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A. Shawky Ibrahim and S.A. Khatab

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

ON FUZZY QUOTIENT SUBGROUPS

A. SHAWKY IBRAHIM
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
KFUPM BOX 181
DHAHRAN, SAUDI ARABIA

AND

S.A. KHATAB
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE, ZAGAZIG UNIVERSITY
ZAGAZIG, EGYPT

ABSTRACT

In this paper we define after Bhattacharya and Mukherjee fuzzy quotient subgroups. Then we describe diagrammetically and otherwise the relation between some finite ascending sequences of fuzzy subgroups and fuzzy quotient subgroups under certain conditions.

1. INTRODUCTION.

Let ϕ be a homomorphism of a group G onto a group H, and let μ be a fuzzy subgroup of G. Rosenfeld [3] proved that the fuzzy subset ν of H which satisfy the sup property and defined by $\nu(y) = \sup_{x \in \phi^{-1}(y)} \mu(x)$ is a fuzzy subgroup of H which Rosenfeld called the image of μ under ϕ.

Now, let μ be a fuzzy normal subgroup of G, i.e., $\mu(xy)=\mu(yx)$ for all $x, y \in G$ [2], and let e be the identity element of G, then the subset $G_\mu = \{g \in G \mid \mu(g) = \mu(e)\}$ is a normal subgrouup of G [2]. Mukherjee and Bhattacharya [2] proved that the relation $\hat{\mu}(xG_\mu) = \mu(x)$ for all $x \in G$ is a fuzzy subgroup of G/G_μ. One can easily verify that $\hat{\mu}$ is the image of μ under the canonical homomorphism $\phi: G \longrightarrow G/G_\mu$. They also proved in [1] that if ν_1 and ν_2 are fuzzy normal subgroups of G such that $\nu_1(e) = \nu_2(e)$ and $\nu_1 \leq \nu_2$, then the relation $\nu_2/\nu_1(xG_{\nu_1}) = \nu_2(x)$ is a fuzzy normal subgroup of G/G_{ν_1} which they called the fuzzy quotient subgroup of ν_2 by ν_1.

The purpose of this work is to investigate more the fuzzy subsets of the factor set G/G_μ and the fuzzy quotient subgroups ν_2/ν_1 for any fuzzy subgroups μ and ν_2 of G which may or may not be fuzzy normal. We use commutative diagrams to describe the relations between the fuzzy quotient subgroup ν_2/ν_1 and the fuzzy subgroups ν_1 and ν_2. Other relations are also studied.

2. PRELIMINARIES.

Definition 2.1 [4]: A fuzzy subset μ of a set S is a function $\mu: S \longrightarrow I = [0,1]$.

Definition 1.2 [3]: A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if $\mu(x^{-1}y) \geq \min(\mu(x), \mu(y))$ for all $x,y \in G$.

Definition 1.3 [2]: A fuzzy subgroup μ of a group G is said to be fuzzy normal if $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Lemma 2.1 [2]: If μ is a fuzzy subgroup of a group G whose identity element is e, then the subset $G_\mu = \{g \in G \mid \mu(g) = \mu(e)\}$ is a subgroup of G. If, in addition, μ is fuzzy normal, then G_μ is a normal subgroup of G.

Lemma 2.2 [2]: Let μ be a fuzzy subgroup of a group G and $x, y \in G$ such that $\mu(x^{-1}y) = \mu(e)$, then $\mu(x) = \mu(y)$.

3. FUZZY QUOTIENT SUBSETS AND NATURAL FUZZY QUOTIENT SUBGROUPS.

Lemma 3.1: Let ν_1 be a fuzzy subgroup of a group G, then the relations η_{11}^{ℓ} and η_{11}^{r} on the quotient set G/G_{ν_1} given by:

$$\eta_{11}^{\ell} = \{(gG_{\nu_1}, \nu_1(g)) \mid g \in G\}$$

and

$$\eta_{11}^{r} = \{(G_{\nu_1}g, \nu_1(g)) \mid g \in G\}$$

are fuzzy subsets of the factor set G/G_{ν_1}.

Proof: All we need to show is that η_{11}^{ℓ} and η_{11}^{r} are well-defined:

If $g_1G_{\nu_1} = g_2G_{\nu_1} \Rightarrow g_1^{-1}g_2 \in G_{\nu_1} \Rightarrow \nu_1(g_1^{-1}g_2) = \nu_1(e) \Rightarrow$ (by Lemma 2.2) $\nu_1(g_1) = \nu_1(g_2) \Rightarrow \eta_{11}^{\ell}(g_1G_{\nu_1}) = \eta_{11}^{\ell}(g_2G_{\nu_1}) \Rightarrow \eta_{11}^{\ell}$ is well-defined $\Rightarrow \eta_{11}^{\ell}$ is a fuzzy subset of the factor set G/G_{ν_1}. Similarly, we can show that η_{11}^{r} is a fuzzy subset of the factor set G/G_{ν_1}. This completes the proof of the lemma.

