King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 108

March 1989

Spaces, Another Generalization

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"$\Sigma^{*\prime}$-spaces, another generalization of σ-spaces."

By

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ASBTRACT

The $\Sigma^{*\prime}$-space is defined by weakening some conditions in the definition of Σ-space. The class of $\Sigma^{*\prime}$-space is found to be larger than the class of $\Sigma^{*\prime}$-space. It is found that if X is a $\Sigma^{*\prime}$-space with point countable pseudobase, then it is a σ-space. Another result, if X is a perfect $\Sigma^{*\prime}$-space, then it is Σ-space. A detailed look at $\Sigma^{\#}$-spaces is also given. Some results are developed which can be used to get a σ-discrete refinement of closed sets from a σ-closure preserving collection of closed sets.

In this paper we give a detailed look at $\Sigma^{\#}$-spaces. The class of $\Sigma^{\#}$-spaces includes all metric spaces, σ-spaces, Σ-spaces, and Σ^{*}-spaces.

$\Sigma^{\#}$-spaces were defined first by E. Michael in [4]. Later A. Okuyama in [7] introduced the class of Σ^{*}-spaces which contains the class of Σ-spaces. Σ-space was defined by K. Nagami in [6].

Here are some of the properties of $\Sigma^{\#}$-spaces:

1) A closed image of a $\Sigma^{\#}$-space is a $\Sigma^{\#}$-space "where this is not true for Σ-spaces".

2) The inverse image of a $\Sigma^{\#}$-space under a quasi-perfect map is a $\Sigma^{\#}$-space. [7]

3) If $\{X_i\}_{i\in\omega}$ is a sequence of strong $\Sigma^{\#}$-spaces, then the product space $X = \prod_{i\in\omega} X_i$ is a strong $\Sigma^{\#}$space.

4) If X is a semi-stratifiable $\Sigma^{\#}$-space, then X is a σ-space.

5) If X is a $\Sigma^{\#}$-space and $\sigma^{\#}$-space, then X is a σ-space. [8].

In addition we define a new space we call it $\Sigma^{*\prime}$-space. $\Sigma^{*\prime}$-space is a generalization of Okuyama's Σ^{*}-space. We give example showing that the class of $\Sigma^{*\prime}$-spaces is strictly larger than the class of Σ^{*}-space. Many important results about Σ^{*}-spaces are also valid when applied on $\Sigma^{*\prime}$-spaces.

All spaces considered in this paper are Hausdorff spaces unless stated otherwise. All functions are continuous. A mapping $f:X \longrightarrow Y$ is quasi-perfect if f is closed and $f^{-1}(y)$ is countably compact, $\forall\, y \in Y$. And if $f^{-1}(y)$ is compact then f is called a perfect mapping. For concepts which are not defined here, see [1].

In section 1 we give basic definitions and results which are needed in this paper. Section 2 is devoted mainly to give properties of $\Sigma^{\#}$-spaces. Section 3 contains main results about $\Sigma^{*\prime}$-spaces.

Section 1.

1.1 Definition: Let $\{\mathcal{F}_n\}$ be a sequence of collections of subsets of a topological space X. We say $\{\mathcal{F}_n\}$ is a Σ-net if, whenver $\{K_n\}$ is decreasing sequence of non-empty closed subsets of X such that $\exists\, x \in X$ and $K_n \subset C(x,\mathcal{F}_n)$, $\forall\, n \in \omega$, then $\bigcap_{n\in\omega} K_n \neq \emptyset$ where $C(x, \mathcal{F}_n) = \bigcap\{F \in \mathcal{F}_n: x \in F\}$ and let $C(x) = \bigcap C(x, \mathcal{F}_n)$, $\forall\, x \in X$.

1.2 Definition: A space X is called Σ-space if it has a Σ-net$\{\mathcal{F}_n\}$ such that $\mathcal{F}_n$ is locally finite closed covering of X, $\forall$ n. Moreover if $\forall$ n, $\mathcal{F}_n$ is closure-preserving (hereditarily closure preserving) closed covering then X is called $\Sigma^{\#}$-space ($\Sigma^{*\prime}$-space).

1.3 Definition: Let $\{\mathcal{F}_n\}$ be a sequence of hereditarily closure preserving closed covering for a space X. X is called a <u>Σ^*-space</u> if $C(x)$ is countably compact $\forall\, x \in X$ and whenever U is an open set containing $C(x)$ there is $k \in \omega$ and $F \in \mathcal{F}_k$ such that $x \in F \subset U$.

If X is a $\Sigma^{\#}$-space such that $\forall\, x \in X$, $C(x) = \{x\}$, then X is called a <u>σ-space</u>.

In all of the above definitions observe that $C(x)$ is countably compact, $\forall\, x \in X$. A Σ-space (Σ^*, $\Sigma^{*\prime}$, $\Sigma^{\#}$-space) is called strong Σ-space (Σ^*, $\Sigma^{*\prime}$, $\Sigma^{\#}$-space) if $C(x)$ is compact.

A space X is called $\sigma^{\#}$-space if there is a sequence $\{\mathcal{F}_n\}$ of closure-preserving collections of closed sets such that for every $x \in X$, $\{x\} = \cap\{F : x \in F \in \mathcal{F}_n\}$ for some $n \in \omega$.

1.4 Definition: A space X is semi-stratifiable if $\forall\, x \in X$, there is a sequence $\{g_i(x)\}$ of neighborhoods of x such that:

a. $\bigcap_{i \in \omega} g_i(x) = \{x\}$.

b. If $x \in g_i(x_i)$, $\forall\, i \in \omega$, then $x_i \longrightarrow x$.

1.5 Lemma: Let X be a $\Sigma^{\#}$-space. Then X has a Σ-net $\{\mathcal{F}_n\}$ which satisfies the following:

a. $\mathcal{F}_n$ is multiplicative, i.e., for every $\mathcal{H}_n$ a subcollection of $\mathcal{F}_n$, $\cap\mathcal{H}_n$ is also in $\mathcal{F}_n$.

b. $\mathcal{F}_n$ is closure-preserving.

c. $\mathcal{F}_{n+1} \supset \mathcal{F}_n$.

d. $C(x, \mathcal{F}_n) \in \mathcal{F}_n$.

<u>Proof</u>: Let $(\mathcal{S}_n)$ be a Σ-net illustrating that X is a $\Sigma^{\#}$-space. Let $\mathcal{F}_1$ be the collection of all intersections of members of $\mathcal{S}_1$. To form $\mathcal{F}_{n+1}$, $n \geq 1$, use the following iterative process:

Suppose $\mathcal{F}_i$, $i \leq n$ are defined and satisfy (a) - (d) above. Let $A_{n+1} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_n \cup \mathcal{S}n+1$. Clearly A_{n+1} is closure preserving. Let $\mathcal{F}_{n+1}$ be the collection of all intersection of members of A_{n+1}. Evidently each $\mathcal{F}_n$ is locally finite, multiplicative, and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Since $C(x, \mathcal{F}_n) = \cap(F \in \mathcal{F}_n: x \in F)$ and $\mathcal{F}_n$ is multiplicative, then $C(x, \mathcal{F}_n) \in \mathcal{F}_n$. It remains to show that the sequence $(\mathcal{F}_n)$ is a Σ-net. Observe $\mathcal{S}_n \subseteq \mathcal{F}_n$, $\forall$ n. So $C(x, \mathcal{S}_n) \supseteq C(x, \mathcal{F}_n)$. So if (K_n) is a decreasing sequene of closed sets with $K_n \subset C(x, \mathcal{F}_n)$, then $K_n \subset C(x, \mathcal{S}_n)$, $\forall$ n. Since $\mathcal{S}_n$ is a Σ-net, $\cap K_n \neq \phi$. Thus $(\mathcal{F}_n)$ is a closure preserving closed Σ-net satisfying conditions (a) - (d) above.

