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Chomology of Schubert Subarieties of GLn /p

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Abstract.

Let GL_n be the group of nxn invertible complex matrices, and P a parabolic subgroup of GL_n. In this paper we give a geometric description of the cohomology ring of a Schubert subvariety Y of GL_n/P. Our main theorem states that the coordinate ring $A(Y \cap Z)$ of the scheme theoretic intersection of Y and the zero scheme Z of the vector field V associated to a principal regular nilpotent element n of gl_n is isomorphic to the cohomology algebra $H^*(Y;\mathbb{C})$ of Y. This theorem was proved in [2] when the space GL_n/P is a Grassmann manifold. We were recently informed that Prof.D.H. Peterson has just proved that GL_n is the only algebraic group G where the cohomology ring of any Schubert subvariety Y of the space G/B is isomorphic to $A(Y \cap Z)$. Here B stands for a Borel subgroup of G.

0. Introduction.

Let X be a nonsingular complex projective variety having the following properties :

(A) there exists an algebraic vector field V with exactly one zero x_0, and

(B) there exists an algebraic $\mathbb{C}^*$- action on X

$$\lambda : \mathbb{C}^* \times X \longrightarrow X , \quad ((t,x) \longrightarrow \lambda(t).x),$$

such that $d\lambda(t).V = t^{p} V$ for some $p > 0$, and for all t in $\mathbb{C}^{*}$, where $d\lambda(t)$ is the asociated tangent action of $\lambda(t)$ on vector fields.

Let Z be the zero scheme of the vector field V , and Y be any V and $\mathbb{C}^{*}$-invariant subvariety of X. It follows from the property (B) that Z is a $\mathbb{C}^{*}$-invariant subscheme of X. Thus, the coordinate ring A(Z) (respectively $A(Y \cap Z)$) of Z (respectively $Y \cap Z$) has a natural graded algebra structure induced from the $\mathbb{C}^{*}$-action λ. Here, $Y \cap Z$ stands for the scheme theoretic intersection of Y and Z. Throughout the rest of the paper these rings A(Z) and $A(Y \cap Z)$ will be regarded as graded algebras with the gradation above, and $H^{*}(W;\mathbb{C})$ will denote the cohomology ring of the variety W with coefficients in the field of complex numbers $\mathbb{C}$. The following theorem is proved in [3], [4].

THEOREM. There exists a graded algebra isomorphism

$$\psi : A(Z) \longrightarrow H^{*}(X;\mathbb{C})$$

which induces a graded algebra homomorphism

$$\bar{\psi} : A(Y \cap Z) \longrightarrow H^{*}(Y;\mathbb{C})$$

commuting with the natural maps

$$A(Z) \longrightarrow A(Y \cap Z), \quad \text{and } H^{*}(X;\mathbb{C}) \longrightarrow H^{*}(Y;\mathbb{C}).$$

For any parabolic subgroup P of a complex reductive algebraic group G , the space G/P has the properties (A) and (B).

Moreover any Schubert subvariety $Y = \overline{B\sigma P}$ of G/P is V and $\mathbb{C}^*$-invariant. Thus, by the Theorem we have a surjective graded algebra homomorphism

$$\bar{\psi} : A(Y \cap Z) \longrightarrow H^*(Y;\mathbb{C}).$$

DEFINITION. The cohomology ring of the Schubert variety Y is said to have a nilpotent description, if $\bar{\psi}$ is an isomorphism.

It is known that the cohomology ring of any Schubert subvariety Y of the Grassmann manifold $G_{k,n}$ has a nilpotent description ([2]). In this paper, we generalize this result to any Schubert subvariety of the partial flag manifold GL_n/P. The paper is organized as follows.In Section 1, we begin with the preliminaries. In Section 2, we investigate a certain ideal in the cohomology ring of GL_n/B associated with a Schubert subvariety $Y= \overline{B\sigma B}$ of GL_n/B. This is done by finding a relation between the functions P_σ constructed by Bernstein, Gelfand, Gelfand in [5] (independently by Demazure in [6]), and the Plücker coordinates. In Section 3, we first prove that if the cohomology ring of any Schubert subvariety of the space G/B has a nilpotent description, then so does the cohomology ring of any Schubert subvariety of G/P. Here P is a parabolic subgroup of a complex reductive linear algebraic group G which contains the Borel subgroup B of G. Then we finally prove that the cohomology rings of the Schubert subvarieties of GL_n/P have nilpotent descriptions.

