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ABSTRACT

We consider an infinite layered strip with a free upper surface while the lower surface is rigidly fixed. By using the Green function, we obtain spectral representation of Love wave operator yielding a representation of delta function in terms of the complete set of eigenfunctions of the problem.

1. INTRODUCTION

In problems of transmission, reflection and diffraction of certain waves, it is quite useful to obtain spectral representation of the operator involved. To do so, we first obtain the Green function associated with the problem and then integrate it around a large circle enclosing all the singularities. An account of this method can be found in Friedman (1956) or Stakgold (1979). Kazi (1975) has used this approach to obtain spectral representation of the Love wave operator for layered media.

In this paper we obtain the spectral representation of the Love-wave operator under somewhat different boundary conditions. Hudson (1962) has shown that if a homogeneous infinite strip is bounded by a free surface and a rigid boundary, then the Love-wave may propagate in this wave-guide type structure. We consider a layered strip with upper surface to be free and lower surface to be rigidly fixed and obtain spectral representation of the operator arising from this problem. This spectral representation may then be used to obtain integral equation formulation and scattering matrix for transmission of Love-waves in a layered strip with free-rigid boundary conditions undergoing horizontal changes.

2. FORMULATION OF THE PROBLEM

We consider an infinite strip consisting of a layer of rigidity μ_1, shear velocity β_1, density ρ_1 and thickness h, overlying a thicker layer of rigidity μ_2, shear velocity

β_2, density ρ_2 and thickness H-h. The upper surface of the layered strip is stress free while the bottom surface is assumed to be rigid.

The equation of motion for the two layers will be

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta_i^2}\frac{\partial^2 v}{\partial t^2}, \quad i = 1,2. \tag{1}$$

Let $v(x,z,t) = V(z)\exp[i(\omega t - kx)]$, where

$$\begin{aligned} V(z) &= V_1(z) \qquad 0 < z < h \\ &= V_2(z) \qquad h < z < H. \end{aligned} \tag{2}$$

Then $V_1(z)$ and $V_2(z)$ satisfy the following equations

$$\frac{d^2V_1}{dz^2} + \sigma_1{}^2 V_1 = 0, \qquad 0 < z < h, \tag{3}$$

where

$$\sigma_1 = (\frac{\omega^2}{\beta_1{}^2} - \lambda), \qquad \lambda = k^2, \quad \beta_1 = \frac{\mu_1}{\rho_1}$$

and

$$\frac{d^2V_2}{dz^2} - \sigma_2{}^2 V_2 = 0 \qquad h < z < H, \tag{4}$$

where

$$\sigma_2{}^2 = \left[\lambda - \frac{\omega^2}{\beta_2{}^2}\right], \qquad \beta_2{}^2 = \frac{\mu_2}{\rho_1}$$

together with the boundary and interface conditions

$$V_1(h) = V_2(h) \tag{5a}$$

$$\mu_1 V_1'(h) = \mu_2 V_2'(h) \tag{5b}$$

$$V_1' = 0 \tag{5c}$$

$$V_2(H) = 0 \tag{5d}$$

3. THE GREEN FUNCTIONS

Let $G(z,\xi;\lambda) = G_{ij}$, $i,j = 1,2$. The subscript i refers to the z-interval while j refers to the ξ-interval and 1,2 refer to the intervals $(0,h)$ and (h,H), respectively (see Fig.2). If we find G_{ij}, the Green function is known completely.

a) If $0 \le \xi \le h$, then G_{11} and G_{12} satisfy the differential equations

$$\frac{d^2 G_{11}}{dz^2} + \sigma_1^2 G_{11} = \delta(z - \xi) \tag{6}$$

and

$$\frac{d^2 G_{21}}{dz^2} - \sigma_2^2 G_{21} = 0 \tag{7}$$

together with the following conditions

$$\frac{\partial G_{11}}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{8a}$$

$$G_{11} = G_{21} \quad \text{at} \quad z = H \tag{8b}$$

$$\mu_1 \frac{\partial G_{11}}{\partial z} = \mu_2 \frac{\partial G_{21}}{\partial z} \quad \text{at} \quad z = h \tag{8c}$$

$$G_{21} = 0 \qquad \text{at} \quad z = H \tag{8d}$$

$$G_{11}(z,\xi+0;\ \lambda) = G_{11}(z+\xi-0;\ \lambda) \tag{8e}$$

and

$$\underset{z\to\xi+0}{\mathrm{Lt}} \frac{\partial G_{11}}{\partial z} - \underset{z\to\xi-0}{\mathrm{Lt}} \frac{\partial G_{11}}{\partial z} = \frac{1}{\mu_1}\ . \tag{8f}$$