If X is a Σ-space, the above proof can be used to get a σ-locally finite Σ-net $(\mathcal{F}_n)$ of locally finite connections of closed sets which satisfies a, c and d in the above lemma.

Throughout this paper whenever we say $(\mathcal{F}_n)$ is a Σ-net for $\Sigma^{\#}$-space (or Σ-space) X we assume the Σ-net $(\mathcal{F}_n)$ satisfy the conditions in the above lemma. We observe the following "If $(\mathcal{F}_n)$ is a Σ-net for a space X, then for any $x \in X$, $(C(x,\mathcal{F}_n)$:

$n \in \omega\}$ is an outer network for $C(x)$, i.e., if O is an open set containing $C(x)$ then there is some n so that $C(x, \mathcal{F}_n) \subseteq O$."

1.6 Lemma: If X is a Σ^*-space then it is a $\Sigma^{*\prime}$-space.

Proof: Let $\{\mathcal{F}_n\}$ be a σ-hereditarily closure preserving collections of closed sets illustrating that X is a Σ^*-space. Let $\{K_n\}$ be a decreasing sequence of closed sets such that $\exists\ x \in X$ and $K_n \subset C(x, \mathcal{F}_n)$, $\forall\ n$. We need to show that $\cap K_n \neq \phi$. If $\cap K_n = \phi$, then $\bigcup_{n \in \omega}(X - K_n) = X - \bigcap_{n \in \omega} K_n = X$ so $C(x) \subset \bigcup_{n \in \omega}(X - K_n)$. Since $C(x)$ is countably compact, $\exists$ m such that $C(x) \subset \bigcup_{n=1}^{m}(X - K_n)$. Observe that $\bigcap_{n=1}^{m} K_n = K_m$ so $C(x) \subset X - \bigcap_{n=1}^{m} K_n = X - K_m$. By the definition of Σ^*-space, there is $j \in \omega$ and $F \in \mathcal{F}_j$ with $x \in F \subset X - K_m$, but $K_j \cap K_m \neq \phi$ and $K_j \subset F$. So $F \cap K_m \neq \phi$. A contradiction. So $\bigcap_{n \in \omega} K_n \neq \phi$. Hence X is a $\Sigma^{*\prime}$-space.

Clearly every $\Sigma^{*\prime}$-space is a $\Sigma^{\#}$-space, and every Σ-space is a Σ^*-space so now we have Σ-space $\longrightarrow$ Σ^*-space $\longrightarrow$ $\Sigma^{*\prime}$-space $\longrightarrow$ $\Sigma^{\#}$-space.

2. $\Sigma^{\#}$-Spaces

2.1 Theorem: Let $f: X \to Y$ be a closed continuous map from a $\Sigma^{\#}$-space X onto a topological space Y. Then Y is a $\Sigma^{\#}$-space.

Proof: Let $\{\mathcal{F}_i\}$ be a sequence of closure-preserving closed covering of X which is also a Σ-net for X. Then $f(\mathcal{F}_i) = \{f(F): F \in \mathcal{F}_i\}$ is a collection of closed sets $\forall$ i. Since, for every subcollection $\mathcal{F}' \subset \mathcal{F}_i$, $f(\cup \mathcal{F}') = \cup\{f(F): F \in \mathcal{F}'\}$ then $f(\mathcal{F}_i)$ is closure-preserving, $\forall$ i. Observe that $\forall x \in X$, $f(x) \in f(C(x, \mathcal{F}_i))$, so $C(f(x), f(\mathcal{F}_i)) \subseteq f(C(x, \mathcal{F}_i))$, this is true because $C(x, \mathcal{F}_i) \in \mathcal{F}_i$. Now if $\{K_i\}$ is a sequence of closed sets in Y such that $K_i \in C(y, f(F_i))$, for some $y \in Y$, then $\cap K_i \neq \emptyset$. This is because for any sequence $\{y_i\}$ such that $y_i \in K_i$, there is $x \in f^{-1}(y)$ and a sequence $\{x_i\}$ such that $f(x_i) = y_n$, $\forall$ i. But $\{\mathcal{F}_i\}$ is a Σ-net, so the sequence $\{x_i\}$ clusters. Consequently the sequence $\{y_i\}$ clusters also. Hence $\cap K_i \neq \emptyset$, i.e., $f(\mathcal{F}_i)$ is a Σ-net for Y. Then Y is a $\Sigma^{\#}$-space.

2.2 Theorem: Let $f: X \longrightarrow Y$ be a closed continuous mapping from a Σ-space X onto Y. Then Y is at least a Σ^*-space.

<u>Proof</u>: Let the sequence $\{\mathcal{F}_n\}$ be a Σ-net for X, illustrating that X is a Σ-space. $\forall$ n, let $f(\mathcal{F}_n) = \{f(F): F \in \mathcal{F}_n\}$. $f(\mathcal{F}_n)$ is hereditarily closure-preserving. To see this suppose that $\{B_\alpha: \alpha < \lambda$ and $B_\alpha \subset f(F_\alpha) \in f(\mathcal{F}_n)\}$ is a collection of closed sets. $\bigcup_{\alpha<\lambda} B_\alpha = \bigcup_{\alpha<\lambda} f(f^{-1}(B_\alpha) \cap F_\alpha)$, since $f^{-1}(B_\alpha \cap F_\alpha)$ is a closed subset of F_α, $\forall$ $\alpha < \lambda$, then $\bigcup_{\alpha<\lambda}(f^{-1}(B_\alpha) \cap F_\alpha)$ is closed in X. But f is closed and $f[\bigcup_{\alpha<\lambda}(f^{-1}(B_\alpha) \cap F_\alpha)] = \bigcup_{\alpha<\lambda} f(f^{-1}(B_\alpha) \cap F_\alpha) = \bigcup_{\alpha<\lambda} B_\alpha$. Hence $f(\mathcal{F}_n)$ is hereditarily closure-preserving closed collection. $\{f(\mathcal{F}_n)\}$ is a Σ-net for Y as in the proof of theorem 2.1. Observe that $C(x, \mathcal{F}_n) \in \mathcal{F}_n$, $\forall$ n, $\forall$ x, so $f(C(x, \mathcal{F}_n))$ is in $f(\mathcal{F}_n)$.

Hence Y is a Σ^*-space.

There is an example in [4] which shows that a closed image of a Σ-space is not a Σ-space. On the other hand, Nagami in [6] was able to get the following theorem.

2.3 Theorem [6]: Let $f: X \longrightarrow Y$ be a quasi-perfect mapping from a space X onto a space Y. Then X is a Σ-space iff Y is a Σ-space.

2.4 Theorem: If $\{X_i\}$ is a sequence of strong $\Sigma^\#$-spaces, then $X = \prod_{i \in \omega} X_i$ is a strong $\Sigma^\#$-space.