I. Preliminaries.

Let GL_n be the group nxn invertible complex matrices, B the group of upper triangular matrices in GL_n, W the symmetric group in 1,2,...,n , $\ell(\tau)$ the length of $\tau \in W$. Let $R = C[x_1,\dots,x_n]$ be the polynomial algebra with the usual grading , and IR the ideal of R generated by the elementary symmetric polynomials in $x_1,\dots,x_n$. W acts on R by permuting the variables. We denote this action by $\sigma.f(x_1,\dots,x_n) = f(x_{\sigma_1},\dots,x_{\sigma_n})$, $\sigma = (\sigma_1,\dots,\sigma_n) \in W$. Let (i,j) denote the transposition of W obtained by changing i with j. We recall the following facts from [5], [6] (see also [9] for more combinatorial approach). For any $1\leq i < j \leq n$, the polynomial $f - (i,j).f$ is divisible by $x_i - x_j$. Thus, the operator $\partial_{(i,j)} : R \longrightarrow R$, $\partial_{(i,j)}(f) = \dfrac{f - (i,j).f}{x_i - x_j}$ is well defined.

Let $i_1,\dots,i_r$ be integers in {1,...,n}, and let $\omega = (i_1,i_1+1) \dots (i_r,i_r+1)$ be any element of W. Then,

(a) if $\ell(\omega) \neq r$, then $\partial_{(i_1,i_1+1)} \dots \partial_{(i_r,i_r+1)} = 0$,

(b) if $\ell(\omega) = r$, then the operator

$\partial_{(i_1,i_1+1)} \dots \partial_{(i_r,i_r+1)}$ depends only on ω and not on the representation in the form $\omega = (i_1,i_1+1) \dots (i_r,i_r+1)$. In this case we put $\partial_\omega = \partial_{(i_1,i_1+1)} \dots \partial_{(i_r,i_r+1)}$. We note that the operator $\partial_\omega : R \longrightarrow R$ preserves the ideal IR , and thus it induces an operator $\bar{\partial}_\omega : R/IR \longrightarrow R/IR$ of homogeneous degree $-\ell(\omega)$. Let ω_0 be the permutation (n,n-1,...,1) in W , and let $P_{\omega_0} = \frac{1}{n!} \left(\prod_{1 \le i < j \le n} (x_i - x_j) \right) \mod (IR)$.

For each ω in W , let $P_\omega = \bar{\partial}_{\omega\,\omega_0}(P_{\omega_0})$, and let $[X_\tau]$ denote the cycle class of the Schubert variety $X_\tau = \overline{B\tau B}$ in $H_*(GL_n/B;\mathbb{C})$. The folowing theorem is proved in [5], [6].

THEOREM 1.1. There exists a graded algebra isomorphism $\beta : R/IR \longrightarrow H^*(GL_n/B;\mathbb{C})$ such that for any ω in W , $\beta(P_\omega) = \mathcal{P}([X_{\omega_0\omega}])$. Here $\mathcal{P}$ stands for the Poincaré duality map

$$\mathcal{P} : H_*(GL_n/B;\mathbb{C}) \longrightarrow H^*(GL_n/B;\mathbb{C}).$$

We shall now discuss the nilpotent case A(Z) for the space GL_n/B . Let U be the group of all lower triangular unipotent matrices in GL_n , and $z_{i,j}$, $1 \le j < i \le n$, the coordinate functions : $z_{i,j}(x) = x_{i,j}$, $x \in U$. Let n be the regular nilpotent nxn matrix , which is in the Jordan form , and let V be the vector field on GL_n/B induced from the one parameter subgroup $\exp(tn)$ of GL_n. V has a unique zero $x_0 = B$, and satisfies the property (B) ([1]). The coordinate ring A(Z) of the zero scheme Z of V in the affine neighbourhood U of x_0 has been computed in [2], and the following description has been obtained. Consider the grading on the polynomial algebra $A(U) = \mathbb{C}[z_{i,j} : 1 \leq j < i \leq n]$ determined by taking $\deg z_{i,j} = i-j$. Then A(Z) is isomorphic, as a graded algebra, to A(U)/ I(Z), where I(Z) is the ideal of A(U) generated by the homogeneous elements

$$z_{i+1,j} - z_{i,j-1} + z_{i,j}(z_{j,j-1} - z_{j+1,j}),$$

where we take $z_{k,r} = 0$ if $k > n$, or $r < 1$, or $r > k$.

