King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 107

November 1988

A Note on Small and Other Singular Sets

R.J. Gardner and W.F. Pfeffer

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

A NOTE ON SMALL AND OTHER SINGULAR SETS†

R.J. Gardner and W.F. Pfeffer

†Remarks relating to a talk given by W.F. Pfeffer on small subsets of the reals at the Conference on Real Analysis and Measure Theory, Capri, 1988.

0. Introduction.

The construction and study of singular subsets of the reals, uncountable subsets with properties so strange that special set-theoretic axioms must often be assumed for their existence, has a long history (see [4]). Despite this, the introduction of a new type of singular set is a relatively rare event, which therefore always causes interest. Very recently, W.F. Pfeffer and K. Prikry discovered two classes of uncountable sets whose construction is based on certain infinite matrices. These sets, the small and slight sets, are so small that they are of universal measure zero and are also perfectly meager. Furthermore, small sets exist in ZFC, while slight sets (which are also small sets) at present are only available under the continuum hypothesis, CH. Perhaps it is surprising, then, that no example of a small set which is not slight is known.

This note began as an attack on this problem. We introduce two new classes, the Borel slight and Borel small sets, and examine the relations between them, the slight and small sets, and other singular sets. We find that each small set is a λ-set, and that no small set can be concentrated on a countable subset; but apart from this, generally speaking, it is consistent to assume that the various classes are distinct.

By working with σ-ideals instead of ideals in our definitions, we find only one more new class, the σ-small sets. We show that each σ-small set is both a small set and a σ-set, and that the existence of uncountable σ-small sets is undecidable in ZFC.

1. Preliminaries.

If A is a set, then $|A|$ denotes the cardinality of A, and $P(A)$ denotes the family of all subsets of A.

By $\mathbb{R}$ and $\mathbb{Q}$ we denote, respectively, the sets of all real and rational numbers equipped with their usual topology. The families of all closed, open, F_σ, G_δ and Borel subsets of $\mathbb{R}$ are denoted by F, G, F_σ, G_δ and B respectively, and we let $A = F_\sigma \cap G_\delta$. For an $X \subset \mathbb{R}$, we employ the symbols $F(X), G(X)$, etc., whose meaning is obvious.

The Zermelo-Fraenkel set theory including the axiom of choice, continuum hypothesis and Martin's axiom are abbreviated as ZFC, CH and MA, respectively.

1.1 Definition. Let E be an algebra of subsets of a set E. An ideal I in E is called

(i) saturated if each disjoint family $E^* \subset E - I$ is countable;

(ii) a discrimination if it is saturated and $E - \cup I^*$ is uncountable for each countable family $I^* \subset I$;

(iii) a weak discrimination if it is saturated and each uncountable family $I^* \subset I$ contains an infinite subfamily I^{**} for which $E - \cup I^{**}$ is uncountable.

1.2 Definition. An uncountable set $X \subset \mathbb{R}$ is called

(i) small or slight if there is, respectively, no discrimination or weak discrimination in $A(X)$;

(ii) Borel small or Borel slight if there is, respectively, no discrimination or weak discrimination in $\mathcal{B}(X)$.

Small and slight sets were introduced in [5] and [6], in a rather more general setting; they were referred to in those papers as ω_1-small and ω_1-slight subspaces of $\mathbb{R}$ (see [6, 2.1, 5.3 and 5.4]).

The following implications are obvious:

$$\begin{array}{ccc} \text{slight} & \Longrightarrow & \text{small} \\ \Downarrow & & \Downarrow \\ \text{Borel slight} & \Longrightarrow & \text{Borel small} \end{array}$$

We make a technical agreement that all countable sets of reals are slight, small, Borel slight and Borel small (cf. [6, 1.1(i) and 5.3]).

2. Relationship to some classical subsets of the reals.

We shall consider sets – sometimes referred to as singular sets – having the following properties: strong and universal measure zero and perfectly meager; and Sierpinski sets, Q-sets, σ-sets, λ- and λ'-sets and concentrated sets. For their definitions and mutual relationship, we refer to [4, 10.1 and 10.2].

A straightforward modification of the arguments employed in [6, 2.3 and Note to 2.6] yields the following observations:

(i) Each Borel small set has universal measure zero and is perfectly meager.

