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Abstract. The Banach-Tarski theorem on polygons in $\mathcal{R}^2$ implies that two polygons are equidecomposable if and only if they are equidissectable. The possibility of strengthening this result in various ways is investigated.

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1. Introduction.

In [1] (the paper containing the celebrated Banach-Tarski "paradox"), S. Banach and A. Tarski proved a theorem which implies the following remarkable result.

THEOREM A (Banach-Tarski) Two polygons in $\mathcal{R}^2$ are equidecomposable if and only if they are equidissectable.

'Equidecomposable' means that one can be partitioned into finitely many disjoint sets, here called pieces, which can be rearranged to form a partition of the other; equidissectability is the classical notion where the pieces are triangles, allowed to overlap on their boundaries.

The question arises, whether this theorem can be strengthened. There are at least two ways in which this might be done. The first is to try to prove the same result for convex bodies in $\mathcal{R}^2$ instead of polygons. Of course, one has to substitute the appropriate generalization of equidissectability. Introduced in [7], this is called convex equidecomposability, and it allows the pieces themselves to be convex bodies which may overlap on their boundaries. Now if one of the polygons is replaced by a convex body, a partial generalization of Theorem A can be proved, by restricting the pieces to be moved by isometries from a discrete group ([2], Theorem 1); but even this is generally false if both polygons are replaced by convex bodies (see [3]).

What Banach and Tarski actually proved is that two polygons in $\mathcal{R}^2$ are equidecomposable if and only if they have the same area. The theorem stated above follows, since the Bolyai-Gerwein theorem says that two polygons have equal areas if and only if they are equidissectable (see [8], Theorem 3.2). It is natural to wonder whether there is a more direct proof of Theorem A. Perhaps — and this is the second way that Theorem A might be improved — the required dissection can be achieved using the same isometries that demonstrate the equidecomposability. Our first result uses Hall's matching theorem to strengthen [2], Theorems 1 and 5, by showing that this is true when the isometries generate a discrete group (see section 2 for unexplained terms).

THEOREM 1 Let F be a finite set of isometries of $\mathcal{R}^d$, which generate a discrete group G. Suppose that K_1 is a polytope and K_2 is a convex body in $\mathcal{R}^d$. The following conditions are equivalent.

(i) K_1 and K_2 are equidecomposable mod. null sets, under isometries from F.

(ii) K_1 and K_2 are equidecomposable mod. first category sets, under isometries from F.

(iii) K_1 and K_2 are convex equidecomposable, under isometries from F.

Theorem 1 was obtained as a contribution to Tarski's famous problem of whether a disc and a square of equal areas are equidecomposable. Recently, in [5], the second author has shown that this is so and furthermore that only translations need be used. On the other hand, a disc and a square are clearly not convex equidecomposable. Consequently Theorem 1 (and its weaker predecessor in [2]) are generally not true if the group G is abelian, which answers the question on p. 8 of [2] negatively. The following example also shows this.

EXAMPLE 1. There are two intervals in $\mathcal{R}$ which are equidecomposable under a certain finite set F of isometries, but which are not equidissectable under isometries from the group generated by F.

The point is that the methods of [5] require a huge number of translations, and the proofs are very complicated. By contrast, Example 1 uses only four isometries, and is based on the relatively simple technique of [4].

Example 1 also provides a simple way to see that the second possible strong form of Theorem A suggested above does not exist. For, by taking the cartesian product of the two intervals in Example 1 with the unit interval, one obtains two squares which are equidecomposable under isometries in F, but not equidissectable by those isometries, or even isometries from the group generated by F. (Again, the much more difficult methods of [5] show more; in fact, any two polygons of equal areas are equidecomposable by translations.)

2. Definitions and notation.

We denote the interior, boundary and cardinality of a set E by $int\,E$, $bd\,E$ and $|E|$, respectively.

Two sets A, B in $\mathcal{R}^d$ are equidecomposable if there are disjoint decompositions $A = \cup_{i=1}^k A_i, B = \cup_{i=1}^k B_i$, and isometries τ_i of $\mathcal{R}^d$ such that $B_i = \tau_i A_i$ for $i = 1, \ldots, k$.

We denote Lebesgue measure in $\mathcal{R}^d$ by λ_d. If, instead of disjointness, the sets above only satisfy $\lambda_d(A_i \cap A_j) = \lambda_d(B_i \cap B_j) = 0$ (respectively, $A_i \cap A_j$ and $B_i \cap B_j$ are of first category), for $1 \leq i \neq j \leq k$, we say A and B are equidecomposable mod. null sets (respectively, equidecomposable mod. first category sets).

If A and B are polytopes in $\mathcal{R}^d$, and the sets A_i and B_i are simplices satisfying, instead of disjointness, $intA_i \cap intA_j = intB_i \cup intB_j = \phi, 1 \leq i \neq j \leq k$, then A and B are equidissectable. Finally, convex bodies A and B are convex equidecomposable if the latter condition on the sets A_i and B_i holds, where these are also convex bodies.

3. Proofs of the results.

Proof of Theorem 1. Since (iii) $\Rightarrow$ (i) or (ii) is obvious, we shall assume (i) or (ii). In either case, we begin by following the argument of [2], Theorem 1. Here it is shown that K_2 must be a polytope, and there is a dissection of K_1 into subpolytopes $P_{jk}, 1 \leq j \leq l, 1 \leq k \leq p_j$, such that:

(a) for each j, all the polytopes P_{jk} are congruent, for $1 \leq k \leq p_j$;

(b) for each j, k, there is an isometry $g_{jk} \in G$ such that the polytopes $Q_{jk} = g_{jk}P_{jk}, 1 \leq j \leq l, 1 \leq k \leq p_j$, form a dissection of K_2;

(c) for each j, $1 \leq j \leq l, \cup\{P_{jk} : 1 \leq k \leq p_j\}$ and $\cup\{Q_{jk} : 1 \leq k \leq p_j\}$ are equidecomposable (mod. null sets or first category sets, as appropriate) under isometries from F.