<u>Proof</u>: For each $i \in \omega$, let $\{\mathcal{F}_{ij}\}$ be the Σ-net associated with X_i which illustrates that X_i is a strong $\Sigma^\#$-space. For each $n = 1,2,\ldots$ define $\mathcal{F}_n$ by

$$\mathcal{F}_n = \mathcal{F}_{1n} \times \mathcal{F}_{2n} \times \ldots \times \mathcal{F}_{nn} \times \prod_{k>n} X_k = \{F_{1n} \times F_{2n} \times \ldots \times F_{nn} \times \prod_{k>n} X_k : F_{1n} \in \mathcal{F}_{1n},\ F_{2n} \in \mathcal{F}_{2n}, \ldots, F_{nn} \in \mathcal{F}_{nn}\}.$$

Note that:

(a) $\forall\, n$, $\mathcal{F}_n$ is a closure-preserving covering of X by closed sets.

(b) $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n \subset \ldots$.

(c) If for each i, we choose $x_i \in X_i$ and form $x \in X$ so that $(x)_i = x_i$ for all i, then $C(x) = \bigcap_{i \in \omega} C(x, \mathcal{F}_i) = \prod_{i \in \omega} C(x_i)$.

We claim that $\{\mathcal{F}_n\}$ illustrates that X is a strong $\Sigma^\#$-space. In light of (a) and (c), it is only necessary to prove that $\{\mathcal{F}_n\}$ is a Σ-net for X. Let $x \in X$. We will show that $\{C(x,\mathcal{F}_n)\}$ is an outer network for $C(x)$. Thus let O be an open neighborhood of $C(x)$ in X. Since $C(x)$ is the product of compact sets, we choose a positive integer m and open sets $O_1, O_2, \ldots, O_m$ so that

(a') $C((x)_i) \subset O_i \subset X_i, \quad 1 \le i \le m.$

(b') $O_1 \times O_2 \times \ldots \times O_m \times \prod_{j>m} X_j \subset O.$

For each $i = 1,2,\ldots,m$, let k_i be a positive integer so that $C((x)_i) \subset C((x)_i, \mathcal{F}_{ik_i}) \subset O_i$. Let $n = \max.\{k_1, k_2, \ldots, k_m, m\}$. Then $C(x) \subseteq C(x, \mathcal{F}_n) = C((x)_1, F_{1n}) \times C((x)_2, \mathcal{F}_{2n}) \times \ldots \times C((x)_n, \mathcal{F}_{nn}) \times \prod_{j>n} X_j \subseteq C((x)_1, \mathcal{F}_{1k1}) \times \ldots \times C((X)_m, \mathcal{F}_{1k_m}) \times \prod_{j>m} X_j \subseteq O_1 \times O_2 \times \ldots \times O_m \times \prod_{j>m} X_j \subset O$. It follows that $\{\mathcal{F}_n\}$ is a Σ-net for X.

Lemma 2.6 was stated in Michael's paper [4], but he did not give the proof. So we present the proof here, and we observe that the proof depends on a result which we generalize in the following lemma.

2.5 Lemma: Let X be a topological space with a Σ-net $\{\mathcal{F}_n\}$ such that, $\forall\, n$, $\mathcal{F}_n$ is point-finite and countable. Assume in addition that $\forall\, x \in X$, $C(x)$ is compact. Then X is Lindelof.

Proof: First we are going to construct a Σ-net $\{\mathcal{H}_n\}$ such that $\forall\, n$, $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $\mathcal{H}_n < \mathcal{F}_n$ and $C(x, \mathcal{H}_n) \in \mathcal{H}_n$, $\forall\, x \in X$. To do this, let $\mathcal{F}_1'$ be the collecltion of all finite intersections of members of $\mathcal{F}_1$. Then $\forall$ n, define $\mathcal{F}_{n+1}' = \mathcal{F}_{n+1} \cup \mathcal{F}_n'$. Let $\mathcal{H}_n$ be the collection of all finite intersections of members of $\mathcal{F}_n'$. Since $\mathcal{F}_n$ is point-finite, then $C(x,\mathcal{F}_n) \in \mathcal{H}_n$ and consequently $C(x, \mathcal{H}_n) \in \mathcal{H}_n$, $\forall$ x, $\forall$ n. Observe that $\mathcal{H}_n$ is countable $\forall$ n. Moreover $\bigcap_{n\in\omega} C(x,\mathcal{F}_n) = \bigcap_{n\in\omega} C(x,\mathcal{H}_n) = C(x)$.

We are now better equipped to prove X is Lindelof. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{O}$ be the set of all finite subcollections of $\mathcal{U}$. $\forall$ $n \in \omega$ let $\mathcal{H}_n' = \{H \in \mathcal{H}_n$: there is $O(H) \in \mathcal{O}$, such that $H \subset \cup O(H)\}$. Let $\mathcal{H}' = \bigcup_{n\in\omega} \mathcal{H}_n'$ and let $\mathcal{V} = \{\cup O(H): H \in \mathcal{H}'\}$. Then $\mathcal{V}$ is a countable cover for X. Countability of $\mathcal{V}$ is clear. To see $\mathcal{V}$ is a cover, let $x \in X$, then there is $O \in \mathcal{O}$ such that $C(x) \subset \cup O$. Since the sequence $\{C(x, \mathcal{H}_n\}$ is an outer network for $C(x)$, there exists $m \in \omega$ such that $C(x, \mathcal{H}_m) \subset \cup O$. But there is $H \in \mathcal{H}_m'$ such that $C(x, \mathcal{H}_m) = H$ and $H \in \mathcal{H}_m$, and so there is $O(H) \in \mathcal{O}$ such that $H \subset \cup O(H)$. Hence $H \in \mathcal{H}'$ and so $\mathcal{V}$ is a countable cover of X. Let $\mathcal{W} = \{u \in \mathcal{U}: u \in O(H)$ for some $H \in \mathcal{H}'\}$. Then $\mathcal{W}$ is a countable subcover of $\mathcal{U}$. Thus X is Lindelof.

2.6 Lemma [4]: In a regular, strong Σ-space every point is contained in a Lindelof, closed G_δ-set.

Proof: Let X be a regular, strong Σ-space with $\{\mathcal{F}_n\}$ a sequence of locally-finite closed covers of X so that $\{\mathcal{F}_n\}$ is a Σ-net for X. Let $x \in X$. $\forall$ $n \in \omega$ set $U_n(x) = X - \cup[\mathcal{F}_n - (\mathcal{F}_n)_x]$. Clearly $U_n(x)$ is open $\forall$ n. Using regularity, we can find a sequence of open neighborhoods of x $\{P_i\}$, such that

$P_1 \subset \overline{P}_1 \subset U_1(x)$ and $P_n \subset \overline{P}_n \subset (P_{n-1} \cap U_n(x))$, for $n > 1$. Observe that $\bigcap_{n\in\omega} P_n = \bigcap_{n\in\omega} \overline{P}_n = P_x$, and so P_x is a closed G_δ-set. $\forall\, n \in \omega$, let $\mathcal{H}_n = \mathcal{F}_n | P_x$. Then $\{\mathcal{H}_n\}$ a Σ-net and $\mathcal{H}_n$ is finite, $\forall\, n \in \omega$. Moreover, $\forall\, y \in P_x$, $C(y, \mathcal{H}_n) = C(y, \mathcal{F}_n) \cap P_x$. Set $D(y) = \bigcap_{n\in\omega} C(y, \mathcal{H}_n) = \bigcap_{n\in\omega} C(y, \mathcal{F}_n) \cap P_x$. This implies that $D(y)$ is a compact subset of P_x. Using Lemma 2.5 on the subspace P_x, yields that P_x is Lindelof. Hence the theorem.

