King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 114

October 1989

Upper Semicontinuous Decompositions of E^3 Into Subarcs of Bing's Sling and Points

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ABSTRACT

An upper semicontinuous decomposition of E^3 into arcs and points whose non-degenerate elements are pairwise disjoint subarcs of Bing's sling is homeomorphic to E^3.

AMS subject classification

(1985, Revision)

Primary 57N12/Secondary 57M99

1. INTRODUCTION

Bing's sling is a certain simple closed curve in Euclidean 3-space E^3 for which there is no homeomorphism h from E^3 onto itself taking it to a circle. Because of the non-existence of h, we say that Bing's sling is a wild simple closed curve. It can be shown that any subarc A of Bing's sling is cellular, that is, each neighborhood of A contains a 3-cell which contains A in its interior.

We will study upper semicontinuous decompositions of E^3 into points and pairwise disjoint subarcs of Bing's sling. In section 4 we prove that such decompositions always yield decomposition spaces that are homeomorphic to E^3.

A decomposition G of a space X is a collection of pairwise disjoint non-empty subsets of X whose union is X. We use H for the set of all non-degenerate elements in G, that is, those with more than one point. E^3/G will denote the decomposition space associated with G. For the definition of a decomposition space we refer to [5].

Recall that a decomposition G of a space X is upper semicontinuous if for each $g \in G$ and for each open set U in X containing g, $\cup\{g' \in G | g' \subset U\}$ is open in X. This paper will deal with upper semicontinuous decompositions G of E^3 whose elements are compact and connected. Such decompositions are called monotone.

We created the Basic Lemma, the elegant repeated applications of which led the proof of the,

<u>Main Theorem</u>: Suppose G is a decomposition of E^3 such that the non-degenerate elements of G are pairwise disjoint subarcs of Bing's sling. Then the decomposition space E^3/G is homeomorphic to E^3.

In section 3 we will discuss the construction of Bing's sling. The construction of Bing's sling was first given in [3]. In section 4 we will prove that H is countable and G is upper semicontinuous.

The following theorems and examples are some of the relevant previous research in this area.

Theorem 1: Suppose G is an upper semicontinuous decomposition of E^3 such that each g in G is cellular, G has only a countable number of non-degenerate elements, and their union is a G_δ set. Then the decomposition space E^3/G is homeomorphic to E^3 [Theorem 1 of [1]].

Theorem 2: Suppose G is an upper semicontinuous decomposition of E^3 such that G has only a countable number of non-degenerate elements and each is a tame arc. Then E^3/G is homeomorphic to E^3.

Example 1: There is an upper semicontinuous decomposition G of E^3 such that H consists of uncountably many tame arcs and $\cup H$ is a compact set but E^3/G is not homeomorphic to E^3. The space E^3/G is called Bing's Dog-bone space. The details are in [2].

Example 2: There is an upper semicontinuous decomposition of the 3-sphere S^3 whose non-degenerate elements are all contained in a cellular arc, but the decomposition space S^3/G is not homeomorphic to S^3. Details are in [4].

2. PRELIMINARIES AND STATEMENT OF RESULTS

In this section we give some definitions and general topology results. Throughout this paper Id will denote the identity map, I the unit interval in E^3 and d will denote the usual metric on E^3.

Definition 1: A set X in E^3 is called cellular if $X = \bigcap_{i=1}^{\infty} D_i$, where the D_i's are 3-cells and Int $D_i \supset D_{i+1}$ for each i. Recall that a 3-cell is a space homeomorphic to the closed unit ball in E^3.

Definition 2: An arc M in E^3 is said to be tame if there exists a homeomorphism ϕ from E^3 onto itself such that $\phi(M) = I$.

Again a simple closed curve M in E^3 is said to be tame if there exists a homeomorphism ψ from E^3 onto itself such that $\psi(M)$ is a polygonal simple closed curve in E^3. In each case if M is not tame we say that it is wild.

Theorem 3: Suppose that $\langle f_i \rangle$ is a sequence of homeomorphisms of E^3 onto itself. Further, suppose that there is a sequence $\langle V_i \rangle$ of open sets such that

(a) $V_i \supset V_{i+1}$

(b) $f_{i+1} = f_i$ on $E^3 \setminus V_i$, and

(c) each component of $f_i(V_i)$ has diameter less than $1/i$.

Then $\langle f_i \rangle$ converges uniformly to a continuous function f.

<u>Theorem 4</u>: Suppose a sequence $\langle f_i \rangle$ of homeomorphisms of E^3 onto itself converges uniformly to a function f. Then f is onto E^3 and f is closed.

The above two theorems are general topology results and not so difficult to prove.

3. THE CONSTRUCTION OF BING'S SLING

Bing's sling is a certain wild simple closed curve in E^3 for which any subarc is cellular. We will use subarcs of Bing's sling for the non-degenerate elements in the decompositions of E^3 that we study. The construction of Bing's sling was first given in [3].

To construct Bing's sling we take first a solid torus T_1. A solid torus is a torus with its interior, that is, it is homeomorphic to the product of a circle and a 2-dimensional disk. Then we divide T_1 into 12 nonoverlapping cylindrical blocks $C_1, C_2, \ldots, C_{12}$, that is, each C_i is homeomorphic to the product of a line segment and a 2-dimensional disk.

In each of these blocks we put 3 solid tubes as shown in the figure below.