Definition 3.1: The fuzzy subsets η_{11}^{ℓ} and η_{11}^{r} defined in Lemma 3.1 are called the left and right fuzzy subsets of the factor set G/G_{ν_1} respectively which are induced by the fuzzy subgroup ν_1 of G.

Lemma 3.2: If ν_1 is a fuzzy normal subgroup of G, then $\eta_{11}^{\ell} = \eta_{11}^{r}$.

Proof: ν_1 is a fuzzy normal subgroup of $G \Rightarrow$ (by Lemma 2.1) G_{ν_1} is a normal subgroup of $G \Rightarrow gG_{\nu_1} = G_{\nu_1}g$ for all $g \in G \Rightarrow \eta_{11}^{\ell} = \eta_{11}^{r}$ which completes the proof of the lemma.

If we let η_{11} denote $\eta_{11}^{\ell} = \eta_{11}^{r}$ in case ν_1 is a fuzzy normal subgroup of G, then we immediately get the following theorem which coincides with Mukherjee's and Bhattacharya's Theorem 3.12 in [2].

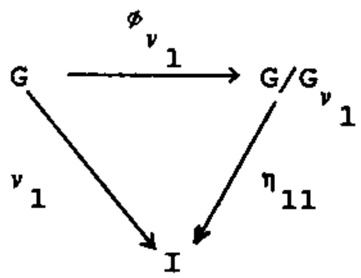
Theorem 3.1: If ν_1 is a fuzzy normal subgroup of G, then η_{11} is a fuzzy normal subgroup of the factor group G/G_{ν_1}.

Definition 3.2: The fuzzy normal subgroup η_{11} defined in Theorem 3.1 is called the Natural Fuzzy Quotient Subgroup of the factor group G/G_{ν_1} which is induced by the fuzzy normal subgroup ν_1 of G.

The relation between ν_1 and η_{11} is described diagrammetically as in the following easily established theorem. We omit details.

Theorem 3.2: Let ν_1 be a fuzzy normal subgroup of a group G and let ϕ_{ν_1} be the canonical homomorphism $\phi_{\nu_1} : G \longrightarrow G/G_{\nu_1}$. Then the following statements are equivalent:

1) η_{11} is the natural fuzzy quotient subgroup of G/G_{ν_1} induced by ν_1.
2) the diagram:



commutes, i.e., $\nu_1 = \eta_{11} \circ \phi_{\nu_1}$.

4. FUZZY QUOTIENT SUBGROUPS.

Theorem 4.1: Let ν_1 and ν_2 be two fuzzy subgroups of a group G such that: i) ν_1 is fuzzy normal, ii) $\nu_1(e) = \nu_2(e)$ and iii) $\nu_1 \leq \nu_2$, then the relation η_{12} defined on the factor group G/G_{ν_1} by $\eta_{12}(gG_{\nu_1}) = \nu_2(g)$ for all $g \in G$ is a fuzzy subgroup of G/G_{ν_1}

Proof: We first show that η_{12} is a well-defined map: Let $xG_{\nu_1} = yG_{\nu_1}$ for some $x, y \in G \Rightarrow x^{-1}y \in G_{\nu_1} \Rightarrow$ (by condition (ii)) $\nu_1(x^{-1}y) = \nu_1(e) = \nu_2(e)$. But, by condition (iii), $\nu_1(x^{-1}y) \leq \nu_2(x^{-1}y) \Rightarrow \nu_2(e) \leq \nu_2(x^{-1}y)$. But $\nu_2(g) \leq \nu_2(e)$ for all $g \in G \Rightarrow \nu_2(x^{-1}y) = \nu_2(e) \Rightarrow$ (by lemma 2.2) $\nu_2(x) = \nu_2(y) \Rightarrow$ $\eta_{12}(xG_{\nu_1}) = \eta_{12}(yG_{\nu_1}) \Rightarrow \eta_{12}$ is a well-defined map and hence a fuzzy subset of G/G_{ν_1}.

Now, $\eta_{12}(g_1^{-1}G_{\nu_1}\, g_2\, G_{\nu_1}) = \eta_{12}(g_1^{-1}\, g_2\, G_{\nu_1}) = \nu_2(g_1^{-1}\, g_2) \geq$ $\min(\nu_2(g_1), \nu_2(g_2)) = \min(\eta_{12}(gG_{\nu_1}), \eta_{12}(g_2G_{\nu_1})) \Rightarrow \eta_{12}$ is a fuzzy subgroup of the factor group G/G_{ν_1}. This completes the proof of the theorem.