The following lemma is a generalization of Lemma 2.6.

2.7 Lemma: In a strong Σ-space every countable compact set is contained in a Lindelof, closed G_δ-set.

Proof: Let X be a strong Σ-space, with Σ-net $\{\mathcal{F}_n\}$ illustrating that X is a strong Σ-space. Let K be a countably compact subset of X. Set $Y = \{\{x\}: x \in X\sim K\} \cup \{K\}$. Y is the quotient space obtained by identifying K to a point. Topologize Y with the quotient topology. Consider the quotient mapping $f:X \longrightarrow Y$. Then f is a quasi-perfect map. By Theorem 2.3 Y is a strong Σ-space. Using Lemma 2.6 there is a Lindelof closed G_δ-set P containing the point $\{K\}$. Denote $f^{-1}(P)$ by Q. Then Q is a closed Lindelof G_δ-subset of X containing K. This completes the proof.

A. Okuyama in [7] introduced a space which is a $\Sigma^{\#}$-space but not a Σ^*-space. In Section 3 we give an example of a $\Sigma^{*\prime}$-space which is not a Σ^*-space. So we have the following relation: Σ^*-space $\Rightarrow \Sigma^{*\prime}$-space $\Rightarrow \Sigma^{\#}$-space and $\Sigma^{*\prime}$-space $\not\Rightarrow \Sigma^*$-space. So

$\Sigma^{\#}$-space $\not\Rightarrow$ Σ^{*}-space.

Section 3

3.1 Definitions: A relation R on space X is a <u>neighbornet</u> for X if $R(x) = \{y \in X: (x,y) \in R\}$ is a neighborhood of x, $\forall\, x \in X$. A sequence $\langle R_n \rangle$ of relations on X is called <u>basic</u> (co-basic) if for each $x \in X$, the sequence $\langle R_n(x) \rangle$ $(\langle R_n^{-1}(x) \rangle)$ is a network at x.

H. Junnila in [3] gave the following important result about producing a discrete collection using certain neighbornets.

3.2 Lemma [3]: Let U be a neighbornet of X such that $R = U \cap U^{-1}$ is an equivalence relation and let V be an open neighbornet contained in U. Denote by H the set $\{x \in X: V^{-1}(x) \subset U(x)\}$. Then the family $\{H \cap R(x): x \in X\}$ is closed and discrete.

3.3 Lemma [3]: A space X is semi-stratifiable iff X has a co-basic sequence of neighbornets.

In this section we will give results about getting a σ-discrete collection of closed sets from a σ-closure preserving collection of closed sets. Then we will use that to prove several results about $\Sigma^{\#}$-spaces and $\Sigma^{*\prime}$-spaces.

3.4 Construction: Let X be a closure-preserving collection of closed sets which cover X. Assume $\mathcal{F}$ is closed under arbitrary intersections. Define the following relations C, K, B and U

on X such that $\forall x \in X$, $C(x) = \cap\{F: x \in F \in \mathcal{F}\} = C(x,\mathcal{F})$, $K(x) = \{z \in X: C(x) = C(z)\}$, $B(x) = \{y \in C(x): C(y) \subsetneq C(x)\}$, and $U(x) = \{y \in X: x \in C(y)\} = C^{-1}(x)$. Let $\mathcal{C} = \{C(x): x \in X\}$ which denote the set of all distinct $C(x,\mathcal{F})$'s. We note the following $\forall x \in X$,

a) $C(x) \in \mathcal{F}$

b) If $y \in C(x) \Rightarrow C(y) \subset C(x)$.

c) K is an equivalence relation on X and $\forall x \in X$, $K(x) \subset C(x)$.

d) $B(x)$ is closed and $B(x) = \cup\{C(y): C(y) \subsetneq C(x)\}$

e) $B(x) \cap K(x) = \emptyset$

f) $U(x) = \cup\{K(y): x \in C(y)\}$

g) $K(x) = U(x) \cap C(x)$

h) If $y \in U(x)$, then $U(y) \subset U(x)$

i) If $y \in B(x)$, then $U(x) \subset U(Y)$

j) Set X be the collection of all distinct members of $\{K(x): x \in X\}$. Index $\mathcal{X}$ be some cardinal Γ such that $\mathcal{X} = \{K_\alpha: \alpha < \Gamma\}$. Since for each K_α there is a unique C_α such that $K_\alpha = K(x_\alpha) \Rightarrow C(x_\alpha) = C_\alpha$. Also for each $a < \Gamma$, define $B(K_\alpha) = B(x_\alpha)$ for some $x_\alpha \in K_\alpha$. Observe that if $\{x,x'\} \subset K_\alpha$, then $B(x) = B(x')$. We can make the association $\alpha \longrightarrow K_\alpha$. Unlike the associations $\alpha \longrightarrow K_\alpha$ and $\alpha \longrightarrow C_\alpha$ which are 1-1, the association $\alpha \longrightarrow B(K_\alpha)$ need not be 1-1. Denote by $\mathcal{B}$ the set of all distinct members of $\{B(K_\alpha): \alpha < \Gamma\}$.

k) Define $\alpha(x) = |\{B \in \mathscr{B}: x \in B\}|$, and define $R(x) = \{y \in B(x): \alpha(y) \geq \omega\}$. Then we observe that $R(x) = \cup\{C(y): y \in R(x)\}$, and hence $R(x)$ is a closed set.

l) If $y \in C(x)$, then $R(y) \subset R(x)$ and if $C(x) = C(y)$ then $R(x) = R(y)$.

m) We will apply the constructions which are described above to each of the covers $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n, \ldots$. Thus we define the following:

Let X be a topological space with $\mathscr{F}$ a σ-closure preserving covers by closed sets i.e., $\mathscr{F} = \bigcup_{n \in \omega}$ and each $\mathscr{F}_n$ is a closure-preserving closed cover of X. We assume that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$, and $\mathscr{F}_n$ is closed under arbitrary intersections, $\forall$ n. We define the following:

For all $n \in \omega$ and $x \in X$, $C_n(x) = \cap(\mathscr{F}_n)_x$

$B_n(x) = \cup\{C_n(y): C_n(y) \subsetneq C_n(x)\}$

$K_n(x) = \{y \in C_n(x): C_n(y) = C_n(x)\}$

$U_n(x) = \cup\{K_n(y): x \in C_n(y)\}$.

Likewise, let $\alpha_n(x)$, $R_n(x)$, $\forall\, n \in \omega$, $\forall\, x \in X$ be defined as in (k).

n) For each $n \in \omega$ set

$\mathcal{K}_n = \{K_n(x): x \in X\} = \{K_{n\alpha}: \alpha < \partial_n\}$

$\mathcal{C}_n = \{C_n(x): x \in X\} = \{C_{n\alpha}: \alpha < \partial_n\}$

$\mathcal{B}_n = \{B_n(x): x \in X\} = \{B_{n\alpha}: \alpha < \partial_n'\}$ for some

cardinals ∂_n, ∂_n' and $\partial_n' \le \partial_n$.

o) $\forall\ n \in \omega$, and $\forall\ x \in X$, $C_{n+1}(x) \subset C_n(x)$, $U_{n+1}(x) \subset U_n(x)$, and $K_{n+1}(x) \subseteq K_n(x)$. This is true because $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

p) $\forall\ x \in X$, $\forall\ n \in \omega$, $B_n(x) \cap C_{n+1}(x) \subset B_{n+1}(x)$.

