following conjecture.

CONJECTURE 2. Chord functions at any two points will suffice for uniqueness.

Reconstruction of density distributions $f(x, y)$ from X-rays is the problem dealt with in computerized tomography, the technique behind the modern CAT scanners so important in medicine. If s is a direction in $\mathbb{R}^2$, an X-ray or chord function in the direction s gives, for each line ℓ parallel to s, the value of the integral

$$\int_{\ell} f(x, y)d\lambda_1,$$

where λ_1 is Lebesgue linear measure in ℓ. For reconstruction of, for example, a slice of a skull, to locate a possible brain tumour, approximate methods must be employed. The reason for this is, as mentioned above, that Theorem 1 fails for density distributions; in fact, even for the intermediate category of measurable sets, no finite set of directions allow the corresponding chord functions to give uniqueness.

To show this, we outline an idea which goes back to G.G. Lorentz in 1949 ([27]; see also [4]). Let $s_1, s_2, \ldots, s_k$ be any finite set of directions in $\mathbb{R}^2$. Let B_1 and W_1 be two disjoint discs of equal radii r with centers on a line parallel to s_1; clearly they have equal chord functions in the direction s_1. Translate B_1 by a suitably large vector $\underline{v}_2$ in the direction s_2, and call this translate W_2. Similarly, let B_2 be W_1 translated by $\underline{v}_2$. The disjoint union of black discs $B_1 \cup B_2$ and the disjoint union of white discs $W_1 \cup W_2$ now have equal chord functions in both the directions s_1 and s_2. Translating again by a suitably large vector $\underline{v}_3$ in the direction s_3, and continuing, we obtain two different measurable sets (each a disjoint union of 2^k discs) which have equal chord functions in all the directions $s_1, s_2, \ldots, s_k$.

This negative result is by no means the end of the story and some interesting results can be expected from the study of chord functions of measurable sets. The reader should also consult A. Volčič's article [40].

2. The equichordal problem.

One of the oldest and most intractable problems in convexity, the equichordal problem was posed independently by Fujiwara [10] in 1916 and by Blaschke et al [5] in 1917. Almost anyone can understand the question, which depends only on the notion of an equichordal point; that is, a point interior to a convex body K such that every chord of K through the point has the same length α. (This α is the corresponding equichordal constant.) We are asked to find a convex body K in $\mathbb{R}^2$ which admits two equichordal points.

The reader who wishes to consult a survey can refer to [23], [24] or [25]. In any case, very little is known.

Most attempts to construct a convex body K with two equichordal points, p and q, began as follows. It is obvious that the equichordal constant must be the same for both p and q. Starting with a point x_1 in the boundary ∂K, not on the line through p and q, the chord through x_1 and q determines another point y_1 in ∂K. Now the chord through y_1 and p gives a third point x_2 in ∂K, and so on. Two sequences $\{x_n\}$ and $\{y_n\}$ of points in ∂K are produced, which one hopes will each converge to single points a and b on the line through p and q (see Fig. 4). Even if this is the case, one only has a skeleton on which a suitable curve must be found.

The only major result concerning the problem is the following, due to E. Wirsing [43] in 1958.

THEOREM 3. If K is a convex body with two equichordal points, then K is unique (given the equichordal constant α) and ∂K is analytic.

The proof is brief and direct, though certainly not trivial.

The equichordal problem now finds its way into most unsolved problem lists in geometry. For example, you will find it in the entertaining paper [34] of C.A. Rogers, whose first piece of advice to those thinking about working on the problem is, 'Don't!'. It is possible, however, to avoid a direct clash, yet still contribute constructively, by changing the problem itself. This was done by V. Klee [23] in 1969, who set the 'equireciprocal problem'.