Definition 4.1: The fuzzy subgroup η_{12} defined in Theorem 4.1 is called the Fuzzy Quotient Subgroup of ν_2 by ν_1.

The following corollary of Theorem 4.1 coincides with Bhattacharya's and Mukherjee's Lemma 5.1 in [1].

Corollary 4.1: Let ν_1 and ν_2 be as defined in Theorem 4.1 with the additional condition that ν_2 is a fuzzy normal subgroup of G. Then η_{12} is a fuzzy normal subgroup of G/G_{ν_1}.

It is worth mentioning that our definition of fuzzy quotient subgroup η_{12} does not require ν_2 to be fuzzy normal which straightens Bhattacharya's and Mukherjee's definition of fuzzy quotient groups given in [1].

Now let η_{11} be the natural fuzzy quotient subgroup of G/G_{ν_1} induced by ν_1 and let $\bar{e}$ be the identity element of G/G_{ν_1}. Let $\mathcal{F}$ be the set of all fuzzy subgroups of G and Γ be the set of all fuzzy subgroups of G/G_{ν_1}. Consider the following two subsets of $\mathcal{F}$ and Γ respectively:

$$\mathcal{F}_{\nu_1} = \{\nu \in \mathcal{F} \mid \nu_1(e) = \nu(e) \text{ and } \nu_1 \leq \nu\}$$

and

$$\Gamma_{\eta_{11}} = \{\eta \in \Gamma \mid \eta_{11}(\bar{e}) = \eta(\bar{e}) \text{ and } \eta_{11} \leq \eta\}.$$

Then we have the following correspondence theorem:

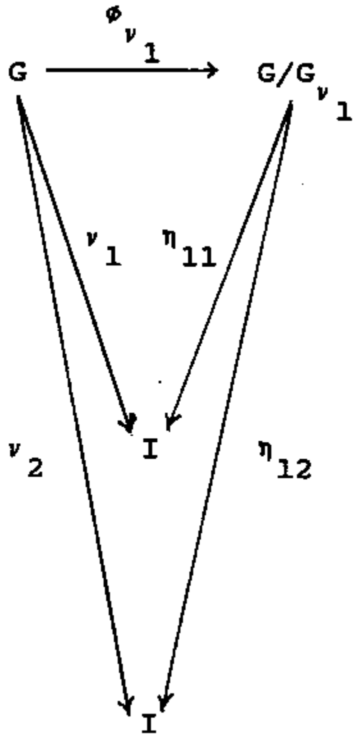
Theorem 4.2 (Correspondence Theorem): Each $\eta \in \Gamma_{\eta_{11}}$ is a fuzzy quotient subgroup of G/G_{ν_1} of some $\nu \in \mathcal{F}_{\nu_1}$ by ν_1. Moreover, there is a one-to-one correspondence between $\mathcal{F}_{\nu_1}$ and $\Gamma_{\eta_{11}}$.

Proof: Let $\eta \in \Gamma_{\eta_{11}}$ and $\nu(g) = \eta(gG_{\nu_1})$ for all $g \in G$, then ν is obviously a fuzzy subgroup of G i.e. $\nu \in \mathcal{F}$. But $\nu(e) = \eta(\bar{e}) = \eta_{11}(\bar{e}) = \nu_1(e)$ and $\nu(g) = \eta(gG_{\nu_1}) \geq \eta_{11}(gG_{\nu_1}) = \nu_1(g)$. Thus $\nu \in \mathcal{F}_{\nu_1}$.

One can also easily check that the map given by $\pi(\nu) = \eta$ whenever $\nu(g) = \eta(gG_{\nu_1})$ defines a one-to-one correspondence between $\mathcal{F}_{\nu_1}$ and $\Gamma_{\eta_{11}}$. This completes the proof of the theorem.