Our assumptions imply that there is a set N such that $(K_1 - N)$ and $(K_2 - N)$ are equidecomposable under isometries from F, where $\lambda_d(N) = 0$ or N is of first category, according as (i) or (ii) holds. For each $x \in (K_1 - N)$, there is a $\tau_x \in F$ such that $\tau_x(x) \in (K_2 - N)$ is the point corresponding to x under the decomposition. Define ψ on $(K_1 - N)$ by letting $\psi(x) = \tau_x(x)$ for each x.

Let us fix a $j, 1 \leq j \leq l$. We shall construct a bipartite graph Γ as follows. The nodes of the two parts of Γ are the polytopes $P_{jk}, 1 \leq k \leq p_j$, and $Q_{jk}, 1 \leq j \leq p_j$, respectively. Two nodes P_{jk_1} and Q_{jk_2} are joined by an edge if there is an $x \in P_{jk_1}$ such that $\phi(x) \in Q_{jk_2}$.

As in [2], Theorem 5, we denote by $\bar{\mu}$ a finitely additive, G-invariant measure defined on all subsets of $\mathcal{R}^d$. If (i) is assumed, we take $\bar{\mu}$ to be an extension of λ_d; if (ii) holds, we suppose that $\bar{\mu}$ satisfies $\bar{\mu}(E) = \lambda_d(E)$ if $\lambda_d(bdE) = 0$, and $\bar{\mu}(H) = 0$ for all first category sets H.

Let I be a subset of $\{1, \ldots, p_j\}$, and let $X = \cup\{P_{jk} : k \in I\} - N$. Then, by (a),

$$\bar{\mu}(X) = |I| \cdot \lambda_d(P_{jk}),$$

since $\bar{\mu}(N) = 0$. Now the set $\psi(X) - N$ is equidecomposable with X, under isometries in F, so must have the same $\bar{\mu}$-measure. Since $\bar{\mu}(P_{jk}) = \lambda_d(P_{jk})$ for all k, it follows by (b) and (c) that $[\psi(X) - N] \cap Q_{jk} \neq \phi$ for all k belonging to a subset J of $\{1, \ldots, p_j\}$ satisfying $|J| \geq |I|$. But $k' \in J$ if and only if $Q_{jk'}$ is joined by an edge in Γ to some P_{jk} with $k \in I$. Consequently Γ satisfies the hypothesis of Hall's matching theorem (see, for example, [6], Theorem 1.1.3), and Γ has a matching.

The existence of this matching shows immediately that the sets $\cup\{P_{jk} : 1 \leq k \leq p_j\}$ and $\cup\{Q_{jk} : 1 \leq k \leq p_j\}$ are equidissectable under isometries from F. Repeating the argument for each $j, 1 \leq j \leq l$, we conclude that (iii) is true.

<u>Construction of Example</u> 1.

Let $u \in [0,1]$ be irrational. From a matching constructed in [4] it follows immediately that $[0,1]$ is equidecomposable with itself under the isometries $\sigma_1(x) = x + u, \sigma_2(x) = x - u, \sigma_3(x) = -x + u$ and $\sigma_4(x) = -x - u + 2$. Suppose that v is a real number such that $1, u$ and v are rationally independent. Then the intervals $[0,1]$ and $[v, v+1]$ are equidecomposable under the isometries $\tau_i(x) = \sigma_i(x) + v, 1 \leq i \leq 4$.

Let G be the group of all isometries of the form $g(x) = \pm x + ku + lv + 2m$, where k, l and m are integers with $(k + l)$ even. We shall prove that $[0,1]$ and $[v, v+1]$ are not equidissectable under isometries from G.

Suppose the contrary. Then there are partitions $0 = a_0 < a_1 < \ldots < a_n = 1$ of $[0,1]$ and $v = b_0 < b_1 < \ldots < b_n = v + 1$ of $[v, v+1]$, such that the intervals $[a_i, a_{i+1}], 0 \leq i < n$, are in one-to-one correspondence with the intervals $[b_j, b_{j+1}], 0 \leq j < n$, via isometries from G. Denote by g_i the isometry in G which maps $[a_i, a_{i+1}]$ to some $[b_j, b_{j+1}]$, for $0 \leq i < n$.

Form a bipartite graph Δ as follows. The nodes of each part of Δ are the numbers $a_i, 0 \leq i \leq n$, and $b_j, 0 \leq j \leq n$, respectively. We join a_i and b_j by an edge if $g_i(a_i) = b_j$ and $i < n$ or $g_{i-1}(a_i) = b_j$ and $i > 0$. Then a_0, a_n, b_0 and b_n have degree one, and all other points have degree two (we allow multiple edges). Therefore a_0 belongs to a component Δ_0 of Δ such that each point in Δ_0 has degree two, unless it is a_0, a_n, b_0 or b_n. So there is a path in Δ_0 beginning at a_0 and ending at a_n, b_0 or b_n.

Since $a_0 = 0$, all points in this path correspond to numbers of the form $(ku+lv+2m)$ with $(k+l)$ even. However, since $1, u$ and v are rationally independent, none of the numbers $a_n = 1, b_0 = v$ or $b_n = v + 1$ can be of this form. This contradiction proves our assertion above.

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