Suppose again that K is a convex body and p an equichordal point. Then each chord $[x, y]$ of K through p satisfies

$$\|x - p\| + \|p - y\| = \alpha.$$

If, instead, each chord of K through p satisfies

$$\|x - p\|^{-1} + \|p - y\|^{-1} = \alpha,$$

we call p an equireciprocal point. Now it so happens that if K is an ellipse, each focus is an equireciprocal point. Klee asked if there are non-elliptical examples. An intriguing and rather complete answer was discovered by K. Falconer [7] in 1983. He showed that the assumption that ∂K is twice differentiable indeed forces an equireciprocal body K to be an ellipse, with the equireciprocal points at the foci. Moreover, he constructed non-elliptical convex bodies, with two equireciprocal points, whose boundaries are only once differentiable.

It does not take much imagination to propose to generalize the equations above, thus:

$$\|x - p\|^k + \|p - y\|^k = \alpha,$$

and define a k-equipower point ($k \neq 0$) to be a point p in the interior of K satisfying this equation for each chord $[x, y]$ of K through p. Further, various considerations lead one naturally to the equation

$$\|x - p\| \cdot \|p - y\| = \alpha$$

to cover the missing value $k = 0$. A surprise is that 0-equipower points were the subject of K. Yanagihara's 1916 paper [44], though not by that name; they have also been called power or equiproduct points. A quick check will confirm that each point interior to a circle is a 0-equipower point (where the constant α now varies according to position). In [22] and [44] it is shown that if K is a convex body with two 0-equipower points, and ∂K is differentiable, then K must be a circle. The analogy with part of Falconer's result is obvious. Noone seems to have taken the trouble to construct non-differentiable convex boundaries with two 0-equipower points, although such presumably exist, to match the equireciprocal case. Nor, as far as I know, has anyone made the following natural conjecture.

CONJECTURE 3. There is no convex body with two k-equipower points, for any $k \geq 1$. Possibly, there are none for any $k \neq 0, -1$.

We shall now describe how measures can be used to prove a result which bears on Conjecture 3.

First, note that the equichordal problem clearly involves chord functions, for the chord function of a convex body K at an equichordal point is by definition a constant. Viewed this way, the chord functions of K at two interior points are being specified. But this knowledge is enough to determine

K uniquely, by an argument which closely resembles the one given in §1 above for chord functions at two exterior points. In short, one can use the Volčič measure μ to retrieve the uniqueness part of Theorem 3.

Now at a k-equipower point, for $k \neq 1$, it is not the chord function at p which is constant, but something which it is convenient to call a k-chord function. Let p be interior to K, and ℓ a line through p, meeting ∂K at x and y. Then the k-chord function of K at p gives for each such ℓ, the value $\|x - p\|^k + \|y - p\|^k$ $(k \neq 0)$, or $\|x - p\| \cdot \|y - p\|$ $(k = 0)$. These functions appear under the name 'generalized chord functions' in [6]; clearly they reduce to the usual chord function when $k = 1$.

To tackle uniqueness questions for k-chord functions, it would be nice to have a suitable measure, such as Volčič's. In fact, it can be shown that

$$\mu_k(E) = \iint_E \frac{dx\, dy}{|y|^{2-k}}$$

defines (with respect to fixed x- and y-axes) such a measure. For $k = 2$ it is of course just Lebesgue measure λ_2, and it remains locally finite provided $k \geq 2$.

Suppose then that K is a convex body and p, q are two interior points. Then if $k \geq 1$, the k-chord functions at p and q determine K uniquely. This implies the uniqueness (up to proportionality factors) of a convex body with two k-equipower points for $k \geq 1$, and explains why Conjecture 3 selects these values of k. For $k \leq 0$, however, the uniqueness proof requires a differentiability condition of order $(-k + 1 + \varepsilon)$ on ∂K. These and other results may be found in [14] (see also [6] and [45]). Probably the ε can be dropped, as in the known special cases $k = 0$ and -1, when the k-chord functions at p and q are known to be constant.

To keep the exposition simple, we restricted the discussion above to convex bodies. In a way it is perhaps more natural to allow bodies which are only starshaped at the relevant interior points. Wirsing's Theorem 3 still holds true, by his own proof; but now the application of Volčič's measure (which, it must be stressed, deals with all chord functions, not just constant ones) needs again a differentiability condition on ∂K. Conjecture 3 is still open for starshaped bodies and the results in [14] mentioned above apply also to them.