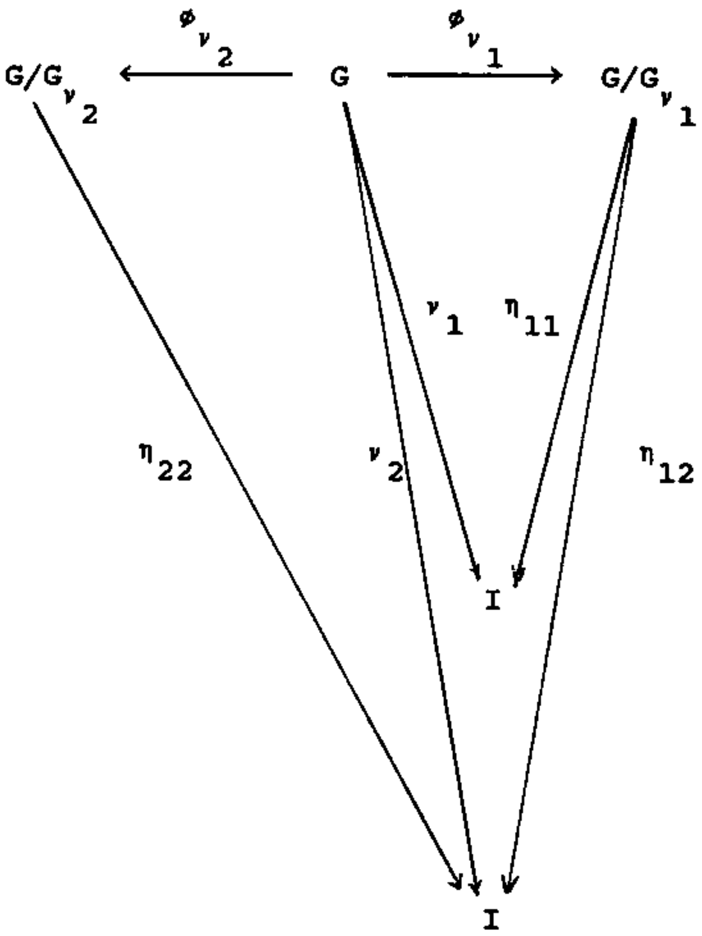
5. MORE COMMUTATIVE DIAGRAMS.

Let η_{11} be the natural fuzzy quotient subgroup induced by $\nu_1 \in \mathcal{F}_{\nu_1}$; and let $\eta_{12} \in \Gamma_{\eta_{11}}$ be the fuzzy quotient subgroup which corresponds to $\nu_2 \in \mathcal{F}_{\nu_1}$. If ϕ_{ν_1} is the canonical homomorphism $\phi_{\nu_1} : G \longrightarrow G/G_{\nu_1}$, then the relation between ν_1, ν_2, η_{11} and η_{12} is such that the diagram



commutes, i.e., $\nu_i = \eta_{1i} \circ \phi_{\nu_1}$, $i = 1, 2$.

If in addition ν_2 is a fuzzy normal subgroup of G, η_{22} is the natural fuzzy quotient of G/G_{ν_2} induced by ν_2 and ϕ_{ν_2} is the canonical homomorphism $\phi_{\nu_2} : G \longrightarrow G/G_{\nu_2}$, then the relation between ν_1, ν_2, η_{11}, η_{12} and η_{22} is such that the diagram