By following stakgold (1979), with some effort, we find,

$$G_{11}(z,\xi;\lambda) = \frac{\gamma_1}{\mu_1\sigma_1} \cos\sigma_1\xi \ \cos\sigma_1 z + \frac{1}{\mu_1\sigma_1}[\sin\sigma_1\xi \ \cos\sigma_1 z \ H(\xi-z)$$

$$+ \cos\sigma_1\xi \ \sin\sigma_1 z \ H(z-\xi)], \tag{9}$$

and

$$G_{21}(z,\xi;\lambda) = -\frac{\cos\sigma_1\xi}{\mu_1\sigma_1}[\gamma_1\cos\sigma_1 h + \sin\sigma_1 h]\frac{\sinh\sigma_2(z-H)}{\sinh\sigma_2(H-h)} \tag{10}$$

where

$$\gamma_1 = \frac{\mu_1\sigma_1 + \mu_2\sigma_2\tan\sigma_1 h \ \coth\sigma_2(H-h)}{\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2\cot\sigma_2(H-h)}\ , \tag{11}$$

and $H(x)$ is the Heaviside unit function.

b) If $h \le \xi \le H$, then G_{22} and G_{12} satisfy the differential equations

$$\frac{d^2G_{12}}{dz^2} + \sigma_1{}^2G_{12} = 0 \tag{12}$$

and

$$\frac{d^2G_{22}}{dz^2} - s_2{}^2G_{22} = \delta(z-\xi)$$

together with following conditions:

$$\frac{\partial G_{22}}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{14a}$$

$$G_{22} = G_{12} \quad \text{at} \quad z = h \tag{14b}$$

$$\mu_2 \frac{\partial G_{22}}{\partial z} = \mu_1, \quad \frac{\partial G_{12}}{\partial z} \quad \text{at} \quad z = h \tag{14c}$$

$$G_{12} = 0 \quad \text{at} \quad z = H \tag{14d}$$

$$G_{22}(z, \xi+0; \lambda) = G_{22}(z, \xi-0; \lambda) \tag{14e}$$

and

$$\underset{z \to \xi+0}{\text{Lt}} \frac{\partial G_{22}}{\partial z} - \underset{z \to \xi-0}{\text{Lt}} \frac{\partial G_{22}}{\partial z} = \frac{1}{\mu_2} \, . \tag{14f}$$

By the usual procedure, we find

$$\begin{aligned} G_{22}(z, \xi; \lambda) = {} & \frac{-\gamma_2}{\mu_2 \sigma_2} \sinh\sigma_2(H-\xi) \sinh\sigma_2(z-H) \\ & - \frac{1}{\mu_2 \sigma_2}[\sinh\sigma_2(H-\xi) \cosh\sigma_2(z-H) \, H(\xi-z) \\ & - \cosh\sigma_2(H-\xi) \sinh\sigma_2(z-H) \, H(z-\xi)], \end{aligned} \tag{15}$$

and

$$\begin{aligned} G_{12}(z, \xi; \lambda) = {} & \frac{1}{\mu_2 \sigma_2} \sinh\sigma_2(H-\xi)[\gamma_2 \sinh\sigma_2(H-h) \\ & - \cosh\sigma_2(H-h)]\frac{\cos\sigma_1 z}{\cos\sigma_1 h} \end{aligned} \tag{16}$$

where

$$\gamma_2 = \frac{\mu_1 \sigma_1 \tan\sigma_1 h \coth\sigma_1(H-h) - \mu_2 \sigma_2}{\mu_1 \sigma_1 \tan\sigma_1 h - \mu_2 \sigma_2 \coth\sigma_2(H-h)} \tag{17}$$

It may be noted that

$$G_{12}(z,\xi;\lambda) = G_{21}(\xi,z;\lambda)$$

$$= \frac{\cos\sigma_1 z \ \sinh\sigma_2(H-\xi)}{\cos\sigma_1 h \ \sin\sigma_2(H-h)\{\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2\coth\sigma_2(H-h)\}} \qquad (18)$$

as is to be expected by the symmetric property of the Green function.