Proof: Observe that for any $k \in \omega$,

$$B_k(x) = \{y \in C_k(x): x \notin C_k(y)\}.$$

Thus if $y \in B_n(x) \cap C_{n+1}(x)$, then $x \notin C_n(y)$. Since $C_{n+1}(y) \subset C_n(y)$, $x \notin C_{n+1}(y)$. But $y \in C_{n+1}(x)$ so $y \in B_{n+1}(x)$.

3.5 Lemma: Let $\mathcal{F}$ be a closure-preserving closed cover of a space X. Then the set $Y = \{C(x,\mathcal{F}): B(x) = \emptyset\}$ is discrete in X.

Proof: Let $y \in X$ and let $U_y = X - \cup\{F \in \mathcal{F}: y \notin F\}$. Then U_y is an open neighborhood of y. Suppose $z \in C(x,\mathcal{F}) \cap U_y$, then $C(z,\mathcal{F}) = C(x,\mathcal{F})$ because $B(x) = \emptyset$. Evidently $C(y,\mathcal{F}) \subseteq C(z,\mathcal{F})$ but $B(x) = B(z) = \emptyset$, so $C(y,\mathcal{F}) = C(z,\mathcal{F})$. Hence $C(y,\mathcal{F}) = C(x,\mathcal{F})$, i.e., whenever U_y intersects $C(x,\mathcal{F})$ for

$C(y,\mathcal{F}) = C(x,\mathcal{F})$, i.e., whenever U_y intersects $C(x,\mathcal{F})$ for some $x \in X$ we get $C(x,\mathcal{F}) = C(y,\mathcal{F}) \Rightarrow U_y$ intersects at most one member of Y. So Y is discrete.

The following theorem is needed to prove some important results. Let $\mathcal{C}$, $\mathcal{B}$, K and $R(x)$ be as defined in 3.4j and 3.4k.

3.6 Theorem: Let X be a perfect space. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two closure preserving covers of closed sets, such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Let $P \subset X$, such that $P = \{x \in X: R_2(x)$ is covered finitely by $K_1\}$. Then P is closed and can be expressed as the union over a σ-discrete collection of closed sets $\{\mathcal{K}_n\}$ that refines $\mathcal{F}_1$ and $C_n(x) \subset C_1(x)$, $\forall\, x \in P$.

Proof: Recall that a topological space is said to be perfect provided that each of its closed subsets is a G_δ-set. Let $\mathcal{K}_1 = \{K_{1\alpha}: a < \Gamma_1\}$, $\mathcal{K}_2 = \{K_{2\alpha}: \alpha < \Gamma_2\}$. Also let $\mathcal{C}_1 = \{C_{1\alpha}: \alpha < \Gamma_1\}$, and $\mathcal{C}_2 = \{C_{2\alpha}: \alpha < \Gamma_2\}$. And for some cardinals Γ_1', Γ_2', let $\mathcal{B}_1 = \{B_{1\alpha}: \alpha < \Gamma_2'\}$ and $\mathcal{B}_2 = \{B_{2\alpha}: \alpha < \Gamma_2'\}$. For every $C_{i\alpha} \in \mathcal{C}_i$, let $\{V_k'(C_{i\alpha})\}$ be a decreasing sequence of open sets such that $\bigcap_{k\in\omega} V_k'(C_{i\alpha}) = C_{i\alpha}$, $i = 1,2$, and for every $B_{i\beta} \in \mathcal{B}_i$, let $\{U_k(B_{i\beta})\}$ be a decreasing sequence of open sets, such that $B_{i\beta} = \bigcap_{k\in\omega} U_k(B_{i\beta})$.

<u>Claim 1</u>: For $i = 1,2$, $x \in X$, if $\alpha_i(x) < \omega$, then there is a sequence of open sets $\{V_k(C_i(x))\}$ such that $C_i(x) = \bigcap_{k\in\omega} V_k(C_i(x))$. And, if $x \in B_{1\alpha}$ for some α, then $V_k(C_i(x)) \subset$

$U_k(B_{i\alpha})$, $\forall$ k.

Proof of Claim 1: Since $\alpha_i(x) < \omega$, so we can express $(\mathcal{B}_i)_x$ as $\{B_{i\alpha_1}, \ldots, B_{i\alpha_n}\}$. For every $j = 1,\ldots,n$ since $x \in B_{i\alpha_j}$, then $C_i(x) \subset B_{i\alpha_j}$ by 3.4d. Note that for every $x \in X$, there is a unique $\alpha < \Gamma_i$ such that $C_{i\alpha} = C_i(x)$. For every $k \in \omega$, define the open set $V_k(C_i(x)) = V_k'(C_{i\alpha}) \cap [\bigcap_{j=1}^{n} U_k(B_{i\alpha_j})]$. Clearly $\bigcap_{k\in\omega} V_k(C_i(x)) = C_i(x)$. If there is $B_{i\alpha} \in (\mathcal{B}_i)_x$, then $\alpha = \alpha_j$ for some $j \in \{1,\ldots,n\}$, then $U_k(B_{i\alpha}) = U_k(B_{i\alpha_j})$, $\forall\, k \in \omega$. Thus $C_i(x) \subset U_k(B_{i\alpha})$, $\forall\, k \in \omega$.

If $x \in X$ but $\alpha_i(x) \geq \omega$, then there is a unique $\alpha < \Gamma_i$ such that $C_{i\alpha} = C_i(x)$. Let $V_k(C_i(x)) = V_k'(C_{i\alpha})$, $\forall\, k \in \omega$.

For every $x \in X$, define the sequence of open sets $\{V_k''(C_2(x))\}$ as follows:

$$V_k''(C_2(x)) = V_k(C_1(x)) \cap V_k(C_2(x)), \quad \forall\, k \in \omega.$$

$\forall\, x \in X$, define $W_k(x) = V_k''(C_2(x)) \cap U_2(x)$, $\forall\, k \in \omega$, where $U_2(x) = \{y \in X : x \in C_2(y)$ as defined in 3.4$\}$.

Now we show P is closed. Let $x \in P$. Then $R_2(x)$ is covered finitely by $\mathcal{X}_1$. If $y \in C_2(x)$ then $R_2(y) \subset R_2(x)$. So $R_2(y)$ is covered finitely by $\mathcal{X}_1$. Thus $y \in P$. Hence $P = \cup\{C_2(x) : x \in P\}$ which is a closed subset of X.

Claim 2: For every $x \in P$, there is $k \in \omega$ such that $W_k^{-1} \subset U_1(x)$, where $W_k^{-1}(x) = \{y \in X: x \in W_k(y)\}$.

Proof of Claim 2: For every $x \in X$, and $k \in \omega$, $W_k(x) \subset U_1(x)$. So $W_k^{-1}(x) \subset C_1(x)$. Let $x \in P$. We will show $\exists\, k \in \omega$ such that $W_k^{-1}(x) \subset U_1(x)$. We need to show that there is $k \in \omega$ so that $\forall\, y \in B_1(x)$, then $x \notin W_k(y)$. Let $y \in B_1(x)$. We have three cases:

Case 1. Suppose $y \in B_2(x) \cap B_1(x) - R_2(x)$. Then $\alpha_2(y) < \omega$. By claim 1, $V_k''(C_2(x)) \subseteq U_k(B_2(x))$, $\forall\, k \in w$. But there is $k_o \in \omega$ such that $x \notin U_{k_o}(B_2(x))$. So $x \notin V''_{k_o}(C_2(y))$. So $\forall\, y \in B_2(x) \cap B_1(x) - R_2(x)$, $x \notin V''_{k_o}\ (C_2(k))$.