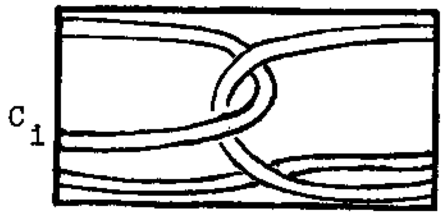
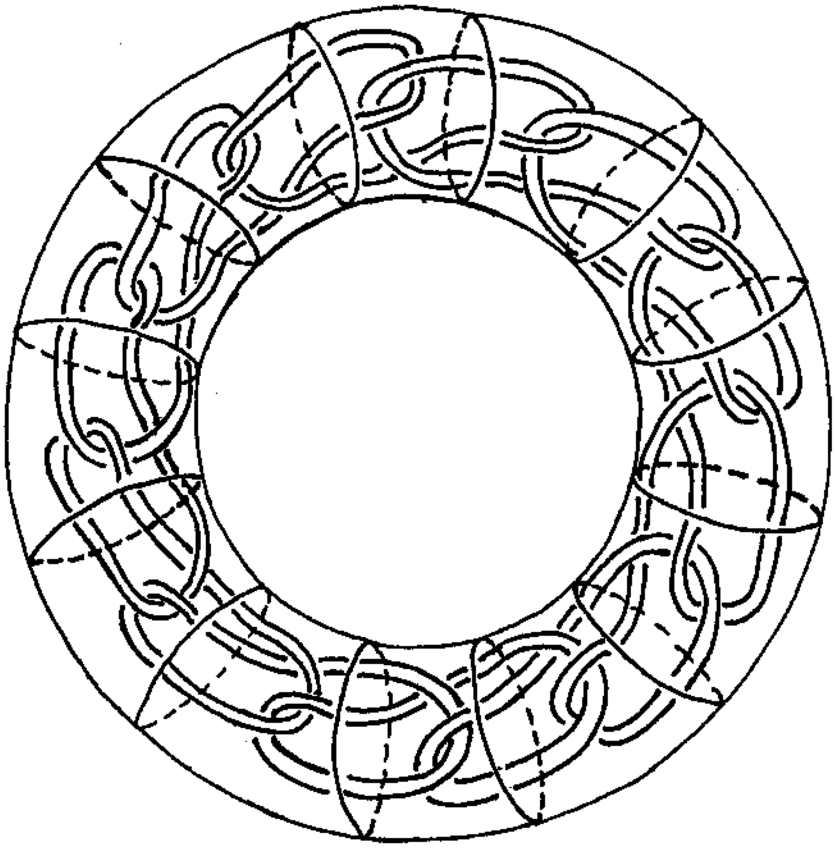


Figure 1

The tubes inside the blocks C_i fit together to make a solid torus T_2 in Int T_1. T_2 has many knots.



$T_2 \subset \text{Int } T_1$

Figure 2

In each component of $T_2 \cap C_i$ we put 4 copies of the block with tubes. The tubes inside these smaller cylinders fit together to make a thinner solid torus T_3.

We continue this process, to get solid tori $T_1 \supset \text{Int } T_1 \supset T_2 \supset \text{Int } T_2 \supset T_3 \supset \ldots,$ so that the diameters of the blocks of T_i approach 0 as i tends to ∞, and so that T_i has 12^i blocks.

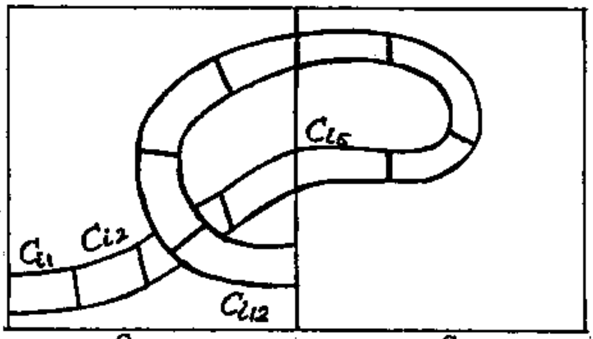


Figure 3

Now with each block C_i of T_1 we associate 12 blocks $C_{i1}, C_{i2}, \ldots, C_{i12}$ of T_2, indexed as shown. Similarly, with each block $C_{i_1 i_2 \ldots i_r}$ of T_r we associate 12 blocks $C_{i_1 i_2 \ldots i_r 1}, C_{i_1 i_2 \ldots i_r 2}, \ldots, C_{i_1 i_2 \ldots i_r 12}$ of T_{r+1} lying in $C_{i_1 i_2 \ldots i_r}$ and the block following $C_{i_1 i_2 \ldots i_r}$ in T_r.

The following figure gives some idea about the construction of Bing's sling.

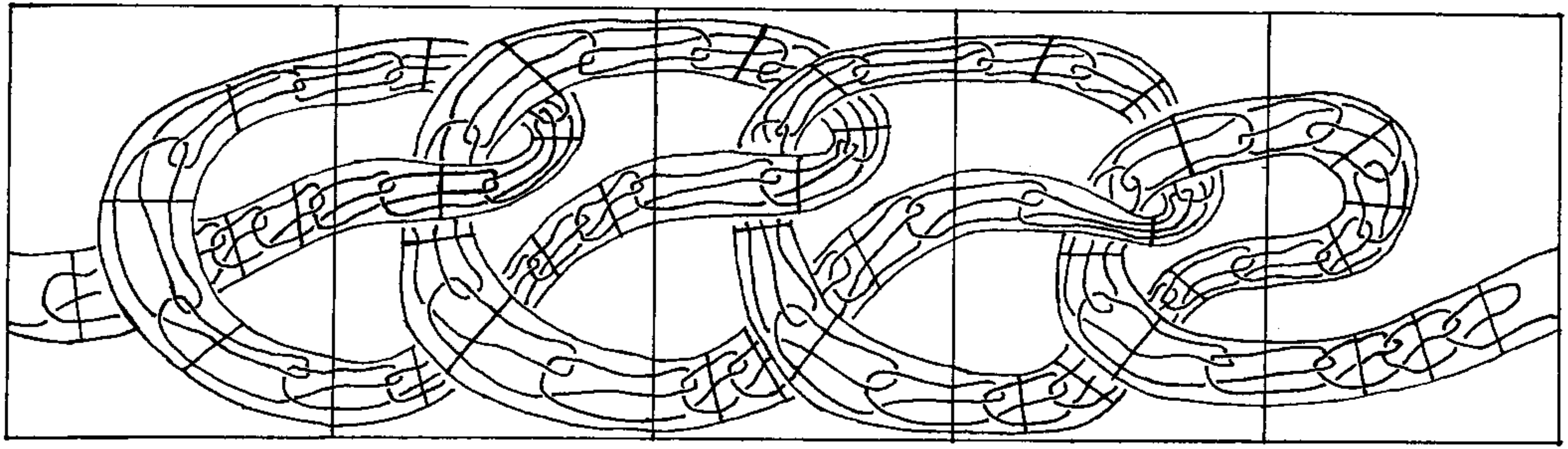


Figure 4

Let $J = \bigcap_{1}^{\infty} T_i$. Then J is a wild simple closed curve that does not pierce any disk, that is, J does not pass through any disk without intersecting the disk in more than one point. This simple closed curve is called Bing's sling.