Perhaps the reader will excuse a final vague thought. Conjecture 3 involves an interplay between uniqueness and existence, and the special case $k = -2$ is reminiscent of an inverse-square law. Could it be that the methods of potential theory will one day bring the solution?

3. Tarski's circle-squaring problem.

Long ago, in 1925, the great logician A. Tarski put forward a question intelligible even to the layman. Is it possible to divide a disc in the plane into a finite number of disjoint pieces, in such a way that these pieces can be rearranged to form a square? Over the years I have actually told many non-mathematicians about the problem. They usually find it difficult to believe that the answer is unknown; mathematicians find this slightly less surprising, especially when reminded that the pieces may be arbitrary sets, possibly even nonmeasurable.

The layman, it seems, often comes up with two questions. Can I use an infinite number of pieces? No; the problem is solved in the affirmative in this case (see [42], chapter 9). Are the disc and the square of the same area? Yes, you may assume this. But to explain why, we must risk losing the layman and talk mathematics.

Suppose A and B are two sets in $\mathbb{R}^n$. We say A and B are equidecomposable, and write $A \approx B$, if (i) $A = \bigcup_{i=1}^{k} A_i$, $B = \bigcup_{i=1}^{k} B_i$ (ii) $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $i = 1, \ldots, k$ and (iii) $B_i = \tau_i A_i$ for each i, where τ_i is an isometry in $\mathbb{R}^n$. Tarski's question asks whether a disc and a square in $\mathbb{R}^2$ are equidecomposable. It is the subject of a beautiful article [41] by Stan Wagon, and is also treated in two books of his ([42], and [25], coauthored by V. Klee). Here we shall explain just enough background to lead to some conjectures.

The problem arose from a magnificent paper [1] of Banach and Tarski in 1924. This contained two main results, gathered together in the next theorem.

THEOREM 4. (1) If P and Q are two polygons in $\mathbb{R}^2$, then $P \approx Q$ if and only if $\lambda_2(P) = \lambda_2(Q)$.

(2) If A and B are any two bounded sets in $\mathbb{R}^n$, $n \geq 3$, with non-empty interiors, then $A \approx B$.

(Recall that λ_n is n-dimensional Lebesgue measure.) The first result is the analogue of the classical theorem of Bolyai and Gerwien, which says that the polygons P and Q are equidissectable if and only if $\lambda_2(P) = \lambda_2(Q)$; this means P may be divided into triangles (simplices are used for polytopes in $\mathbb{R}^n$) which can be rearranged, with only boundaries possibly overlapping, to form Q. The interesting direction in the Banach-Tarski theorem on polygons is that if $P \approx Q$, then $\lambda_2(P) = \lambda_2(Q)$. To prove this, they showed that λ_2 can be extended to a finitely additive, isometry invariant measure defined on $P(\mathbb{R}^2)$, the class of all subsets of $\mathbb{R}^2$. Nowadays we would deduce this quickly from the Hahn-Banach theorem. The italicized properties of the extended measure immediately give the required implication. Indeed, there is no need to restrict P and Q to be polygons here; if L, M are λ_2-measurable and $L \simeq M$, then $\lambda_2(L) = \lambda_2(M)$, which explains the answer to the layman's second question above.

The second result in Theorem 1 is in marked contrast. In the view of J. Mycielski, it is 'the most surprising result of theoretical mathematics'. Surely many would agree. Theoretically, a ball in $\mathbb{R}^3$ can be taken apart into a finite number of pieces (5, according to a finer analysis) which can then be fitted together to make two balls, each the same size as the original. More relevant to our discussion of Tarski's problem is that in three or more dimensions the ball can be 'cubed'. In fact <u>any</u> ball and <u>any</u> cube are equidecomposable, whatever their size.

Some mathematicians worry about all this, and the phenomenon is usually called the Banach-Tarski paradox. Those versed in measure theory should lose no sleep, however. The pieces used in the decompositions such as the duplication of the ball in $\mathbb{R}^3$ must obviously be nonmeasurable, and the 'paradox' implies that in $\mathbb{R}^n$, $n \geq 3$, λ_n does not extend in the way λ_2 does in $\mathbb{R}^2$.