(ii) Each Q-set is small.

2.1 Proposition. Each small set is a λ-set.

Proof. Let X be a small set, and let $C \subset X$ be countable. Then $\mathcal{I} = \{A \in \mathcal{A}(X): A \cap C = \emptyset\}$ is a saturated ideal in $\mathcal{A}(X)$, and hence it contains a countable subfamily $\mathcal{I}^*$ such that $B = X - \cup\, \mathcal{I}^*$ is countable. As $B \in G_\delta(X)$ and $C \subset B$, it follows that $C \in G_\delta(X)$.

2.2 Remark. The converse of Proposition 2.1 is generally false. Indeed, under CH, there is a Sierpinski set X which is simultaneously a σ-set and a λ'-set (see [4, 10.2, 2.2 and 10.1]). As the outer Lebesgue measure of X is positive, X cannot be Borel small.

2.3 Proposition. Let X be an uncountable set which is concentrated on a countable set $C \subset X$. Then X is not small.

Proof. The family $\mathcal{I} = \{A \in \mathcal{A}(X): A \cap C = \emptyset\}$ is a saturated ideal in $\mathcal{A}(X)$, and each $A \in \mathcal{I}$ is countable; for $X - A$ belongs to $G_\delta(X)$ and contains C. As X is uncountable, $\mathcal{I}$ is a discrimination in $\mathcal{A}(X)$.

2.4 Corollary. The following statements are consistent with ZFC:

(i) There is a set X which is Borel small but not small.

(ii) There is a small set Y which is not a λ'-set.

Proof. By [3], it is consistent with ZFC to have a Q-set Y, with $|Y| = \omega_1$, which is concentrated on $\mathbb{Q}$. Thus Y is a small set, and $X = Y \cup \mathbb{Q}$ is neither a small set nor a λ-set. As Y and $\mathbb{Q}$ belong to $\mathcal{B}(X)$, an argument completely analogous to the proof of [6, 2.4(iii)] shows that X is Borel small.

2.5 Remark. By [6, 2.4(iii)], a set of reals which is a countable union of its relatively closed small subsets is also small. Corollary 2.4(ii) demonstrates that the assumption of relative closedness is essential.

A major question left open in [6] is whether there are small sets which are not slight. As to the relations between these and the other properties, we have the following observations. Let FM be the model in [3] in which the set Y from 2.4 exists. Consider the statement

(S) In FM, each discrete space of cardinality ω_1 is slight.

If (S) is true, then it is easy to see that Y, and hence $X = Y \cup \mathbb{Q}$, is Borel slight; but X is not small. If, on the other hand, (S) is false, then Y is small but not slight, and X is Borel small but not Borel slight.

2.6 Question. Is there a universal measure zero and perfectly meager set which is not Borel small?

A set $X \subset \mathbb{R}$ is said to have Baire order ω_1 if for each $\alpha < \omega_1$ there is a set $B \in \mathcal{B}(X)$ of rank larger than α.

2.7 Proposition (CH). There is a slight set of Baire order ω_1.

Proof. Let E be an Erdös-Hajnal-Milner matrix in $\mathbb{R}$ (see [2, 14.1] or [6, 5.7]). By [6, 5.11], there is a countable point-separating family $\mathcal{H} \subset \mathcal{P}(\mathbb{R})$ such that for each $B \in \mathcal{B} \cup E$ there are subfamilies $\mathcal{H}_n$ and $\mathcal{H}^n$ of $\mathcal{H}$, $n = 1, 2, \ldots,$ with

$$\bigcap_{n=1}^{\infty} \bigcup \mathcal{H}_n = \bigcup_{n=1}^{\infty} \bigcap \mathcal{H}^n = B ,$$

(cf. [7]). If $e: \mathbb{R} \to 2^H$ is the Marczewski evaluation map (see [9]), then by [6, 5.12] the set $X = e(\mathbb{R})$ is slight (as usual, we have identified 2^H with a compact subset of $\mathbb{R}$). According to [1, Theorem 12], the Baire order of H in $\mathbb{R}$ is ω_1, and hence so is the Baire order of X.