We briefly indicate how to define a homeomorphism h from a circle $\sum$ onto $J = \bigcap_{i=1}^{\infty} T_i$. Let S_i be the center simple closed curve of T_i. Divide $\sum$ into 12 equal arcs $J_1, .., J_{12}$ and let h_1 be a homeomorphism from $\sum$ onto S_1 taking J_i to $S_1 \cap C_i$. Then divide each J_i into 12 equal arcs. Let h_2 be a homeomorphism from $\sum$ onto S_2 taking each subsegment J_{i_j} of J_i to $C_{i_j} \cap S_2$ for $i,j = 1,2,...,12$. We continue this process indefinitely and get a sequence $\langle h_i \rangle$ of homeomorphisms from $\sum$ into E^3. The sequence $\langle h_i \rangle$ converges uniformly to a homeomorphism h from $\sum$ onto J. This proves that J is a simple closed curve.

4. PROOF OF THE MAIN THEOREM

Before we prove our Main Theorem we need to prove that H is countable and G is upper semicontinuous. Then we state and prove our Basic Lemma which is developed in this paper for the proof of the Main Theorem.

Theorem 5: H is countable.

Proof: Let J be Bing's sling. J is a separable metric space, so J has a countable basis B.

Suppose H is uncountable. Now the non-degenerate elements in H are compact sets. Since each $g \in H$ is homeomorphic to a subarc of the simple closed curve J, each $g \in H$ has interior points in J. Thus, there is a nonempty basis element B_g such that $B_g \subset g$. Since the elements of H are pairwise disjoint, $\{B_g | g \in H\}$ is a collection of pairwise disjoint sets. Hence $\{B_g | g \in H\}$ is an uncountable subset of the basis for J. This is a contradiction because the basis B for J is countable.

Hence H is countable. We write $H = \{g_i | i = 1, 2, \ldots\}$.

Theorem 6: The diameter of g_i approaches zero as i tends to infinity.

Proof: Suppose there exists $\varepsilon > 0$ such that $\operatorname{diam} g_i \geq \varepsilon$ for infinitely many i's, say for $i_1, i_2, \ldots$. There exist a_{i_j}, b_{i_j} in g_{i_j} with $d(a_{i_j}, b_{i_j}) > \varepsilon/2$. Since J is compact, there exist subsequences $\langle a_{i_{j_k}} \rangle$, $\langle b_{i_{j_k}} \rangle$ that converge to a, b respectively. Since $d(a_{i_{j_k}}, b_{i_{j_k}}) > \varepsilon/2$, $d(a, b) \geq \varepsilon/2$, so $a \neq b$. But now the $g_{i_{j_k}}$'s are pairwise disjoint subarcs of the simple closed curve J and for any large k, $g_{i_{j_k}}$ contains a point near a and a point near b, which is impossible from the geometry of a simple closed curve.

Theorem 7: G is an upper semicontinuous decomposition of E^3.

Proof: Given $g \in G$, let U be an open set in E^3 with $g \subset U$. Consider $A = \cup\{g' \in G | g' \cap (E^3 \setminus U) \neq \phi\}$. We want to show that A is closed, so that $E^3 \setminus A = \cup\{g' \in G | g' \subset U\}$ will be open in E^3, thus showing that G is upper semicontinuous. Any convergent sequence of points from $E^3 \setminus U$ converges to a point in $E^3 \setminus U$, but A may contain points from U.

Suppose $\langle a_n \rangle$ is a convergent sequence of points in A with $a_n \in U$. Then each a_n belongs to some non-degenerate element g_{i_n} which meets $E^3 \backslash U$. But if $\{g_{i_n} | a_n \in U\}$ were finite, then $\cup\{g_{i_n} | a_n \in U\}$ would be a closed set and the sequence $\langle a_n \rangle$ would converge to some point in $\cup\{g_{i_n} | a_n \in U\}$ which is contained in A, because $g_{i_n} \cap (E^3 \backslash U) \neq \phi$, so that $g_{i_n} \subset A$ from each i_n.

So, suppose $\{g_{i_n} | a_n \in U\}$ is countably infinite. But $\langle a_n \rangle$ converges to a_0 for some $a_0 \in E^3$. If a_0 belongs to some non-degenerate element g_{i_n} to which a_n belongs then $a_0 \in A$ since $g_{i_n} \subset A$.

Suppose a_0 does not belong to any non-degenerate element to which some a_n belongs and that $a_0 \in U$. Let $d(a_0, E^3 \backslash U) = \varepsilon > 0$. Since a_n converges to a_0, there exists n_0 such that for each $n \geq n_0$, $d(a_0, a_n) < \varepsilon/2$. But diam g_{i_n} approaches 0 as n tends to infinity. So, for $\varepsilon/2 > 0$, there exists m_0 such that for each $n \geq m_0$, diam $g_{i_n} < \varepsilon/2$. Let $N_0 = \max\{n_0, m_0\}$. Then for each $n \geq N_0$, and for each $x \in g_{i_n}$, $d(a_0, x) \leq d(a_0, a_n) + d(a_n, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, $g_{i_n} \subset U$. Thus, for each

$n \geq N_0$, $g_{i_n} \subset U$, which is a contradiction because the g_{i_n}'s meet $E^3 \backslash U$. Hence, $a_0 \in E^3 \backslash U \subset A$.

Finally, let $\langle a_n \rangle$ be a convergent sequence of points of A for which $a_i \in E^3 \backslash U$ for infinitely many i and $a_i \in U$ for infinitely many i. Now, the subsequence $\langle a_{n_i} \rangle$ with $a_{n_i} \in E^3 \backslash U$ converges to some point in $E^3 \backslash U$ since $E^3 \backslash U$ is closed. Since $\langle a_n \rangle$ is a convergent sequence, the subsequence and the sequence converge to the same point in $E^3 \backslash U \subset A$. Hence A is closed.