It seems that something quite dramatic happens in the step from two to three dimensions. The first to understand the underlying reason was von Neumann, who saw that it involves the structure of the group of isometries. The key property is called amenability. This has a purely algebraic definition, but most readers of this paper will be happier with the following one. A group G is called <u>amenable</u> if there is a finitely additive, left-invariant probability measure defined on all subsets of the group. The importance of this concept will become clear when we state the next theorem, due to J. Mycielski (see [42, Theorem 11.20).

THEOREM 5. Let G be a group of isometries in $\mathbb{R}^n$. Then λ_n can be extended to a finitely additive G-invariant measure on $P(\mathbb{R}^n)$ if and only if G is amenable.

This result helps enormously to explain Theorem 1. It can be shown that all Abelian, and indeed all solvable groups are amenable. Since the group of all isometries in $\mathbb{R}^2$ is solvable, the extension of λ_2 exists, preventing 'paradoxes'. On the other hand, the group of all isometries in $\mathbb{R}^n$, $n \geq 3$, contains a free subgroup of rank two, and one can prove this implies it is not amenable, prohibiting the extension of λ_n. Notice that Theorem 5 applies to any subgroup G of the group of all isometries in the same way.

To describe our conjectures we shall need just one more concept, due to G.T. Sallee [35] in 1969. Two convex bodies H, K are called <u>convex equidecomposable</u> ($H \overset{c}{\approx} K$) if H can be divided into convex pieces that may be rearranged (with only boundaries possibly overlapping) to form K (see Fig. 5). This is not only a natural idea, but a genuine generalization of the classical notion. For, if H is actually a polytope, and $H \overset{c}{\approx} K$, it is easy to see that K must also be a polytope, and H and K are equidissectable. In particular, a disc and a square are not convex equidecomposable.

We can now state the first conjecture.

CONJECTURE 4. Let P be a polytope and K a convex body in $\mathbb{R}^n$. If $P \approx K$ under isometries $\tau_1, \ldots, \tau_k$ from an amenable group, then $P \overset{c}{\approx} K$.

Quite a lot of work has gone into formulating this simple conjecture. It is a modification of an idea from [35], in which P was also allowed to be a general convex body. Some restriction is necessary, however, as examples found in 1985 (see [12]) show. The main point of the conjecture is that it includes Tarski's problem, in the sense that if it is true, the circle cannot be squared, in view of the remarks above. But many other natural-sounding conjectures would also be subsumed. For example, it would imply that the ball and cube in $\mathbb{R}^n$ are not equidecomposable if only translations are used (since the group of translations in $\mathbb{R}^n$ is Abelian). Furthermore, it would even say something about polyhedra in $\mathbb{R}^3$. Since the cube and the regular tetrahedron are not equidissectable, by Dehn's solution of Hilbert's third problem, it would follow that these polyhedra are not equidecomposable via isometries in an amenable group.

The conjecture is true if K is a polygon in $\mathbb{R}^2$; this follows from Theorem 1(1) and the Bolyai-Gerwein theorem. Beyond this, evidence rests at present on another known special case, namely, when the group $\langle \tau_1, \ldots, \tau_k \rangle$ generated by the isometries $\tau_1, \ldots, \tau_k$ is discrete. This is proved in [13]; note that discrete groups are known to be amenable. In fact, the proof of Theorem 1 of [13] shows that under the hypotheses of Conjecture 4, $P \overset{c}{\simeq} K$ under the <u>same</u> isometries $\tau_1, \ldots, \tau_k$.