Since no uncountable analytic set is perfectly meager, all analytic Borel small sets are countable. Whether there are uncountable coanalytic Borel small sets is undecidable in ZFC. Indeed, both the existence of an uncountable coanalytic Q-set, and the assumption that each uncountable projective set contains a perfect subset, are consistent with ZFC (see [4, 4.3] and [8]).

2.8 Proposition. Each set $Y \subseteq \mathbb{R}$ with $|Y| = \omega_1$ is a continuous image of a small set, and under CH, of a slight set.

Proof. We proceed by a standard argument (see [4, 9.3]). Let $\{y_\alpha: \alpha < \omega_1\}$ be an enumeration of Y, and let $\{x_\alpha:\alpha < \omega_1\}$ be an enumeration of a small set X. By [6, 2.4(i)], $Z = \{(x_\alpha, y_\alpha): \alpha < \omega_1\}$ is a small subspace of $\mathbb{R}^2$, for the projection to the first coordinate is a continuous injection of Z onto X. The projection to the second coordinate is a continuous injection of Z onto Y. Applying the fact that the irrationals are homeomorphic to its square, it is easy to see that Z is a continuous injective image of a small set. Using [6,5.6(i)], the second part is proved analogously.

3. σ-small sets.

A σ-ideal in an algebra of sets E is an ideal I in E such that the union of each countable subfamily of I belongs to I.

3.1. Proposition. Let $\mathcal{E}$ be an algebra of subsets of an uncountable set E which contains all singletons, and let I be a σ-ideal in $\mathcal{E}$. Then I is a discrimination in $\mathcal{E}$ if and only if it is a weak discrimination in $\mathcal{E}$.

Proof. As the converse is trivial, let I be a weak discrimination in $\mathcal{E}$, and let I^* be a countable subfamily of I. Letting $A = \cup I^*$, it suffices to show that $E - A$ is uncountable. Since this is so if A is countable, we assume that A is uncountable. By our assumptions, $\{A - \{x\}: x \in A\}$ is an uncountable subfamily of I. Thus there are distinct $x_n \in A$, $n = 1, 2, \ldots$, such that $E - \cup_n(A - \{x_n\}) = E - A$ is uncountable.

3.2 Definition. A set $X \subset \mathbb{R}$ is called σ-small if there is a saturated σ-ideal in $A(X)$, but no such ideal is a discrimination in $A(X)$.

We note that the analogously defined "Borel σ-small" sets coincide with the Borel small sets of Definition 1.2(ii). Indeed, for an $X \subset \mathbb{R}$, $B(X)$ always contains a saturated σ-ideal ($B(X)$ itself), and if I is a discrimination in $B(X)$, then so is the σ-ideal in $B(X)$ generated by I. Moreover, in view of Proposition 3.1, no new subsets of $\mathbb{R}$ will be introduced by defining "σ-slight" and "Borel σ-slight" sets.

If X is a σ-set, then $A(X) = B(X)$, and so $A(X)$ itself is a saturated σ-ideal in $A(X)$. The next example shows that saturated σ-ideals in $A(X)$ may exist even if $X \subset \mathbb{R}$ is not a σ-set.

3.3 Example (CH). By [4, 5.7] there is a σ-set Y concentrated on a countable set $C \subset \mathbb{R}$. Let $X = Y \cup C$ and $I = \{A \in A(X): A \cap C = \emptyset\}$. Since Y is a σ-set and $G_\delta(Y) \subset G_\delta(X)$, we see that I is a saturated σ-ideal in $A(X)$. As $C \notin G_\delta(X)$, X is not a σ-set.

3.4 Lemma. Let $X \subset \mathbb{R}$, and let I be a σ-ideal in $A(X)$. Then each $A \in I$ is a σ-set.

Proof. If $A \in I$, then $A(A) \subset A(X)$, and so $A(A) \subset I$. In particular, $F(A) \subset I$ and since I is a σ-ideal, $F_\sigma(A) \subset I$ also. Thus

$$F_\sigma(A) \subset G_\delta(X) \cap P(A) = G_\delta(A),$$

and A is a σ-set.

3.5 Corollary. Let $X \subset \mathbb{R}$, and let I be a saturated σ-ideal in $A(X)$. Then X is totally imperfect.

Proof. If X contains a non-empty perfect set, then it contains an uncountable disjoint collection of such sets. As I is saturated, it contains one of them. This contradicts Lemma 3.4.