Consequently, E^3/G is an upper semicontinuous decomposition of E^3.

Now we have to construct homeomorphisms of E^3 onto itself which shrink each non-degenerate element of G into a set of small diameter. With this end in mind, we state and prove the following basic lemma.

Basic Lemma:

Given a positive ε and an open set U containing $\cup H$, there exists a homeomorphism h_ε of E^3 onto itself which shrinks each $g \in G$ into a set $h_\varepsilon(g)$ of diameter less than ε, is identity on $E^3 \backslash U$ and takes each component of U onto itself.

Proof: Let $\varepsilon > 0$ be given and let U be an open set such that $U \supset \cup H$.

Let $H_\varepsilon = \{g \in H | \text{diam } g \geq \varepsilon/2\}$. Now, H_ε is finite by the result in Section 3.2. If $H_\varepsilon = \phi$, then we need only take h_ε to be the identity, so suppose $H_\varepsilon \neq \phi$. Consider any g in H_ε. Fix a point q in $J \setminus \cup H_\varepsilon$ and let $O_g = \{x | d(x,g) < \frac{1}{2} d(g,(\cup H_\varepsilon \cup \{q\}) \setminus g)\}$. Then O_g is an open set in E^3 containing g and not meeting any other element of H_ε or the point q.

Let $V_g = \cup\{g' \in G | g' \subset O_g \cap U\}$. Then V_g is open since G is an upper semicontinuous decomposition of E^3. Further, by the definition of O_g, $\{V_g | g \in H_\varepsilon\}$ is a collection of pairwise disjoint open sets. Also, for each $g \in H_\varepsilon$, $g \subset V_g \subset U$, for any element $g' \in G$ either $g' \cap V_g = \phi$ for each $g \in H_\varepsilon$ or $g' \subset V_g$ for some $g \in H_\varepsilon$ and $q \notin V_g$ for all $g \in H_\varepsilon$. Now let us fix a g in H_ε and let r be the minimum of $\frac{1}{2} d(g, E^3 \setminus V_g)$ and $\varepsilon/4$. There exists a positive integer n_1 such that the solid torus T_{n_1} has blocks of diameter less than r.

Now, let B_1 be the union of all blocks of T_{n_1} meeting g. Then B_1 is connected since g and the blocks are all connected. $B_1 \subset V_g$, by the choice of r, so not all blocks of T_{n_1} lie in B_1 because the point q lies in J, hence in a block of T_{n_1}, but $q \notin V_g$. Thus, B_1

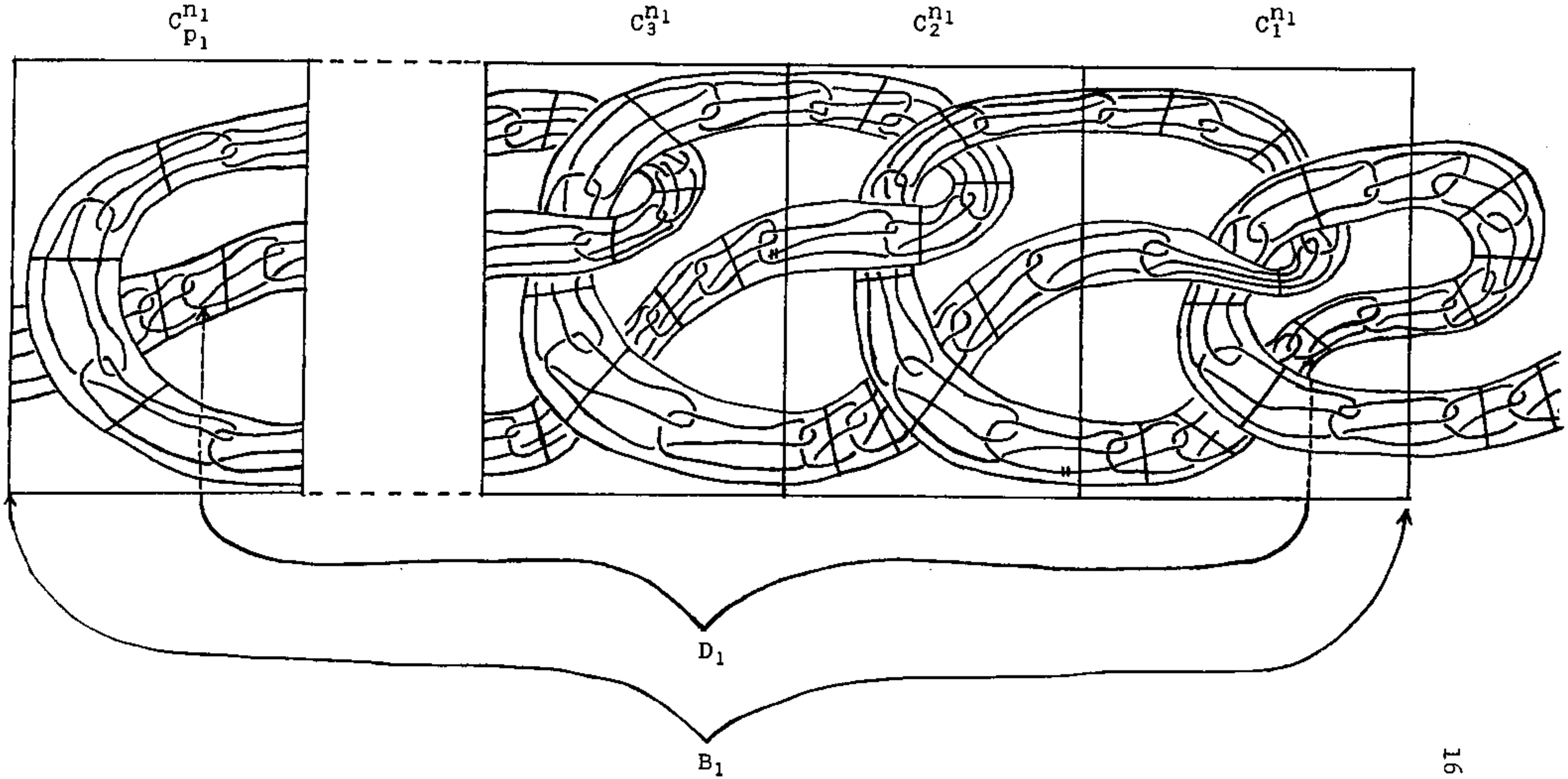


Figure 5

is a connected union of some but not all blocks of T_{n_1} and so is a 3-cell.