Let I_k, $k = 1,2,\dots,n-1$, denote the set of sequences of integers $(i_1,\dots,i_k)$ such that $1 \leq i_1 < i_2 < \dots < i_k \leq n$, and let W_k be the set of all permutations $(\mu_1,\dots,\mu_n)$ in W such that $(\mu_1,\dots,\mu_k) \in I_k$ and $(\mu_{k+1},\dots,\mu_n) \in I_{n-k}$. For any $(i_1,\dots,i_k)$ in I_k there exists a unique permutation in the form $(i_1,\dots,i_k,i_{k+1},\dots,i_n)$ in W_k. We denote this permutation by $\sigma(i_1,\dots,i_k)$. For $(i_1,\dots,i_k)$ in I_k, let $[i_1,\dots,i_k]$ denote the function in A(Z), which is induced from the Plücker coordinate

$$\left| z_{i_m,j} \right|, \quad m = 1,2,\dots,k,$$

where $1 \leq i_1 < \dots < i_k \leq n$, and $1 \leq j \leq k$.

Here, and troughout the rest of the paper, we take $z_{k,r} = 0$, if $k > n$, or $r > k$, or $r < 1$. The following theorem is proved in [2].

THEOREM 1.2. The homomorphism $\varphi : R \longrightarrow A(U)$ determined by

$$\varphi(x_i) = z_{i+1,i} - z_{i,i-1} \quad , \ i = 1, \ldots, n,$$

induces a graded algebra isomorphism

$$\bar{\varphi} : R/IR \longrightarrow A(Z).$$

Moreover for any $(i_1, \ldots, i_k)$ in I_k we have

$$\bar{\varphi}\,(P_{\sigma(i_1,\ldots,i_k)}) = [i_1, \ldots, {}_k].$$

II. A Certain ideal associated with a Schubert variety in the cohomology of GL_n/B

We keep the notation of Section 1 , and moreover for a given sequence of distinct integers $(j_1, \ldots, j_k)$, $(j_1, \ldots, j_k)^<$ (respectively $(j_1, \ldots, j_k)^>$) denotes the sequence $(j_{\tau_1}, \ldots, j_{\tau_k})$, where $j_{\tau_1} < \ldots < j_{\tau_k}$ (respectively , $j_{\tau_1} > \ldots > j_{\tau_k}$) for some permutation $\tau = (\tau_1, \ldots, \tau_k)$ of $\{1,2,\ldots,k\}$. We recall the following well known formula , which is due to Monk [10] (see also [5], [6], and [9]).

THEOREM 2.1. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a permutation in W , and $k = 1,2,\ldots,n-1$. Then the following identity holds in R/IR.

$P_\mu x_k = \sum \operatorname{sgn}(j-k)\, P_{\mu(j,k)}$, where the sum is over all $j \neq k$ such that $\ell(\mu(j,k)) = \ell(\mu) + 1$.

For $k = 1,2,\dots,n-1$, let $p_k : W \longrightarrow W_k$ denote the projection map $p_k(\mu_1, \dots, \mu_n) = \sigma((\mu_1, \dots, \mu_k)^<)$

$$= ((\mu_1, \dots, \mu_k)^<, (\mu_{k+1}, \dots, \mu_n)^<)$$

We note that the Bruhat ordering $<$ on W ($\tau \leq \mu$ if and only if $B\tau B \subseteq \overline{B\mu B}$ in GL_n/B) induces an ordering on W_k , which we will also denote by $<$. Recall that for $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_n)$ in W_k, $\mu \leq \nu$ (in W_k) if and only if $\mu_i \leq \nu_i$ for $i = 1, \dots, k$.

LEMMA 2.1. let $\mu = (\mu_1, \dots, \mu_n)$ be a permutation in W which satisfies $\mu_1 > \dots > \mu_k$ and $\mu_{k+1} > \dots > \mu_n$. We have the following equality in R/IR.

$$P_\mu = P_{p_k(\mu)}\, x_1^{k-1} x_2^{k-2} \cdots x_{k-1}\, x_{k+1}^{n-k-1} x_{k+2}^{n-k-2} \cdots x_{n-1}$$

$$+ \sum m_\tau P_\tau \text{ , where the sum is over all } \tau \text{ in W such that}$$

$p_k(\mu) < p_k(\tau)$ in W_k .