3.6 Corollary. Each σ-small set is a σ-set.

Proof. If X is a σ-small set, then $A(X)$ contains a saturated σ-ideal I which is not a discrimination in $A(X)$. This implies that there is an $A \in I$ such that $B = X - A$ is countable. Thus by Lemma 3.4, X is a union of two σ-sets from $A(X)$, and the corollary follows.

3.7 Corollary. Each σ-small set is small, and among σ-sets, the σ-small, small and Borel small sets coincide. Moreover, it is consistent with ZFC that there is a small set which is not σ-small.

Proof. If X is σ-small, then by Corollary 3.6, $A(X) = B(X)$ is a σ-algebra in X. Thus if I is a discrimination in $A(X)$, then so is the σ-ideal in $A(X)$ generated by I; a contradiction. By [6, 2.8], uncountable small sets exist in ZFC, while by Corollary 3.6 and [4, 4.3], it is consistent with ZFC that no uncountable σ-small sets exist. The corollary follows.

3.8 Remark. By Proposition 2.7, under CH, there is a slight set which is not a σ-set, and hence not a σ-small set.

3.9 Proposition. Let $X \subset \mathbb{R}$.

(i) If X is σ-small, then so is each $Y \in A(X)$.

(ii) If $X_1, X_2 \in A(X)$ are σ-small and $X = X_1 \cup X_2$, then X is σ-small.

Proof. (i) If I_X is a saturated σ-ideal in $A(X)$, then $I_Y = \{A \in I_X : A \subset Y\}$ is a saturated σ-ideal in Y. If J_Y is a σ-ideal in $A(Y)$ which is a discrimination in $A(Y)$, then $J_X = \{A \in A(X) : A \cap Y \in J_Y\}$ is a σ-ideal in $A(X)$ which is a discrimination in $A(X)$.

(ii) If I_i is a saturated σ-ideal in $A(X_i)$, $i = 1, 2$, then the family of all finite unions of elements from $I_1 \cup I_2$ is a saturated σ-ideal in $A(X)$. If J is a saturated σ-ideal in $A(X)$, then $J_i = \{A \in J : A \subset X_i\}$ is a saturated σ-ideal in $A(X_i)$, $i = 1, 2$. As J_i is not a discrimination in $A(X_i)$, $i = 1, 2$, we see that J is not a discrimination in $A(X)$.

3.10 Proposition. It is undecidable in ZFC whether there are uncountable σ-small sets.

Proof. We have noted that by [4, 4.3], it is consistent with ZFC that all σ-small sets are countable. On the other hand, MA + $\neg$CH implies the existence of uncountable Q-sets, and the proposition follows from Corollary 3.7.

References

1. R.H. Bing, W.W. Bledsoe and R.D. Mauldin, Sets generated by rectangles, Pacific J. Math., 51(1974) 27-36.

2. P. Erdős, A. Hajnal and E. Milner, On the complete subgraphs of graphs defined by systems of sets, Acta Math. Hung., 17(1965) 159-229.

3. W.G. Fleissner and A.W. Miller, On Q sets, Proc. Amer. Math. Soc., 78(1980) 280-284.

4. A.W. Miller, Special subsets of the real line, Handbook of Set-Theoretic Topology, ed. K. Kunen and J.E. Vaughan, North-Holland, Amsterdam, 1984, 201-233.

5. W.F. Pfeffer and K. Prikry, A topological concept of smallness, Proc. Natl. Acad. Sci. USA, 84(1987) 3957-3958.

6. W.F. Pfeffer and K. Prikry, Small spaces, Proc. London Math., Soc., to appear.

7. F. Rothberger, On families of real functions with a denumerable base, Ann. of Math., 45(1944) 397-407.

8. R. Solovay, Real valued measurable cardinals, Proc. Symp. Pure Math. 13, Part I, ed. D. Scott, 1971, 397-428.

9. E. Szpilrajn (Marczewski), The characteristic function of a sequence of sets and some of its applications, Fund. Math., 31(1938) 207-223.

Department of Mathematical Sciences,
King Fahd University of Petroleum and Minerals,
Dhahran 31261,
Saudi Arabia

and Department of Mathematics,
UC Davis,
Davis,
California 95616,
U.S.A.