Now the blocks of T_{n_1} meeting g are ordered in the same way as ordered in T_{n_1}. Let them be $C_1^{n_1}, C_2^{n_1}, \ldots, C_{p_1}^{n_1}$ in this ordering, so that $B_1 = \bigcup_{i=1}^{p_1} C_i^{n_1}$.

Since $g \subset \text{Int } T_{n_1}$ and every block of T_{n_1} meeting g lies in B_1, we have $g \subset \text{Int } B_1 \subset B_1$. Since $B_1 \subset V_g$, the only element of G of diameter greater than or equal to $\varepsilon/2$ meeting B_1 is g itself.

Now, $g \subset \text{Int } B_1$ and $\text{Int } B_1$ is open. Since G is an upper semicontinuous decomposition of E^3, $W_1 = \cup\{g' \in G \mid g' \subset \text{Int } B_1\}$ is open in E^3. Then $g \subset W_1 \subset \text{Int } B_1$ and there exists a positive integer $m_1 > n_1$ such that each block of the solid torus T_{m_1} has diameter less than $\frac{1}{2}\, d(g, E^3 \backslash W_1)$.

As before, the union of all blocks of T_{m_1} meeting g is a 3-cell D_1. Since $m_1 > n_1$, all blocks of T_{m_1} have diameter less than $\varepsilon/4$. Also, $D_1 \subset W_1$ by properties of the diameters of blocks of T_{m_1}. Also, just as before $g \subset \text{Int } D_1$. Thus $g \subset \text{Int } D_1 \subset D_1 \subset W_1 \subset \text{Int } B_1 \subset B_1 \subset V_g$.

Consider the 3-cell $C_1^{n_1} \cup C_2^{n_1}$. Now, let h_1 be a homeomorphism from E^3 onto itself such that $h_1 = \text{Id}$ (the identity) on $E^3\setminus \mathrm{Int}(C_1^{n_1} \cup C_2^{n_1})$, h_1 takes all the blocks of D_1 meeting $C_1^{n_1}$ into the interior of $C_2^{n_1}$, and h_1 takes all blocks of D_1 lying in $C_1^{n_1} \cup C_2^{n_1}$ into $C_2^{n_1}$. The reason that such an h_1 exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of T_{m_1} leaves $C_1^{n_1}$, it does not go beyond $C_2^{n_1}$ before returning. Some other blocks of T_{m_1} not lying in D_1 but contained in $C_1^{n_1}$ may be pulled into the interior of $C_2^{n_1}$ or may be stretched by h_1 so that their images meet the interior of $C_2^{n_1}$.

Thus $h_1(g) \subset \mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$, so that $g \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1}))$. Because $h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}))$ is open in E^3, $W_2 = \cup\{g' \in G \mid g' \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1}))\}$ is open in E^3 and $g \subset W_2 \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}))$. Then there exists a positive integer $n_2 > m_1$ such that each block of the solid torus T_{n_2} has diameter less than $\frac{1}{2}\, d(g, E^3\setminus W_2)$. Since $T_{n_2} \subset \mathrm{Int}\, T_{m_1}$, the diameter of each block of T_{n_2} is less than $\varepsilon/4$.

Now, consider the 3-cell $B_2 = \bigcup_{i=1}^{p_2} h_1(C_i^{n_2})$, where $C_1^{n_2}, C_2^{n_2}, \ldots, C_{p_2}^{n_2}$ are all the blocks of T_{n_2} meeting g which are ordered in the same way as ordered in T_{n_2}. Then $B_2 \subset h_1(W_2)$, because the blocks of T_{n_2} meeting g lie in W_2. Moreover, $h_1(g) \subset \mathrm{Int}\, B_2$, because

$g \subset \text{Int}(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{p_2}^{n_2})$. Thus, $h_1(g) \subset \text{Int } B_2 \subset B_2 \subset h_1(W_2) \subset \text{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \text{Int } B_1 \subset B_1 \subset V_g$.

Let $W_2' = \cup\{g' \in G | g' \subset \text{Int} \bigcup_{i=1}^{p_2} C_i^{n_2}\}$. Then $g \subset W_2' \subset \text{Int} \bigcup_{i=1}^{p_2} C_i^{n_2}$ so $h_1(g) \subset h_1(W_2') \subset \text{Int } B_2$. There exists a positive integer $m_2 > n_2$ such that each block of the solid torus T_{m_2} has diameter less than $\frac{1}{2}\, d(g, E^3 \backslash W_2')$. Since $T_{m_2} \subset \text{Int } T_{n_2}$, all blocks of T_{m_2} have diameter less than $\varepsilon/4$.

Now, let D_2 be the image under h_1 of the union of all blocks of T_{m_2} meeting g. The interior of this union contains g and because the blocks of T_{m_2} meeting g lie in W_2', the union lies in W_2', so that $h_1(g) \subset \text{Int } D_2 \subset D_2 \subset h_1(W_2') \subset \text{Int } B_2 \subset B_2 \subset h_1(W_2) \subset \text{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \text{Int } B_1 \subset B_1 \subset V_g$.

Consider the 3-cell $h_1(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$, where $\ell_2 < p_2$, $h_1(C_{\ell_2}^{n_2}) \subset \text{Int } C_3^{n_1}$ and $h_1(C_{\ell_2}^{n_2} \cup C_{\ell_2+1}^{n_2} \cup \ldots \cup C_{p_2}^{n_2}) \subset \text{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. There is a homeomorphism h_2 from E^3 onto itself such that $h_2 = \text{Id}$ on $E^3 \backslash \text{Int } h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$, h_2 takes all the blocks of D_2 meeting $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2 - 1}^{n_2})$ into the interior of $h_1(C_{\ell_2}^{n_2})$, and h_2 takes all blocks of D_2 lying in $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$ into

$h_1(C_{\ell_2}^{n_2})$. The reason that such an h_2 exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of $h_1(T_{m_2})$ leaves $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2-1}^{n_2})$, it does not go beyond $h_1(C_{\ell_2}^{n_2})$ before returning. Some other blocks of $h_1(T_{m_2})$ not lying in D_2 but contained in $h_1(C_1^{n_2})$ may be pulled into the interior of $h_1(C_{\ell_2}^{n_2})$ or may be stretched by h_2 into $h_1(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$. Certainly, $h_2(h_1(g)) \subset \text{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$.