4. SPECTRAL REPRESENTATION

We use the formula (Stakgold (1979) p.416):

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} G(z,\xi;\lambda)d\lambda = - \Sigma \, \phi^{(n)}_{(z)} \, \overline{\phi^{(n)}}(\xi) = - \frac{\delta(z-\xi)}{\mu(\xi)}, \quad (19)$$

where $\{\phi^{(n)}(z)\}$ is the orthonormalized complete set of eigenfunctions associated with this problem. To carry out the integration involved in the formula, we proceed as follows:

(a) Let $I_{11} = \lim_{R\to\infty} - \frac{1}{2\pi i} \oint G_{11}(z,\xi;\lambda)d\lambda$

$$= \lim_{R\to\infty} - \frac{1}{2\pi i} \oint_{|\lambda|=R} \Big[\frac{\gamma_1}{\mu_1\sigma_1} \cos\sigma_1\xi \ \cos\sigma_1 z$$

$$+ \frac{1}{\mu_1\sigma_1}\left\{\sin\sigma_1\xi \ \cos\sigma_1 z \ H(\xi - z) + \cos\sigma_1\xi \ \sin\sigma_1 z \ H(z-\xi\right\}\Big]d\lambda \tag{20}$$

where $\gamma_1 = \dfrac{\mu_1\sigma_1 + \mu_2\sigma_2 \tan\sigma_1 h \coth\sigma_2(H-h)}{\mu_1\sigma_1 \tan\sigma_1 h - \mu_2\sigma_2 \coth\sigma_2(H-h)}$.

We note that the only singularities of the integrand are the poles which are roots of the equation

$$\mu_1\sigma_1 \tan\sigma_1 h - \mu_2\sigma_2 \coth\sigma_2(H - h) = 0 \tag{21}$$

All these poles are simple and are located on the real axis. The equation (21) satisfied by the poles is, in fact dispersion relation satisfied by Love-waves travelling in an infinite strip with free-rigid boundary conditions (Hudson (1962)).

The evaluation of the integral, therefore gives

$$I_{11} = \sum_{n=1}^{\infty} \frac{-\mu_1\sigma_1^{(n)} + \mu_2\sigma_2 \tan\sigma_1^{(n)}h \coth\sigma_2^{(n)}(H-h)}{\mu_1\sigma_1^{(n)} \frac{\partial}{\partial\lambda}\left[\mu_1\sigma_1 \tan\sigma_1 h - \mu_2\sigma_2 \coth\sigma_2(H-h)\right]_{\lambda=\lambda_n}} \times$$

$$\times \cos\sigma_1^{(n)}\xi \ \cos\sigma_1^{(n)} \tag{22}$$

where $\sigma_i^{(n)} = \left[\frac{\omega^2}{\beta_i^2} - \lambda_n\right]^{1/2}$, $i = 1,2;$ $\{\lambda_n\}$, $n = 1,2,\ldots$ being the infinite set of roots of equation (21).

If we put the phase velocity $c_n = \frac{\omega}{\lambda_n^{1/2}}$ and group velocity

$u_n = 2\lambda_n^{1/2} \frac{d\omega}{d\lambda_n}$, so that

$$\left.\frac{\partial\sigma_1}{\partial\lambda}\right|_{\lambda=\lambda_n} = -\frac{1}{2\sigma_1^{(n)}} \frac{\left(1 - u_n c_n\right)}{\beta_1^{2}}$$

and

$$\left.\frac{\partial\sigma_2}{\partial\lambda}\right|_{\lambda=\lambda_n} = \frac{1}{2\sigma_2^{(n)}} \left(1 - \frac{u_n c_n}{\beta_1^{2}}\right),$$

we arrive at

$$I_{11} = \sum_{n=1}^{\infty} D_n^2 \frac{\cos\sigma_1^{(n)}\xi}{\cos\sigma_1 h} \frac{\cos\sigma_1 z}{\cos\sigma_1 h}, \tag{23}$$

where $D_n = 2\left\{\sigma_2^{(n)}\right\}^{1/2}\left\{\frac{\beta_1^{-2} - u_n^{-1} c_n^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right\}^{1/2} \times$

$$\times \frac{\sinh \sigma_2 (H-h)}{\left\{\sinh 2\sigma_2^{(n)}(H-h) - 2\sigma_2^{(n)}(H-h)\right\}^{1/2}}. \tag{24}$$