Case 2. Suppose $y \in R_2(x) \cap B_1(x)$. Since $R_2(x)$ is finitely covered by $\mathfrak{X}_1$, then $R_2(x) \cap B_1(x)$ is covered minimally by $\mathfrak{X}' = \{K_{1\alpha_1}, \ldots, K_{1\alpha_s}\}$. We observe that $x \notin \bigcup_{i=1}^{s} C_{1\alpha_i}$. So $\forall\, i$, $i = 1,\ldots,s$, $\exists\, k_i \in \omega$ such that $x \notin V''_{k_i}(C_{1\alpha_i})$. Since for every $z \in K_{1\alpha_i}$, $C_1(z) = C_{1\alpha_i}$ and $V''_{k_i}(C_1(z)) = V_{k_i}''(C_{1\alpha_i})$, so $x \notin V_{k_i}''(C_1(z))$, $\forall\, z \in K_{1\alpha_i}$, $i = 1,\ldots,s$. Let $k = \max\{k_o, k_1, \ldots, k_s\}$. $\forall\, y \in R_2(x) \cap B_1(x)$, $x \notin V_k''(C_1(y))$. This is true because the sequence $(V_k''(c))$ is a decreasing sequence $\forall\ c \in \mathscr{C}_2$.

Case 3. For $y \in B_1(x) - B_2(x)$, then $x \notin U_2(y)$. So $x \notin W_k(y)$, $\forall k \in \omega$.

Then Claim 2 is proven.

For every $n \in \omega$, let $H_n = \{x \in X': W_n^{-1}(x) \subset U_1(x)\}$. Since $W_n(y) \subset U_2(y) \subset U_1(y)$, $\forall y \in X$, then by Lemma 3.2, the collection $\mathcal{H}_n = \{H_n \cap K_{1\alpha}: a < \Lambda_1\}$ is a discrete collection of closed sets refining $\mathcal{F}_1$. Since $\forall x \in X'$, $\exists n$ such that $x \in H_n$, then $X' = \bigcup_n \mathcal{H}_n$. Thus X' is expressed as a σ-discrete collection of closed sets refining $\mathcal{F}_1$.

3.7 Lemma: Let $\mathcal{F}$ be a heriditarily closure preserving cover by closed sets for the space X. Let $\mathcal{K}$ be the partition associated with $\mathcal{F}$. Let $Y = \{X_i: i \in \omega\}$ be an infinite subset of X such that $\forall i \in \omega$, $x_i \in K_{\alpha_i} \in \mathcal{K}$ and $K_{\alpha_i} \neq K_{\alpha_j}$ iff $i \neq j$. Then Y contains an infinite discrete subset.

Proof: Let $\mathcal{F}' = \bigcup_{i \in \omega} (\mathcal{F})_{x_i}$. Then $|\mathcal{F}'| \geq \omega$, for if $\mathcal{F}'$ is finite then the number of distinct C_α's that can be derived from $\mathcal{F}'$ is finite. Since there is a 1-1 correspondence between $\mathcal{K}$ and $\mathcal{C}$, the number od distinct K_α's that can be derived from $\mathcal{F}'$ is also finite. Thus $|\mathcal{F}'| < \omega$ which is not possible, so $|\mathcal{F}'| \geq \omega$. To complete the proof, we consider two cases.

Case 1. If $\left|\{X_i : \left|(\mathcal{F})_{x_i}\right| < \omega\}\right| \geq \omega$ then we can produce a subsequence $G = \{X_{i_j} \in Y : j \in \omega\}$ such that $\forall\, j \in \omega$, $F_j \in \mathcal{F}'$, and $x_{i_j} \in F_j$ so that if $j \neq j'$, $i_j \neq i_{j'}$ and $F_j \neq F_{j'}$. Then G is a discrete infinite subset of Y.

Case 2. If $\left|\{x_i \in Y : \left|(\mathcal{F})_{x_i}\right| < \omega\}\right| < \omega$, then we have for infinitely many $x_i \in Y$, $(\mathcal{F})_{x_i}$ is an infinite collection. Here also we can choose a sequence of points $\{x_{i_j} : j \in \omega\}$ in Y such that $x_{i_j} \neq x_{i_{j'}}$, if $i_j \neq i_j'$ and $\forall\, j \in \omega$, $\exists\, F_j \in \mathcal{F}'$ such that $x_{i_j} \in F_j$ and $F_j \neq F_j'$ if $i_j \neq i_j'$. As in Case 1, G is a discrete infinite subset of Y.

3.8 Lemma: Let X be a $\Sigma^{*\prime}$-space with $\{\mathcal{F}_n\}$ the associated σ-heriditarily closure preserving collections of closed sets. For every $x \in X$, and every $n \in \omega$, there is $m \in \omega$ such that $B_m(x)$ is finitely covered by members of $\mathcal{K}_n$.

Proof: Let $x \in X$, $n \in \omega$. Assume for every $m \in \omega$, $B_m(x)$ is not covered by any finite subcollection of $\mathcal{K}_n$. $\exists\, K_{n\alpha_1} \in \mathcal{K}_n$ with $x_1 \in B_1(x) \cap K_{n\alpha_1}$. Also $\exists\, K_{n\alpha_2} \in \mathcal{K}_n$ with $x_2 \in B_2(x) \cap K_{n\alpha_2}$ and $K_{n\alpha_1} \neq K_{n\alpha_2}$. So by induction there is

$K_{n\alpha_m} \in \mathcal{X}_n$ and $K_{n\alpha_i} \neq K_{n\alpha_m}$ $\forall\, i < m$, such that $x_m \in B_m(x) \cap K_{n\alpha_m}$. Set $Y = \{x_i\colon i \in \omega\}$. Then by Lemma 3.7 y has an infinite discrete subset $G = \{x_{i_j} \in Y\colon j \in \omega\}$. On the other hand $x_{i_j} \in C_{i_j}(x)$, $\forall\, j \in \omega$. Since X is a $\Sigma^{*\prime}$-space, the set G must have a cluster point. A contradiction. Thus $\exists\, m \in \omega$ such that $B_m(x)$ is finitely covered in $\mathcal{X}_n$.

3.9 Theorem: Let X be a perfect $\Sigma^{*\prime}$-space. Let $\{\mathcal{F}_n\}$ be a σ-heriditarily closure preserving collection of closed sets illustrating that X is a $\Sigma^{*\prime}$-space. For $n, m \in \omega$, $m > n$, set $X_{nm} = \{x \in X\colon B_m(x)$ is finitely covered by $\mathcal{X}_n\}$. Then the followings are true:

1. X_{nm} is closed, $\forall\, n, m \in \omega$, and can be expressed as a union of a σ-discrete collection of closed sets that refines $\mathcal{F}_n$.