Continuing inductively in this way, we get the following:

(1) integers $n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \ldots < n_j < m_j < \ldots < n_{p_1-2} < m_{p_1-2} < n_{p_1-1} < m_{p_1-1}$,

(2) homeomorphisms $h_1, h_2, \ldots, h_{p_1-1}$ of E^3 onto itself, and for $j = 2, 3, \ldots, p_1$ putting $\psi_{j-1} = h_{j-1} \circ h_{j-2} \circ \ldots \circ h_2 \circ h_1$,

(3) open sets $W_j = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \text{Int}(C_j^{n_1} \cup \ldots \cup C_{p_1}^{n_1})\}$,

(4) 3-cells $B_j = \bigcup_{i=1}^{p_j} \psi_{j-1}(C_i^{n_j})$, where $C_1^{n_j}, C_2^{n_j}, \ldots C_{p_j}^{n_j}$ are the blocks of T_{n_j} meeting g which are ordered in the same way as ordered in T_{n_j}.

(5) open sets $W_j' = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \text{Int } B_j\}$,

(6) 3-cells D_j which are the image under ψ_{j-1} of the union of all blocks of T_{m_j} meeting g.

(7) 3-cells $\bigcup_{i=1}^{\ell_j} C_i^{n_j}$, where $\ell_j < P_j$.

These integers, homeomorphisms, open sets and 3-cells satisfy the following conditions:

(a) Each block of the solid torus T_{n_j} has diameter less than $\frac{1}{2}\, d(g, E^3\setminus W_j)$,

(b) Each block of the solid torus T_{m_j} has diameter less than $\frac{1}{2}\, d(g, E^3\setminus W_j')$,

(c) h_j takes $\psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j})$ onto itself,

(d) h_j shrinks the part of D_j meeting $\psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j-1}^{n_j})$ into the interior of $\psi_{j-1}(C_{\ell_j}^{n_j})$,

(e) h_j takes the part of D_j lying in $\psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j})$ into $\psi_{j-1}(C_{\ell_j}^{n_j})$,

(f) $h_j = \text{Id}$ on $E^3\setminus\text{Int } \psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j})$,

(g) $\psi_{j-1}(\bigcup_{i=1}^{\ell_j} C_i^{n_j}) \subset \text{Int}(C_j^{n_1} \cup C_{j+1}^{n_1})$,

(h) $\psi_{j-1}(C_{\ell_j}^{n_j}) \subset \text{Int } C_{j+1}^{n_1}$,

(i) $\psi_{j-1}(C_{\ell_j}^{n_j} \cup C_{\ell_j+1}^{n_j} \cup \dots \cup C_{p_j}^{n_j}) \subset \text{Int}(C_{j+1}^{n_1} \cup C_{j+2}^{n_1} \cup \dots \cup C_{p_1}^{n_1})$, From the definition of the 3-cells D_j and conditions (d), (e), (h) and (i) we have $\psi_j(g) \subset \text{Int}(C_{j+1}^{n_1} \cup \dots \cup C_{p_1}^{n_1})$. Thus $\psi_{p_1-1}(g) \subset \text{Int}(C_{p_1}^{n_1})$ and so has diameter less than $\varepsilon/4$. Let $h_g = \psi_{p_1-1}$. Then h_g is a homeomorphism from E^3 onto itself such that $h_g = \text{Id}$ on $E^3 \setminus V_g$ and the diameter of $h_g(g)$ is less than $\varepsilon/4$.

Now, we want to prove that h_g maps any other g' in H meeting $B_1 \subset V_g$ into a set of diameter less than ε. So, suppose that $g' \in H$ and that g' meets $C_1^{n_1} \cup C_2^{n_1}$ but that g' does not meet W_2. Then by definition (2) we have $h_1(g')$ does not meet $h_1(W_2)$. Now, by definitions (4) and (7) and condition (a) we have $h_1(C_1^{n_2} \cup \dots \cup C_{\ell_2}^{n_2}) \subset h_1(W_2)$. Since, $h_1(g')$ does not meet $h_1(W_2)$, it does not meet $\text{Int}\, h_1(C_1^{n_2} \cup \dots \cup C_{\ell_2}^{n_2})$. Now, by condition (f) in the construction of the homeomorphism h_g, we get $h_2 = \text{Id}$ on $E^3 \setminus \text{Int}\, h_1(C_1^{n_2} \cup \dots \cup C_{\ell_2}^{n_2})$. Hence, h_2 does not move any point of $h_1(g')$. So, $h_2(h_1(g')) = h_1(g')$.

By definition (3), we have $W_3 = \cup\{g' \in G \mid h_2(h_1(g')) \subset \text{Int}(C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1})\}$ and $W_2 = \cup\{g' \in G \mid h_1(g') \subset \text{Int}(C_2^{n_1} \cup \dots \cup C_{p_1}^{n_1})\}$. Consider any $g' \subset W_3$. Then $h_2(h_1(g')) \subset \text{Int}(C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1})$. Suppose that

$h_1(g') \not\subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. Then $h_1(g') \cap h_1(W_2) = \phi$, by definition (3). By condition (f) we have $h_2 = \mathrm{Id}$ on $E^3 \setminus \operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$. Again by definitions (4) and (7) and condition (a) we have $\operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2}) \subset h_1(W_2)$ so $E^3 \setminus \operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2}) \supset E^3 \setminus h_1(W_2)$. Thus $h_2(h_1(W_2)) = h_1(W_2)$ so $h_1(g') \cap h_2(h_1(W_2)) = h_1(g') \cap h_1(W_2) = \phi$ and hence h_2 does not move any point of $h_1(g')$. Then $h_2(h_1(g')) = h_1(g') \not\subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ and thus $h_1(g') = h_2(h_1(g')) \not\subset \operatorname{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$, which is a contradiction because $g' \subset W_3$. Hence $h_1(g') \subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ and thus $g' \subset W_2$ and $W_3 \subset W_2$. So $h_2(h_1(W_3)) \subset h_2(h_1(W_2))$. Since $h_1(g') \cap h_2(h_1(W_2)) = \phi$, we have $h_1(g') \cap h_2(h_1(W_3)) = \phi$. Now, by definitions (4), (7) and condition (a) we have

$$\operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3}) \subset B_3 = \bigcup_{i=1}^{p_3} (h_2 \circ h_1)(C_i^{n_3}) \subset$$