<u>Proof</u>. By using the Monk's formula for the successive multiplications

$P_{p_k(\mu)} x_1$, $(P_{p_k(\mu)} x_1) x_1$, ... , $(P_{p_k(\mu)} x_1^{k-2}) x_1$,

$(P_{p_k(\mu)} x_1^{k-1}) x_2 , \ldots , (P_{p_k(\mu)} x_1^{k-1} x_2^{k-2}) , \ldots$

$\ldots , P_{p_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1}$, it is not difficult to see that at each stage of the multiplication , there appears, in the sum, only one P_ζ with $p_k(\zeta) = p_k(\mu)$, and all the remaining P_ν satisfy $p_k(\mu) < p_k(\nu)$. (Note that we start with the permutation $p_k(\mu)$, where the first k elements appear in ascending order). Thus we get an expression in the form

$$P_{p_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1} = P_{(\mu_1,\ldots,\mu_k,\mu_n,\ldots,\mu_{k+1})} + \sum m_\xi P_\xi$$

where $m_\xi \in Z$, and the sum is over all ξ in W such that $p_k(\mu) < p_k(\xi)$. We repeat this process , multiplying $P_{p_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1}$ first by x_{k+1} , then x_{k+1}^2, ..., and then x_{k+1}^{n-k-1} ,... ,finally x_{n-1} . It is clear that by arguing as above we obtain the claim. ■

LEMMA 2.2. For any permutation $\mu = (\mu_1,\ldots,\mu_n)$ in W , and $k = 1,2,\ldots,n-1$, the following holds in R/IR

$$P_\mu = f\, P_{p_k(\mu)} + \sum m_\tau P_\tau ,$$ where the sum is over all τ in W such that $p_k(\mu) < p_k(\tau)$ in W_k .

<u>Proof</u>. It follows from Lemma 2.1 that

$$P_{((\mu_1,\ldots,\mu_k)^>,(\mu_{k+1},\ldots,\mu_n)^>)} = P_{p_k(\mu)}\, g + \sum m_\xi P_\xi ,$$

where $g = x_1^{k-1} x_2^{k-2} \ldots x_{k-1} x_{k+1}^{n-k-1} \ldots x_{n-1}$. Since the operator $\partial_{(i,i+1)}$ has the property that $\partial_{(i,i+1)}(P_{(\xi_1,\ldots,\xi_n)}) = P_{(\xi_1,\ldots\xi_{i+1},\xi_i,\ldots,\xi_n)}$, if $\xi_i > \xi_{i+1}$, and $= 0$, otherwise, we can pass from $P_{(\,(\mu_1,\ldots,\mu_k)^{>},(\mu_{k+1},\ldots,\mu_n)^{>}\,)}$ to P_μ by using $\partial_{(i,i+1)}$ in an appropriate way. We note that in doing this we need to use only those $\partial_{(i,i+1)}$, where $i \neq k$. On the other hand for $i \neq k$ we have

(a) $\partial_{(i,i+1)}(P_{p_k(\mu)}\, g) = P_{p_k(\mu)}\partial_{(i,i+1)}(g)$, because $P_{p_k(\mu)}$ is a symmetric polynomial $x_1,\ldots,x_k$, and does not depend on the remaining variables $x_{k+1},\ldots,x_n$.

(b) $p_k(\partial_{(i,i+1)}(P_\xi)) = p_k(P_\xi)$.

Thus the assertion follows.■

For a given permutation μ in W, let J_μ be the ideal of R/IR generated by P_σ, $\sigma \not\leq \mu$, and let $\mathcal{G} = \bigcup_{k=1}^{n-1} W_k$ denote the set of the so called Grassmannian permutations of $\{1,2,\ldots,n\}$.

THEOREM 2.2. For any permutation μ in W, J_μ is the ideal generated by P_τ, where $\tau \not\leq \mu$, and τ is in $\mathcal{G}$.

<u>Proof</u>. The assertion is true for $\mu=\omega_0=(n,n-1,\ldots,1)$. For every permutation $\mu\neq\omega_0$ there exists a permutation ν and $k \in \{1,\ldots,n\}$

such that $\mu=\nu(k,k+1)$ and $\ell(\nu)=\ell(\mu)+1$. Thus, it is sufficient to prove the following implication: If the assertion is true for ν, then it is true for μ. Let $\mathcal{I}(\mu)$ be the set of all permutations σ such that $\sigma \not\leq \mu$. It suffices to show that for every $\omega\in\mathcal{I}(\mu)-\mathcal{I}(\nu)$ the polynomial P_ω belongs to the ideal J_μ . This is true for $\omega=\nu$. To end, it is sufficient to prove the following implication: If P_ξ belongs to the ideal J_μ , then for every ω such that $p_k(\xi) > p_k(\omega)$, the polynomial P_ω belongs to the ideal J_μ .