Therefore if we write

$$\phi_1^{(n)}(z) = D_n \frac{\cos\sigma_1 z}{\cos\sigma_1 h}, \tag{25}$$

we have

$$I_{11} = \sum_{n=1}^{\infty} \phi_1^{(n)}(\xi)\phi_2^{(n)}(z) \tag{26}$$

b) Next, we consider

$$I_{12} = \lim_{n\to\infty} \frac{-1}{2\pi i} \times$$

$$\times \oint_{|\lambda|=R} \frac{\cos\sigma_1 z \sinh\sigma_2(H-\xi)}{\cos\sigma_1 h \sinh\sigma_2(H-h)\{\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2 \coth\sigma_2(H-h)\}} d\lambda \tag{27}$$

Again, the poles of the integrand are the same as those in case of I_{11} and so a similar process yields

$$I_{12} = \sum_{n=1}^{\infty} D_n^2 \frac{\cos\sigma_1 z}{\cos\sigma_1 h} \frac{\sinh\sigma_2(H-\xi)}{\sinh\sigma_2(H-h)}, \tag{28}$$

where D_n is given by equation (24). Thus taking the same $\phi_1^{(n)}(\xi)$ as in (25) and

$$\phi_2^{(n)}(\xi) = D_n \frac{\sinh\sigma_2(H-\xi)}{\sinh\sigma_2(H-h)}, \tag{29}$$

we may write

$$I_{12} = \sum_{n=1}^{\infty} \phi_1^{(n)}(z)\ \phi_2^{(n)}(\xi) \tag{30}$$

c) Next, we have

$$I_{22} = \lim_{R\to\infty} -\frac{1}{2\pi i} \oint_{|\lambda|=R} G_{22}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} \left[\frac{\gamma_2}{\mu_2\sigma_2} \sinh \sigma_2(H-\xi)\ \sinh\sigma_2(z-H)\right.$$

$$+ \frac{1}{\mu_2\sigma_2}[\sinh\sigma_2(H-\xi)\ \cosh\sigma_2(z-H)H(\xi-z)$$

$$\left. - \cosh\sigma_2(H-\xi)\ \sinh\sigma_2(z-H)H(z-\xi)\right]d\lambda, \tag{31}$$

where $\gamma_2 = \dfrac{\mu_1\sigma_1\tan\sigma_1 h\ \coth\sigma_2(H-h) - \mu_2\sigma_2}{\mu_1\sigma_1\ \tan\sigma_1 h - \mu_2\sigma_2\ \coth\sigma_2(H-h)}$.

There are no branch points as before and thus calculating residues at poles we get

$$I_{22} = \sum_{n=1}^{\infty} D_n^2 \frac{\sinh \sigma_2^{(n)}(H-\xi)}{\sinh \sigma_2^{(n)}(H-h)} \quad \frac{\sinh\sigma_2^{(n)}(H-z)}{\sinh\sigma_2^{(n)}(H-h)}$$

$$= \sum_{n=1}^{\infty} \phi_2^{(n)}(\xi)\phi_2^{(n)}(z). \tag{32}$$

d) Since $G_{21}(z,\xi;\lambda) = G_{12}(\xi,z;\lambda)$, we also have

$$I_{21} = \sum_{n}^{\infty} \phi_1^{(n)}(z)\phi_2^{(n)}(\xi) = I_{12}. \tag{33}$$

We therefore obtain the following representations of the delta function:

$$\delta(z-\xi) = \sum_{n=1}^{\infty} \mu(\xi)\ \phi^{(n)}_{(z)}\ \phi^{(n)}_{(\xi)}, \tag{34}$$

where $$\phi^{(n)}(z) = \begin{cases} \phi_1^{(n)}(z) = D_n \dfrac{\cos \sigma_1 z}{\cos\sigma_1 h}, & 0 \le z \le h, \\ \phi_2^{(n)}(z) = D_n \dfrac{\sinh \sigma_2(H-\xi)}{\sinh \sigma_2(H-h)}, & h \le z \le H, \end{cases} \tag{35}$$

$$\mu(z) = \begin{cases} \mu_1, & 0 \le z < h, \\ \mu_2, & h < h \le H. \end{cases} \tag{36}$$

This completes the spectral representation of the Love-wave operator in an infinite strip satisfying free-rigid boundary conditions.

In further work, we intend to consider scattering of Love-waves in an infinite strip satisfying free-rigid boundary conditions using the spectral representation achieved in the present paper.

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