2. $X = \bigcup_{m \in \omega} X_{nm}$, $\forall\, n \in \omega$

3. X is a Σ-space.

Proof: 1) Let $m, n \in \omega$, and $m > n$. We observe that $\mathcal{F}_n \subset \mathcal{F}_m$, and if $B_m(x)$ is finitely covered by $\mathcal{X}_n$, then $R_m(x)$ is also covered finitely by $\mathcal{X}_n$. Then $X_{nm} \subset X_{nm}'$, where $X_{nm}' = \{x \in X\colon R_m(x)$ is covered finitely by $\mathcal{X}_n\}$. But X_{nm} is closed, since if $x \in X_{nm}$ then $\forall\, y \in C_m(x)$, $B_m(y)$ is finitely covered by $\mathcal{X}_n$. So $C_m(x) \subset X_{nm}$ i.e., $X_{nm} =$

$\cup\{C_m(x): x \in X_{nm}\}$. By Theorem 3.6 $X_{nm}' = \bigcup_{k \in \omega}(\cup \mathcal{F}_{nmk})$, where $\mathcal{F}_{nmk}$ is discrete collection of closed sets that refines $\mathcal{F}_n$, $\forall k \in \omega$. $\forall k \in \omega$, set $\mathcal{K}_{nmk} = \mathcal{F}_{nmk}|X_{nm}$. Then $X_{nm} = \bigcup_{k \in \omega}(\cup \mathcal{K}_{nmk})$ and $\mathcal{K}_{nmk}$ is a discrete collection of closed sets and $\mathcal{K}_{nmk} < \mathcal{F}_n$.

2) Let $x \in X$, we will show $\forall n \in \omega$, there exists $m \in \omega$, such that $x \in X_{nm}$. Let $n \in \omega$, by Lemma 3.8 there is $m' \in \omega$ such that $B_{m'}(x)$ is finitely covered by $\mathcal{X}_n$. Thus $x \in X_{nm'}$. Hence $X = \bigcup_{m \in \omega} X_{nm}$.

3) For every i, $m \in \omega$, let $\{\mathcal{K}_{imk}\}$ be a sequence of discrete collections derived as a result of part 1 of this theorem. For every $n \in \omega$, define $\mathcal{S}_n$ as follows:

$$\mathcal{S}_n = \{H \in \mathcal{K}_{imk}: 1 \le i \le n \text{ and } 1 \le m \le n \text{ and } 1 \le k \le n\} \cup \{X\}.$$

Then $\mathcal{S}_n$ is a locally finite cover of closed sets, $\forall n \in \omega$. Moreover $\{\mathcal{S}_n\}$ is a Σ-net for X. This can be seen if we let $x \in X$, and let $x_n \in C(x, \mathcal{S}_n) = C_n(x)$. Let $C_n'(x) = C(x, \mathcal{F}_n)$. There exists $N \in \omega$ such that $\forall n \ge N$, $C_n(x) \subset C_n'(x)$. Then the sequence $\{x_n\}$ clusters. So $\{\mathcal{S}_n\}$ is a Σ-net for X. Then X is a Σ-space.

3.10 Theorem: Let X be a $\Sigma^{\#}$-space with a point countable pseudobase $\mathcal{U}$. For every $n, m \in \omega$, set $X_{nm} = \{x \in X: B_m(x)$ is finitely covered by $\mathcal{X}_n\}$. Set $X' = \{x \in X: \forall n \in \omega$, there is $m \in \omega$ such that $x \in X_{nm}\}$. Then X' can be expressed as a union of σ-locally finite collection $\{\mathcal{X}_n\}$, such that for each $n \in \omega$, $\exists\, m \in \omega$ with $\mathcal{X}_m < \mathcal{C}_n$.

Proof: Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ be the associated σ-closure preserving collection of closed sets illustrating that X is a $\Sigma^{\#}$-space. For every $n \in \omega$, let $\mathcal{C}_n$, $\mathcal{K}_n$, $\mathcal{B}_n$ be the collections associated with $\mathcal{F}_n$, as defined in 3.4.

Without loss of generality assume $\mathcal{U}$ is closed under finite intersections, and $X \in \mathcal{U}$.

For every $n \in \omega$, and every $C_{n\alpha} \in \mathcal{C}_n$, define the collection $\mathcal{W}_{n\alpha} = \{\mathcal{U}_{n\alpha}' \subset \mathcal{U}\colon \mathcal{U}_{n\alpha}'$ is a finite minimal cover for $C_{n\alpha}\}$. By a well-known result in [5] $\mathcal{W}_{n\alpha}$ is countable. So set $\mathcal{W}_{n\alpha} = \{\mathcal{U}_{n\alpha}^{i}\colon i \in \omega\}$. For every $k \in \omega$ define the open set $V_k'(C_{n\alpha}) = \bigcap_{i=1}^{k}(\cup\, \mathcal{U}_{n\alpha}^{i})$. For every $x \in X$, there is a unique $\alpha < \gamma_n$, such that $x \in K_{n\alpha} \subset C_{n\alpha}$. So define $V_k'(C_n(x)) = V_k'(C_{n\alpha})$. Let $V_k(C_n(x)) = \bigcap_{i=1}^{n} V_k'(C_i(x))$, where $V_k(C_1(x)) = V_k'(C_1(x))$, $\forall\, x \in X$, $k \in \omega$, $n \in \omega$.

For every $x \in X$, $i, \ell, k \in \omega$, $i < \ell$, we define the following open set $W_{i\ell k}(x) = U_\ell(x) \cap V_k(C_i(x))$.

<u>Claim</u>: Let $x \in X'$, let $n \in \omega$. Then there are $i, \ell, k \in \omega$ such that $x \notin W_{i\ell k}(y)$, $\forall\, y\, B_n(x)$.

Proof of Claim: Let $x \in X'$, $n \in \omega$ consider $B_n(x) \cap C(x)$. $B_n(x) \cap C(x)$ is a compact closed set. Hence $B_n(x) \cap C(x)$ has a finite minimal cover 0 such that $0 \subset \mathcal{U}$ and $x \notin O = \cup 0$. $O \cup U_n(x) \supset C(x)$. So $\exists M \in \omega$ such that $C_j(x) \subset O \cup \mathcal{U}_n(x)$, $\forall j \geq M$. Let $i = \max\{n,M\}$, then 0 is a finite minimum cover for $B_i(x) \cap B_n(x)$. Since $x \in X'$, there is $\ell \in \omega$, $\ell \geq i$ such that $x \in X_{i\ell}$ i.e., $B_\ell(x)$ is finitely covred by $\mathcal{K}_i$. Let $K_{i\alpha_1},\ldots,K_{i\alpha_s}$ be a minimal collection covering $B_\ell(x) \cap B_n(x)$. $C_{i\alpha_q} \subset B_i(x) \cap B_n(x)$ for $q = 1,\ldots,s$. There is a finite minimum subfamily 0_q such that $0_q \subset 0$ and $C_{i\alpha_q} \subset \cup 0_q$, $\forall q = 1,2,\ldots,s$. So $0_q \in \mathcal{W}_{i\alpha_q}$, $\forall q = 1,\ldots,s$, and there is $k_q \in \omega$ with $V_{k_q}(C_{i\alpha_q}) \subset \cup 0_q$. Let $k = \max\{k_1,\ldots,k_s\}$. Then $x \notin V_k(C_{i\alpha_q})$, $\forall q = 1,\ldots,s$. Thus for every $y \in B_n(x)$, if $y \notin B_\ell(x) \Rightarrow x \notin U_\ell(y)$. And if $y \in B_\ell(x) \cap B_n(x)$, there is a unique $q \in \{1,\ldots,s\}$ such that $y \in K_{i\alpha_q} \subset C_{i\alpha_q}$. So $x \notin V_k(C_{i\alpha_q})$. Thus $\forall y \in B_n(x)$, $x \notin W_{i\ell k}(y)$. Hence the claim.