By Lemma 2.2 we get

$P_\omega = f\,P_{p_k(\omega)} + \sum m_\xi P_\xi$, where the summation is over ξ such that $p_k(\xi) > p_k(\omega)$, $m_\xi \in Z$, and $f\in R/IR$. We know that the terms in the sum on the right hand side are in J_μ. Moreover it is not hard to check that $\omega \in \mathcal{I}(\mu)-\mathcal{I}(\nu)$ if and only if $p_k(\omega) \in \mathcal{I}(\mu)-\mathcal{I}(\nu)$. Therefore $f\,P_{p_k(\omega)} \in J_\mu$, and the proof is complete.■

III. The nilpotent description of the cohomology ring of a Schubert subvariety of GL_n/P

Let G be a complex reductive linear algebraic group, B a Borel subgroup of G, and P a parabolic subgroup of G which contains B. Let n be regular nilpotent element of the Lie algebra g of G which is taken from the Lie algebra b of B , and $\breve{V}$

(respectively V) the vector field induced from the C-action $\exp(tn)$ on G/B (respectively G/P). By the Jacobson-Morosov Lemma (see [7]) $\tilde{V}$ (respectively V) satisfies the property (B), and in fact the above C^*-action is induced from a one parameter subgroup of B via the left multiplication. We also note that $\tilde{V}$ (respectively V) has only one zero x_0 = B (respectively P). Thus we can talk about the nilpotent description of any B-invariant subvariety of G/B (respectively G/P).

PROPOSITION 3.1. If the cohomology ring of any Schubert subvariety of G/B has a nilpotent discription, then the cohomology ring of any Schubert subvariety of G/P has also a nilpotent description.

Proof. Let $\tilde{Z}$ (respectively Z) denote the zero scheme of $\tilde{V}$ (respectively V), and let $Y_\sigma = \overline{B\sigma P}$ be the Schubert subvariety of G/P . Let $\pi: G/B \longrightarrow G/P$ denote the natural projection map. It is well known that the inverse image scheme $\pi^{-1}(Y_\sigma)$ of Y_σ is a Schubert subvariety $X_{\sigma\tau} = \overline{B\sigma\tau B}$ of G/B , and the restriction map $\rho := \pi| : X_{\sigma\tau} \longrightarrow Y_\sigma$ is a P/B fibration (see [8], for example). Thus the fibre product map $(Y_\sigma \cap Z) \times_{Y_\sigma} X_{\sigma\tau} \longrightarrow Y_\sigma \cap Z$ induced by ρ is also a P/B fibration. This implies that $(Y_\sigma \cap Z) \times_{Y_\sigma} X_{\sigma\tau}$ is B-equvariantly isomorphic to $(Y_\sigma \cap Z) \times P/B$, because dim $Y_\sigma \cap Z=0$. Since ρ is a surjective B- equvariant map, the fixed point scheme

$((Y_\sigma \cap Z) \times_{Y_\sigma} X_{\sigma\tau})^{\mathbf{C}}$ of the $\mathbf{C}$- action induced by $\exp(tn)$ on $(Y_\sigma \cap Z) \times_{Y_\sigma} X_{\sigma\tau}$ is isomorphic to $X_{\sigma\tau} \cap \breve{Z}$. This gives us $(Y_\sigma \cap Z) \times (P/B)^{\mathbf{C}} \cong X_{\sigma\tau} \cap \breve{Z}$. Let ρ_1 denote the map $X_{\sigma\tau} \cap \breve{Z} \longrightarrow Y_\sigma \cap Z$, induced by the projection $\rho : X_{\sigma\tau} \cap \breve{Z} \longrightarrow Y_\sigma \cap Z$. It follows from above that the comorphism $(\rho_1)^* : A(Y_\sigma \cap Z) \longrightarrow A(X_{\sigma\tau} \cap \breve{Z})$ is an inclusion . On the other hand, we have the following commutative diagram of graded algebra homomorphisms

$$\begin{array}{ccc} A(X_{\sigma\tau} \cap \breve{Z}) & \xrightarrow{\ \simeq\ } & H^*(X_{\sigma\tau};\mathbf{C}) \\ {\scriptstyle (\rho_1)^*}\uparrow & & \uparrow \\ \bar{\psi}:\ A(Y_\sigma \cap Z) & \longrightarrow & H^*(Y_\sigma;\mathbf{C}) \end{array},$$

(see [1], for example). It follows from the diagram that $\bar{\psi}$ is injective, and therefore it is an isomorphism.■

THEOREM 3.1. The cohomology ring of any Schubert subvariety of GL_n/B has a nilpotent description.