Conjecture 4 is a weaker version of the question on p. 8 of [13]. At the Conference on Real Analysis and Measure Theory in Capri, 1988, I made the conjecture in an even stronger form, asking if (as in the discrete case) the same isometries can be used for the convex equidecomposability. This was disproved before the end of the following talk! M. Laczkovich quickly realized that his paper [26] does this. It shows that if $P = K = [0, 1]$, and u is irrational, then $P \simeq K$ under the isometries $\tau_1(x) = x + u$, $\tau_2(x) = x - u$, $\tau_3(x) = -x + u$ and $\tau_4(x) = -x - u + 2$, but that this is not true if only measurable pieces are allowed. So, Conjecture 4 is formulated with this example and certain modifications of it in mind. Taking the product of

Laczkovich's example with [0, 1], we get a result for the unit square in $\mathbb{R}^2$ which shows that the Banach-Tarski theorem on polygons cannot be strengthened too much. In fact, I should say that this example of Laczkovich has somewhat shaken my belief in the truth of Conjecture 4.

Perhaps it is the possibility that the pieces in Tarski's problem may be nonmeasurable that has discouraged many who work in measure theory from attempting a solution. But equidecomposability with arbitrary pieces and with convex pieces are only two extremes in a spectrum of variations. We could just allow measurable pieces, or only Borel pieces, and still almost nothing is known. For example, the following conjecture is open.

<u>CONJECTURE 5</u>. Let P be a polytope and K a convex body in $\mathbb{R}^n$. If $P \simeq K$ with λ_n-measurable pieces under isometries $\tau_1, \ldots, \tau_k$ from an amenable group, then $P \overset{c}{\simeq} K$, under the same isometries $\tau_1, \ldots, \tau_k$.

Further variants are obtained when (cf. [13], Theorem 5) sets of measure zero, or sets of first category, are neglected. Rather than pursue this line of thought, let us strip away the geometrical features of Conjecture 4, for this yields another unsolved question in measure theory.

<u>CONJECTURE 6</u>. Suppose A, B are λ_n-measurable sets in $\mathbb{R}^n$. If $A \simeq B$ under isometries from an amenable group, then $A \simeq B$ with λ_n-measurable pieces.

Both W. Just [19] and I have independently formed the opinion that the techniques of ergodic theory might be brought to bear on these questions. In a way this is of course a guess. But an argument used in [26] already shows the guess is not completely misplaced.

4. Bang's plank problem.

We shall call the closed region between two parallel hyperplanes in $\mathbb{R}^n$ a slab; alternative terms, particularly in $\mathbb{R}^2$, are plank or strip. Suppose we cover a convex body K in $\mathbb{R}^n$ by k slabs S_i. For each slab, we measure its relative width w_i/W_i, where w_i is the width of the slab and W_i the width of K in the same direction (see Fig. 6). Then is it true that $\sum_i w_i/W_i \geq 1$?

Imagine the slabs S_i are all parallel and adjacent, and cover K as efficiently as possible. Then, of course, the sum of their widths is equal to the width of K in the same direction, giving equality above. So Th. Bang's simply stated and basic problem asks us to show that we can do no better by adjusting the directions of the slabs. It remains unsolved, though it appeared in the 1951 paper [2]. In fact, it is a stronger and affine invariant form of an even earlier question, known as Tarski's plank problem. The main purpose of [2] was to present the following solution to the latter.

THEOREM 6. Let K be a convex body in $\mathbb{R}^n$, S_i a slab for $i = 1, \ldots, k$, and $K \subset \cup_i S_i$. Then $\sum_i w_i/w \geq 1$, where w is the minimum width of K.

Bang's proof was well received, since it was elementary yet ingenious, and solved a problem already 18 years old. Initially, it surprised me that Tarski should have come up with two of the handful of outstanding problems in elementary geometry. Later, I learned that his plank problem also arose from his work on equidecomposability. The connection is as follows. Let R be a rectangle with sides of length x and $1/x$. Then, from the Bolyai-Gerwien theorem, we know that R is equidissectable with the unit square. Let $\tau(x)$, the 'degree of equivalence', be the natural number which is the least number of polygons which can be used in the dissection. Then the behaviour of $\tau(x)$ is linked (see [37]) to covers of the unit square by slabs of width x.