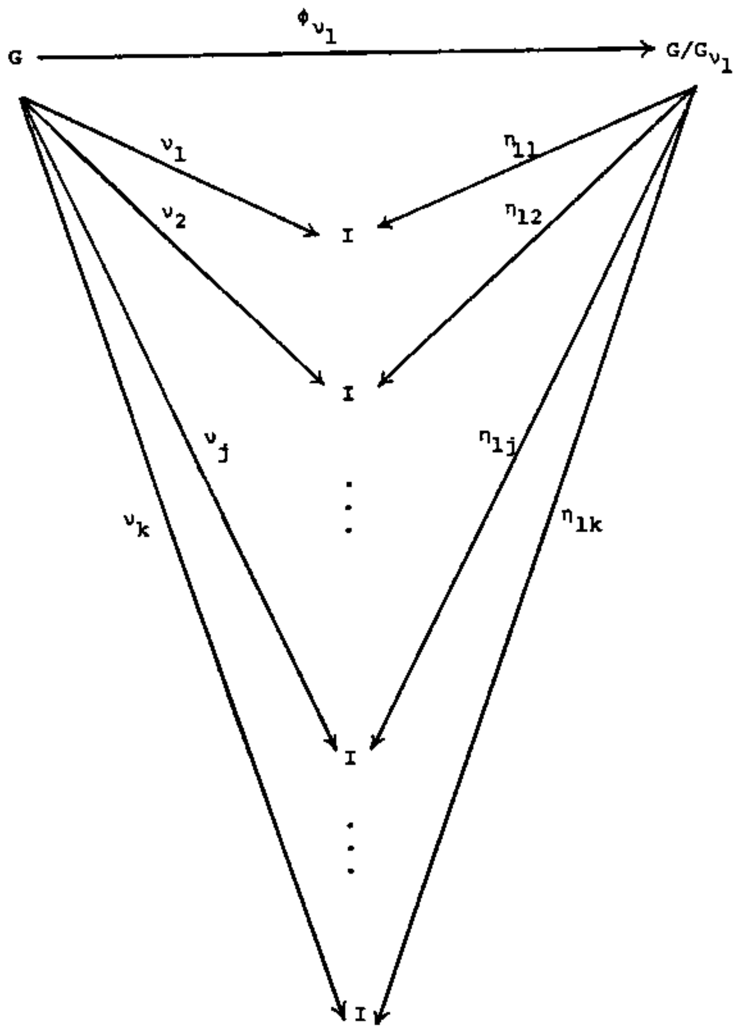


commutes i.e., $\nu_1 = \eta_{11} \circ \phi_{\nu_1}$ and $\nu_2 = \eta_{12} \circ \phi_{\nu_1} = \eta_{22} \circ \phi_{\nu_2}$

The following two theorems are the consequence of the above discussion which show that the above diagrams can be extended to involve finite sequences of fuzzy subgroups. We omit details.

Theorem 5.1: For each finite ascending sequence $(\nu_j)_{1\leq j\leq k}$ of fuzzy subgroups in $\mathcal{F}_{\nu_1}$, there is a finite ascending sequence

$\{\eta_{1j}\}_{1\le j\le k}$ of fuzzy quotient subgroups in $\Gamma_{\eta_{11}}$ such that the following diagram



commutes, i.e., $\nu_j = \eta_{1j} \circ \phi_{\nu_1}$ $1 \le j \le k$. Moreover,

$$(G/G_{\nu_1})_{\eta_{1j}} = \{gG_{\nu_1} \mid \eta_{1j}(gG_{\nu_1}) = \eta_{1j}(\bar{e})\} = G_{\nu_j}/G_{\nu_1}, \quad 1 \le j \le k.$$

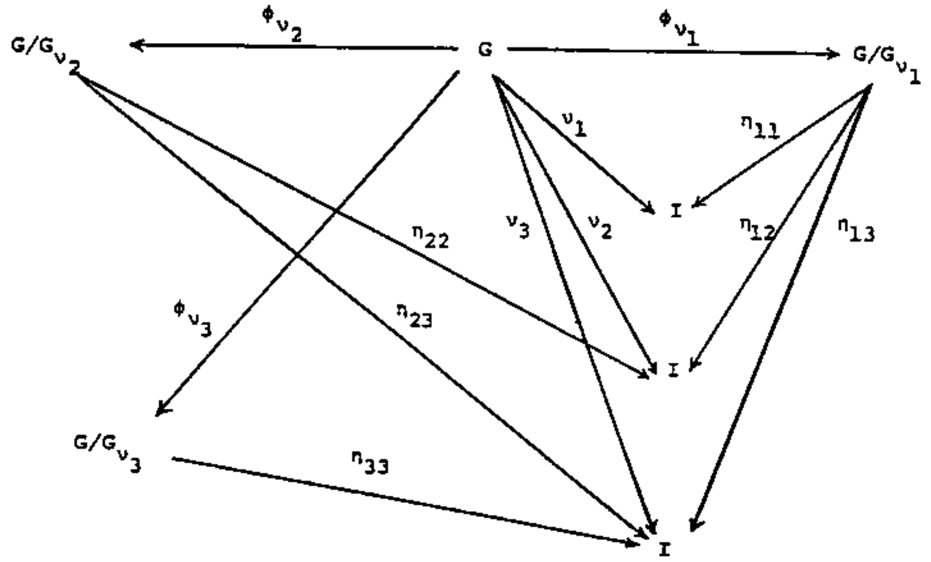
Theorem 5.2: For each finite ascending sequence $\{\nu_j\}_{1\le j\le k}$ of fuzzy normal subgroups in $\mathcal{G}_{\nu_1}$ there is a finite double-indexed sequence $\{\eta_{ij}\}_{1\le i\le j\le k}$ of fuzzy quotient subgroups with $\eta_{ij} \in \Gamma_{\eta_{ii}}$ such that:

1) $\eta_{ii} \le \eta_{ii+1} \le \dots \le \eta_{ik} \quad 1 \le i \le k$

2) $\text{Image}(\eta_{ij}) = \text{Image}(\nu_j) \quad 1 \le i \le j \le k$

3) If $\phi_{\nu_j} : G \longrightarrow G/G_{\nu_j} \quad 1 \le j \le k$ is the canonical homomorphism, then we have $\nu_j = \eta_{ij} \circ \phi_{\nu_i} \quad 1 \le i \le j \le k.$

A commutative diagram for $1 \le i \le j \le 3$ is as follows:



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