<u>Proof</u>. Let $X_\omega = \overline{B\omega B}$ be the Schubert subvariety of Gl_n/B associated to ω in W, and let J_ω be the ideal of $A(U)= \mathbf{C}[Z_{i,j} : 1\le j< i\le n]$ generated by those Plücker coordinates

$\left|z_{i_m,j}\right|$, where $m=1,\dots,k$, $1\le j\le k$, $(i_1,\dots,i_k)\in I_k$, and $\sigma(i_1,\dots,i_k) \nleq \omega$ in W. It is well known that J_ω is the ideal of the Schubert variety X_ω in the affine neighbourhood U of $x_0=B$ (see [8] Theorem 9.1, for example). This implies that if f is in $\bar{J}_\omega$, then f= 0 in $A(X_\omega \cap Z)$, where $\bar{J}_\omega$ is the ideal of A(Z) generated by $[i_1,\dots,i_k]$ such that $\sigma(i_1,\dots,i_k) \nleq \omega$ in W. Here $[i_1,\dots,i_k]$ denotes the function in A(Z) induced from the Plücker coordinate $\left|z_{i_m,j}\right|$.

By using Theorem 1.2, and Theorem 2.2 , we obtain $j^*(\bar{\varphi}(P_\tau)) = 0$ whenever $\tau \nleq \omega$ in W. Here j stands for the natural inclusion $X_\omega \cap Z \longrightarrow Z$, and $\bar{\varphi}$ is the isomorphism $R/IR \cong A(Z)$ given in Theorem 1.2. It follows from this fact that the vector space $A(X_\omega \cap Z)$ is spanned by the set $\{ j^*(\bar{\varphi}(P_\xi)) : \xi \leqq \omega \}$. Since $\{ P_\sigma : \sigma\in W \}$ is a basis of R/IR , we get $\dim_C A(X_\omega \cap Z) \leqq$ cardinality $\{ \xi \in W: \xi \leqq \omega \} = \dim_{\mathbf{C}} H^*(X_\omega;\mathbf{C})$. Thus the surjective map $\bar{\psi}: A(X_\omega \cap Z) \longrightarrow H^*(X_\omega;\mathbf{C})$ is an isomorphism. ■

COROLLARY. The cohomology ring of any Schubert subvariety of the partial flag manifold GL_n/P has a nilpotent description.

<u>Proof</u>. It follows from Proposition 3.1, and Theorem 3.2. ■

REFERENCES

[1] E. Akyildiz, SL_2 actions and cohomology of Schubert varieties, Proceedings of Classical Algebraic Structures, Banach Center Publication Series, vol.26, to appear.

[2] E. Akyildiz and Y. Akyildiz, The relations of Plücker coordinates to Schubert calculus, J. Diff. Geometry, 29 (1989), 135-142.

[3] E. Akyildiz, J.B. Carrell, and D.I. Lieberman, Zeros of holomorphic vector fields on singular spaces and intersection rings of Schubert varieties , Compositio Math. 57 (1986), 237-248.

[4] E. Akyildiz and J.B. Carrell, Cohomology of projective varieties with regular SL_2 actions, Manuscripta Math. 58 (1987), 473-486.

[5] I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand, Schubert cells and cohomology of the space G/P, Russian Math. Surveys, 28 (1973), 1-26.

[6] M. Demazure, Desingularisation des varietes de Schubert generalisees, Ann. Scient. Ec. Norm. Sup., 4^e serie, t.7(1974), 53-88.

[7] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of complex semisimple Lie group, Amer. J. Math. 81 (1959), 973-1032.

[8] V. Lakshmibai and C.S. Seshadri, Geometry of G/P-V, J. Algebra 100 (1986), 462-557.

[9] A. Lascoux and M.P. Schützenberger, Symmetry and flag manifolds, Springer Lecture Notes in Math. 996 (1983), 118-144.

[10] D. Monk, The geometry of flag manifolds, Proc. London Math. Soc., 9 (1959), 253-286.

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