For our purposes it is instructive to contemplate a special case of Theorem 6 which was known to H. Moese in 1932. Suppose K is a disc of diameter 1 in $\mathbb{R}^2$, covered by k slabs S_i. Regard K as a subset of the coordinate plane $\{z = 0\}$ in $\mathbb{R}^3$, and let H be a hemisphere in $\mathbb{R}^3$ whose vertical projection is K. To each slab S_i there corresponds a curved strip C_i on the surface of H which projects vertically onto S_i. Denoting as before the width of S_i by w_i, the surface area $|C_i|$ of C_i is $w_i \cdot |H| = \pi w_i/2$. This follows from the famous observation of Archimedes, that the surface area on a sphere between two parallel planes is in the same proportion to the whole, as the distance between the planes is to the diameter of the sphere. Since the sets C_i cover the surface of H, we have $\sum_i |C_i| \geq \pi/2$, and hence $\sum_i w_i \geq 1$ as required.

Now G. Hajós observed (see [33] by A. Rényi) that if there exists a probability measure μ in the convex body K, such that for each slab S_i

$$\mu(S_i \cap K) = w_i/W_i, \qquad (*)$$

then Bang's conjecture is true for K, since then

$$\textstyle\sum_i w_i/W_i = \sum_i \mu(S_i \cap K) \geq \mu(K) = 1$$

whenever the slabs S_i cover K. And in fact such a measure does exist at least for one special case. If K is a disc in $\mathbb{R}^2$, we can take μ to be the normalized projection of surface area measure in the hemisphere over K. The argument of H. Moese above is thus seen to be an instance of Hajós' observation.

In 1952, D. Ohmann [30] claimed to have constructed such a measure μ in an arbitrary convex body K. Unfortunately, these measures necessarily take negative values as well as positive ones, and so Bang's conjecture does not follow at all. Despite this the approach is so appealing that it seemed a good idea to investigate whether alternative constructions might provide non-negative measures μ with the required property. So, let Θ be a set of directions in $\mathbb{R}^n$, and define a relative width measure μ in a convex body K in $\mathbb{R}^n$ for Θ to be a countably additive, non-negative, Borel probability measure in K, satisfying (*) whenever S_i is a slab orthogonal to some $\theta \in \Theta$.

At first sight, the definition above seems too restrictive. The inequality $\mu(S_i \cap K) \leq w_i/W_i$ is enough for our purposes; but it is easily shown that this implies (*). Further, it would appear that we only require a finitely additive measure μ defined on the algebra generated by sets of the form $S \cap K$, where S is a slab orthogonal to some $\theta \in \Theta$. Such a measure, satisfying (*), may however be extended to a relative width measure.

After finding the following theorem, I was told that G. Schwarz [36] had already proved it, in a paper couched in the language of probability theory. (But see [15], Theorem 4, for a considerably stronger result.)

THEOREM 7. There is a relative width measure ν_n in the unit ball B^n for S^{n-1} (all directions) in $\mathbb{R}^n$ if and only if $n = 1, 2$, or 3.

We have seen the case $n = 2$ above. For $n = 1$, just take $\frac{1}{2}\lambda_1$ on $[-1,1]$, and for $n = 3$, as Archimedes knew, we have normalized surface area measure in B^3. The interesting part is the non-existence of relative width measures in B^n for all directions for $n \geq 4$. This is proved using Fourier transform techniques.

Theorem 7 still does not conclusively show that relative width measures cannot be applied to Bang's problem. In 1957, D. Ohmann, in another paper [31] on the subject, demonstrated that to solve the question it suffices to consider a convex body K in $\mathbb{R}^n$ covered by n slabs, each orthogonal to one of the coordinate directions. For, suppose there is a cover of a convex body H in $\mathbb{R}^d$ by k slabs T_i with $\sum_i w_i/W_i < 1$. If $k \geq d$, consider H as a set in $\mathbb{R}^k$, covered by k slabs S_i' which project orthogonally onto the corresponding T_i. Fatten the set H slightly to obtain a convex body K', and adjust the directions of the slabs S_i' slightly to obtain slabs S_i'' whose orthogonal directions are linearly independent, in such a way that the sum of the relative widths is still less than one. Now use the affine invariance of relative widths to transform K' and S_i'' to a K and slabs S_i orthogonal to the coordinate directions. If $k < d$, we use projections to get the result.