In the proof of the claim, we observe that $W_{i\ell k}(x) \subset U_\ell(x) \subset U_i(x) \subset U_n(x)$. $\forall n, i, \ell, k \in \omega$, such that $n \leq i \leq \ell$ set $H_{ni\ell k} = \{x \in X' : W^{-1}_{i\ell k}(x) \subset U_n(x)\}$. Observe that $X' = \bigcup_{i,\ell,k\in\omega} H_{ni\ell k}$. For every $n,i,\ell,k \in \omega$, $n \leq i \leq \ell$, set $\mathcal{H}_{ni\ell k} = \{H_{ni\ell k} \cap K_{n\alpha} : \alpha < \gamma_n\}$. Observe that $\mathcal{H}_{ni\ell k} < \mathcal{G}_n$, $\forall n,i,\ell,k \in \omega$, $n \leq i \leq \ell$. By Lemma 3.2 $\mathcal{H}_{ni\ell k}$ is a discrete collection of (relative) closed subsets of X'. $\forall\, m \in \omega$, let $\mathcal{H}_m =$ $\{H \in \mathcal{H}_{ni\ell k}: n + i + \ell + k \leq m$, for every $n, i, \ell, k \in \omega\}$. Let $\mathcal{H}_m$ be closed under arbitrary intersection and $\mathcal{H}_m \subset \mathcal{H}_{m+1}$. $\{\mathcal{H}_m\}$ is a sequence of locally finite collections of (relative to X') closed sets in X'. We observe that the sequence $\{\mathcal{H}_m\}$ refines the sequence $\{\mathcal{E}_n\}$, and if for some $x \in X$, $\exists\, x_m \in C(x, \mathcal{H}_m)$, $\forall\, m \in \omega$, then the sequence $\{x_m\}$ clusters in X.

3.11 Corollary: Every $\Sigma^{*\prime}$-space with a point countable pseudobase is a σ-space.

Proof: Let X be a $\Sigma^{\#\prime}$-space with a point countable pseudobase $\mathcal{U}$, then by Theorem 3.9, $\bigcup_{m \in \omega} X_{nm} = X$, for every $n \in \omega$. So using the terminology of Theorem 3.10, $X' = X$. Hence X is a Σ-space. By theorem 1.1 in [8] X is a σ-space.

Several important and historical results about σ-spaces, semi-stratifiable spaces and stratifiable spaces can be proven using the ideas of this paper. Examples of such theorems are:

1. "Negata and Sewiec [9]: For a topological space X, the following are equivalent:

   i) X has a σ-closure preserving network of closed sets.

   ii) X has a σ-discrete network of closed sets.

2. "If X is a semi-stratifiable $\Sigma^{\#}$-space, then X is a σ-space." To prove this use Lemma 3.3.

3. "T. Shiraki [8]. If X is a $\Sigma^{\#}$-space and $\sigma^{\#}$-space then X is a σ-space."

4. "R. Heath [2]: If X is a stratifiable space then it is a σ-space.

3.12 Example: Let $A = P(\omega)$ = power set of the set of natural numbers. Add one distinct point to A so the space $X = A \cup \{\alpha\}$ with the topology on X be defined as follows: $\{x\}$ is open for every $x \in A$ and U is a neighborhood of α iff $X - U$ is countable. X is a $\Sigma^{*\prime}$-space which is not a Σ^{*}-space. To show that X is a $\Sigma^{*\prime}$-space we will produce a σ-hereditarily closure preserving collection of closed sets which is a Σ-net.

$\forall\, a \in A$, a is a subset of ω. For each $k \in \omega$ define $F_k = \{a \in A: k \in a\} \cup \{\alpha\}$, $E_k = \{a \in A: k \notin a\} \cup \{\alpha\}$. Clearly each F_k, E_k is closed in X. Let $\mathcal{F} = \{E_k, F_k: k \in \omega\}$. Then clearly $\mathcal{F}$ is a hereditarily closure-preserving collection of closed sets. Moreover $\forall\, a \in A$, $C(a,\mathcal{F}) = \bigcap\{F \in \mathcal{F}: a \in F\} = \{a, \alpha\}$, and $C(\alpha,\mathcal{F}) = \{\alpha\}$. Let $\mathcal{F}_i = \mathcal{F}$, $\forall\, i \in \omega$, then $\{\mathcal{F}_i\}$ is a sequence of hereditarily closure-preserving collections of closed sets which is a Σ-net. So X is a $\Sigma^{*\prime}$-space. In [7] it was shown that X is not a Σ^{*}-space. W. Fleissner gave the following lemma in a private letter sent to me in December 1984. I will use it to give a sufficient condition for a $\Sigma^{*\prime}$-space to be a Σ^{*}-space.

3.13 <u>Lemma (Fleissner)</u>: Let $\mathcal{F}$ be a hereditarily closure preserving collection of closed subsets of a space X of countable tightness. Then the collection $\mathscr{E}$ (as defined in 3.4) is hereditarily closure preserving.

<u>Proof</u>: Assume not, that $\mathcal{E} = \{E_C: C \in \mathscr{E}'\}$ is a family of closed sets such that $E_C \subset C$, $\mathscr{E}' \subset \mathscr{E}$ and $z \in \overline{\cup\mathcal{E}} - \cup\mathcal{E}$. By countable tightness, we may assume that $\mathcal{E}$ is countable. For $E \in \mathcal{E}$, define $\mathcal{F}(E) = \{F \in \mathcal{F}: E \subset F\}$. Set $\mathcal{E}_0 = \left\{E \in \mathcal{E}: |\mathcal{F}(E)| \geq \omega\right\}$,

$\mathcal{E}_1 = \left\{E \in \mathcal{E}: |\mathcal{F}(E)| < \omega\right\}$. Let $f:\mathcal{E}_0 \longrightarrow \mathcal{F}$ be one-to-one such

that $f(E) \in \mathcal{F}(E)$. Enumerate $\cup\left\langle\mathcal{F}(E): E \in \mathcal{E}_1\right\rangle$ as $\left\langle F_n : n \in \omega\right\rangle$.

For $E \in \mathcal{E}_1$ set $f(E) = F_n$ where $n = \max\left\langle i: E \subset F_i\right\rangle$. Then

$\left\{\cup\left\langle E \in \mathcal{E}: f(E) = F\right\rangle : F \in \text{range of } f\right\}$ demonstrates that $\mathcal{F}$ is

not hereditarily closure preserving.

3.14 <u>Theorem</u>: A $\Sigma^{*\prime}$-space with countable tightness is Σ^*-space.

Proof: Let X be a $\Sigma^{*\prime}$-space with countable tightness. Let $\{\mathcal{F}_n\}$ be a sequence of hereditarily closure preserving collections of closed sets illustrating that X is a $\Sigma^{*\prime}$-space. By lemma 3.13, $\forall\, n$, $\mathscr{E}_n$ is hereditarily closure preserving closed collection. Moreover, if U is an open set containing $C(x)$ for some $x \in X$, then becuase $\{\mathcal{F}_n\}$ is a Σ-net for X,

$C(x)$ for some $x \in X$, then becuase $\{\mathcal{F}_n\}$ is a Σ-net for X, there is $k \in \omega$ such that $C(x, \mathcal{F}_k) \subseteq U$. But $C(x, \mathcal{F}_k) \in \mathcal{E}_k$, and $C(x)$ is countable compact. Then X is Σ^*-space.

Some questions relating to this subject are still open. Here are some of them:

Question 1: Is every $\Sigma^\#$-space which has a point countable pseudobase a $\Sigma^{*\prime}$-space?

Question 2: Is every $\Sigma^\#$-space which is perfect a $\Sigma^{*\prime}$-space?

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