$h_2(h_1(W_3))$. Hence, $h_1(g') \cap \operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3}) = \phi$. Again by condition (f) we have $h_3 = \mathrm{Id}$ on $E^3 \setminus \operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3})$ and thus $h_3(h_1(g')) = h_1(g')$. Similarly, $h_4, h_5, \ldots, h_{p_1-1}$ do not move any point of $h_1(g')$. Hence $h_g(g') = h_1(g')$. Since $h_1 = \mathrm{Id}$ on $E^3 \setminus \operatorname{Int}(C_1^{n_1} \cup C_2^{n_1})$, $h_1(g') \subset g' \cup C_1^{n_1} \cup C_2^{n_1}$, and because g' meets $C_1^{n_1} \cup C_2^{n_1}$, $\operatorname{diam}(g' \cup C_1^{n_1} \cup C_2^{n_1}) \leq \operatorname{diam} g' + \operatorname{diam}(C_1^{n_1} \cup C_2^{n_1}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $\operatorname{diam} h_g(g') = \operatorname{diam} h_1(g') < \varepsilon$.

The construction of J shows that g is the only element of H which meets a block of T_{n_1} between $C^{n_1}_{p_1-1}$ and $C^{n_1}_2$. To prove this suppose $g' \neq g$ is an element in H which meets $C^{n_1}_1$. We want to show that g' does not meet $C^{n_1}_3$. So, if possible suppose that g' meets $C^{n_1}_3$. Now $g' \cap g = \phi$ and both g and g' are compact. Then $d(g,g') > 0$. There exists j such that each block of the solid torus T_j has diameter less than $\frac{1}{2}\, d(g,g')$ and thus any block of T_j meets at most one of g' and g. Since g' meets $C^{n_1}_3$, there exists a block C of T_j lying in $C^{n_1}_3$ that meets g' only. But g is connected, lies in $\bigcap\limits_1^\infty T_i$ and so meets all the blocks between $C^{n_1}_1$ and $C^{n_1}_{p_1}$ of T_{n_1}. Therefore g must pass through all the blocks of T_j lying in $C^{n_1}_3$, in particular C. This is a contradiction. Hence g' does not meet $C^{n_1}_3$. Similarly, any other $g' \neq g$ in H meeting $C^{n_1}_{p_1}$ can not meet $C^{n_1}_{p_1-2}$. Hence g is the only element of H which meets a block of T_{n_1} between $C^{n_1}_{p_1-1}$ and $C^{n_1}_2$.

Now suppose that $g' \neq g$ in H is such that it meets $C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1}$. Then g' does not meet $C^{n_1}_{p_1-2}$. Now $g' = (g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \cup (g' \cap (E^3\backslash \mathrm{Int}\, B_1))$. From the construction of h_g we see that $h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \subset C^{n_1}_{p_1}$. So, $\mathrm{diam}\, h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \leq \mathrm{diam}\, C^{n_1}_{p_1} < \varepsilon/4$.

Since $B_1 \subset V_g$ and g' meets B_1 we have $g' \subset V_g$ and therefore diam $g' < \varepsilon/2$. Hence diam $h_g(g' \cap (E^3 \setminus \mathrm{Int}\, B_1))$ $< \varepsilon/2$, because $h_g = \mathrm{Id}$ on $E^3\setminus \mathrm{Int}\, B_1$. Consequently, diam $h_g(g') \leq$ diam $h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1}))$ +

diam $h_g(g' \cap (E^3\setminus \mathrm{Int}\, B_1)) \leq \varepsilon/4 + \varepsilon/2 < \varepsilon$.

Finally, from definitions (3), (4) and (7) and conditions (a) and (c) we see that if g' belongs to H and meets $C^{n_1}_1 \cup C^{n_1}_2$ and W_2 then either $g' \cap W_j = \phi$ and $g' \subset W_{j-1}$ for some j in $\{3,2,\ldots,p_1-1\}$ or $g' \subset W_j$ for all j. So we have the following two general cases, which will complete the proof that h_g takes any g' in H meeting B_1 into a set of diameter less than ε.

<u>General Case 1</u>: Suppose that g' belongs to H, that g' meets $(C^{n_1}_1 \cup C^{n_1}_2)$ and W_2, that g' does not meet W_j but that $g' \subset W_{j-1}$ for some j in $\{3,2,\ldots,p_1-1\}$. Then by definition (3) we have $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$. Now, by definitions (4) and (7) and condition (a) we have $\psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j}) \subset \psi_{j-1}(W_j)$. Since $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$, it does not meet $\mathrm{Int}\, \psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j})$. Now, by condition (f) in the construction of the homeomorphism h_g, we get $h_j = \mathrm{Id}$ on $E^3\setminus \mathrm{Int}\, \psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j})$. Hence h_j does not

move any point of $\psi_{j-1}(g')$. So, $\psi_j(g') = h_j(\psi_{j-1}(g'))$ $= \psi_{j-1}(g')$.