It transpires, then, that we need only seek relative width measures in K in $\mathbb{R}^n$ for the set of n coordinate directions. For $n = 2$, this turns out to be possible; one can construct such a measure using suitable multiples of λ_1 on certain line segments inside K. This yields the known result (see [29]) that Bang's conjecture is true for 2 slabs.

Unfortunately, the measures do not always exist in $\mathbb{R}^3$, even for the 3 coordinate directions. An example is the tetrahedron with vertices at the origin, (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Apparently, relative width measures cannot, after all, be applied to Bang's problem to achieve a full solution. But even so, some interesting measure theory has been produced; the proofs of the above and other results may be consulted in [15]. Perhaps a different approach using measures may crack the problem, but somehow I doubt it. Even the case of covers in $\mathbb{R}^3$ by 3 slabs, each orthogonal to a coordinate direction, is unresolved and presents difficulties which seem, to me, of a combinatorial and/or algebraic nature.

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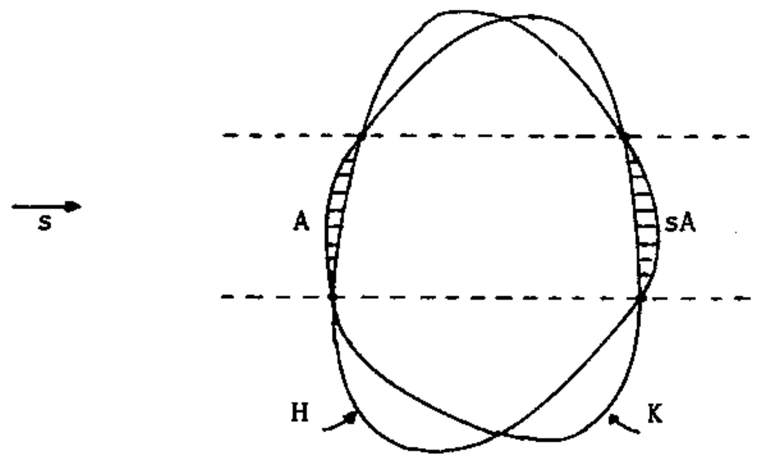

Fig. 1. The Cavalieri Principle

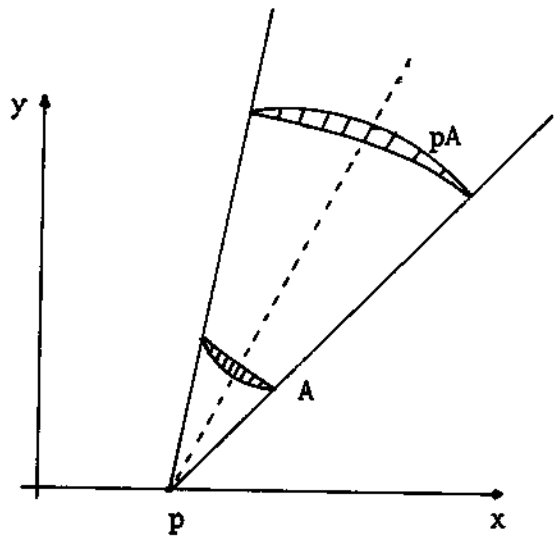

Fig. 2. The Volčič principle

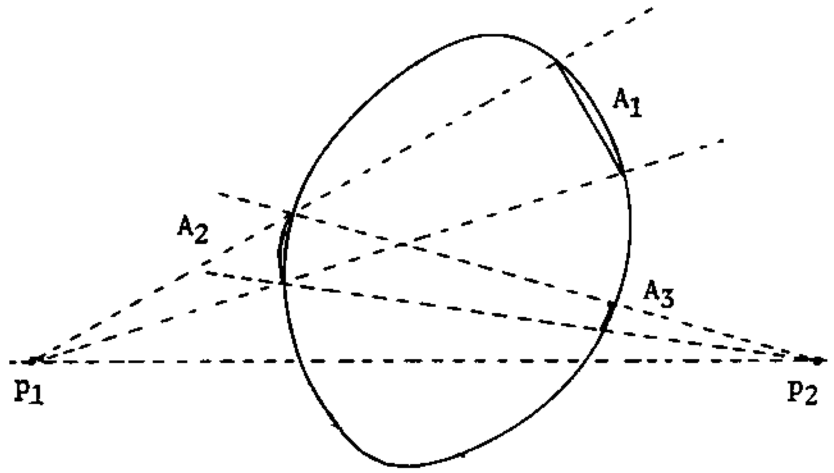

Fig. 3. Chord functions at two points

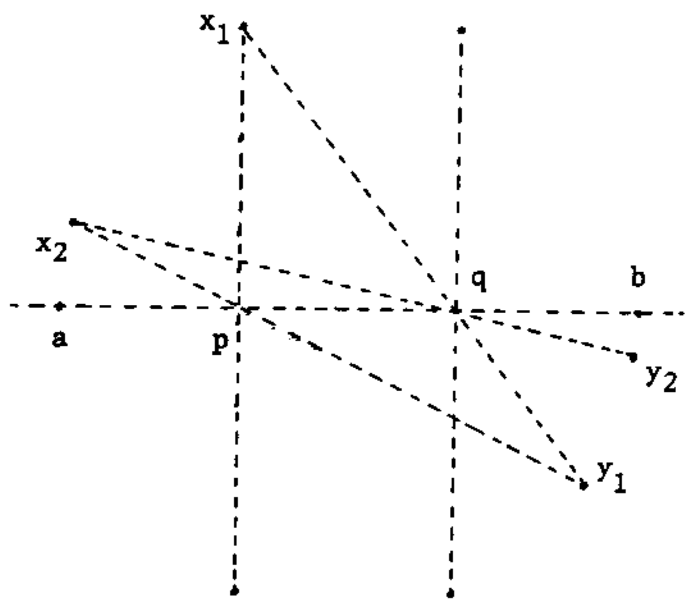

Fig. 4. Framework for an equichordal body

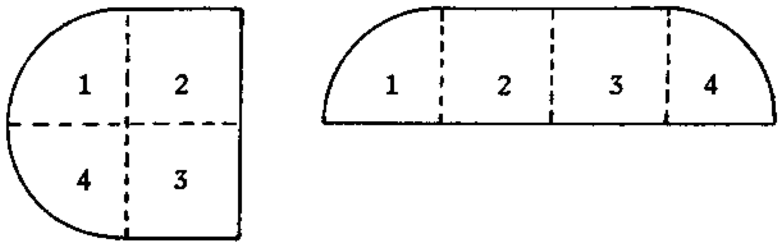

Fig. 5. Convex equidecomposable convex bodies

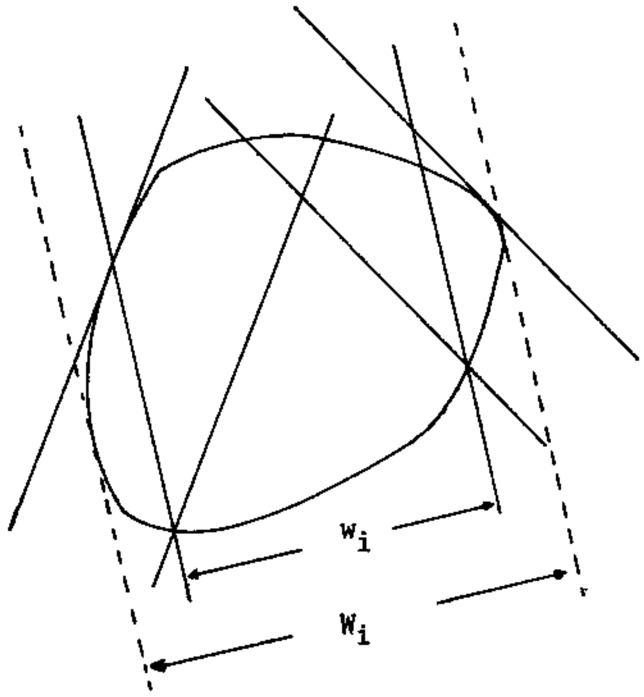

Fig. 6. Relative width of a slab