By definition (3), we have $W_{j+1} = \cup\{g' \in G \mid \psi_j(g') \subset \text{Int}(C_{j+1}^{n_1} \cup \dots \cup C_{p_1}^{n_1})\}$ and $W_j = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \text{Int}(C_j^{n_1} \cup \dots \cup C_{p_1}^{n_1})\}$. Consider any $g' \subset W_{j+1}$. Then $\psi_j(g') \subset \text{Int}(C_{j+1}^{n_1} \cup \dots \cup C_{p_1}^{n_1})$. Suppose that $\psi_{j-1}(g') \not\subset \text{Int}(C_j^{n_1} \cup \dots \cup C_{p_1}^{n_1})$. Then $\psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \phi$, by definition (3). By condition (f) we have $h_j = \text{Id}$ on $E^3 \backslash \text{Int}\ \psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j})$. Again by definitions (4) and (7) and condition (a) we have $\text{Int}\ \psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j}) \subset \psi_{j-1}(W_j)$ so $E^3 \backslash \text{Int}\ \psi_{j-1}(C_1^{n_j} \cup \dots \cup C_{\ell_j}^{n_j}) \supset E^3 \backslash \psi_{j-1}(W_j)$. Thus $h_j(\psi_{j-1}(W_j)) = \psi_{j-1}(W_j)$ so $\psi_{j-1}(g') \cap h_j(\psi_{j-1}(W_j))$ $= \psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \phi$ and hence h_j does not move any point of $\psi_{j-1}(g')$. Then $h_j(\psi_{j-1}(g')) = \psi_{j-1}(g') \not\subset \text{Int}(C_j^{n_1} \cup \dots \cup C_{p_1}^{n_1})$ and thus $\psi_{j-1}(g') = h_j(\psi_{j-1}(g')) \not\subset \text{Int}(C_{j+1}^{n_1} \cup \dots \cup C_{p_1}^{n_1})$, which is a contradiction because $g' \subset W_{j+1}$. Hence $\psi_{j-1}(g') \subset \text{Int}(C_j^{n_1} \cup \dots \cup C_{p_1}^{n_1})$ and thus $g' \subset W_j$ and $W_{j+1} \subset W_j$. So, $h_j(\psi_{j-1}(W_{j+1})) \subset h_j(\psi_{j-1}(W_j))$, that is $\psi_j(W_{j+1}) \subset \psi_j(W_j)$. Since $\psi_{j-1}(g') \cap \psi_j(W_j) = \phi$, we have $\psi_{j-1}(g') \cap \psi_j(W_{j+1}) =$ $= \phi$. Definitions (4) and (7) and condition (a) imply that $\text{Int}\ \psi_j(C_1^{n_{j+1}} \cup \dots \cup C_{\ell_{j+1}}^{n_{j+1}}) \subset B_{j+1} = \bigcup_{i=1}^{p_{j+1}} \psi_j(C_i^{n_{j+1}}) \subset$

$\psi_j(W_{j+1})$. Hence, $\psi_{j-1}(g') \cap \text{Int}\, \psi_j(C_1^{n_{j+1}} \cup \dots \cup C_{\ell_{j+1}}^{n_{j+1}}) = \phi$. Again by condition (f) we have $h_{j+1} = \text{Id}$ on $E^3 \backslash \text{Int}\, \psi_j(C_1^{n_{j+1}} \cup \dots \cup C_{\ell_{j+1}}^{n_{j+1}})$ and thus $h_{j+1}(\psi_{j-1}(g')) = \psi_{j-1}(g')$. Similarly, $h_{j+2}, h_{j+3}, \dots, h_{p_1-1}$ do not move any point of $\psi_{j-1}(g')$. Hence $h_g(g') = \psi_{j-1}(g')$.

Now, by condition (f), for any k we have $h_k = \text{Id}$ on $E^3 \backslash \text{Int}\, \psi_{k-1}(C_1^{n_k} \cup \dots \cup C_{\ell_k}^{n_k})$. So, condition (g) implies that h_k only moves points lying in $\text{Int}(C_k^{n_1} \cup C_{k+1}^{n_1})$. We proved before that g is the only element of H which meets a block of T_{n_1} between $C_{p_1-1}^{n_1}$ and $C_2^{n_1}$. Now, g' meets $C_1^{n_1} \cup C_2^{n_1}$ so it can not meet $C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1}$. Since $g' \subset W_{j-1}$, $\psi_{j-2}(g') \subset \text{Int}(C_{j-1}^{n_1} \cup \dots \cup C_{p_1}^{n_1})$ by definition (3). Since $h_1 = \text{Id}$ on $E^3 \backslash C_1^{n_1} \cup C_2^{n_1}$, by conditions (f) and (g) we see that $\psi_{j-2} = h_{j-2} \circ \dots \circ h_2 \circ h_1$ only moves points in $C_1^{n_1} \cup \dots \cup C_{j-1}^{n_1}$ so that $\psi_{j-2}(g') \subset \text{Int}\, C_{j-1}^{n_1}$. Since h_{j-1} only moves points in $C_{j-1}^{n_1} \cup C_j^{n_1}$, we have $h_g(g') = \psi_{j-1}(g') = h_{j-1}(\psi_{j-2}(g')) \subset C_{j-1}^{n_1} \cup C_j^{n_1}$. Consequently $\text{diam}\, h_g(g') \leq \text{diam}\, C_{j-1}^{n_1} + \text{diam}\, C_j^{n_1} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

General Case 2: If g' in H is such that g' meets $C_1^{n_1} \cup C_2^{n_1}$ but $\psi_{j-1}(g') \subset \text{Int}(C_j^{n_1} \cup \dots \cup C_{p_1}^{n_1})$ for each $j = 2,3,\dots,p_1-1$, then $h_g(g') = \psi_{p_1-1}(g')$ and $\psi_{p_1-2}(g') \subset$

$\text{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Thus $h_{p_1-1}(\psi_{p_1-2}(g')) \subset \text{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Hence $h_g(g') = \psi_{p_1-1}(g') \subset \text{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Consequently $\text{diam}\, h_g(g') \leq \text{diam}\, C^{n_1}_{p_1-1} + \text{diam}\, C^{n_1}_{p_1} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

Since H_ε is finite, we let $H_\varepsilon = \{g_1, g_2, \ldots, g_r\}$. Now, let $h_\varepsilon = h_{g_1} \circ h_{g_2} \circ \ldots \circ h_{g_r}$. Since B_1 is connected and contained in U it lies in the component of U containing g. Then using the conditions (f) and (g) we see that h_j maps B_1 onto itself and therefore any component of the open set U onto itself. Using this property in the construction of h_g and the definition of h_ε, it follows that h_ε maps any component of the open set U onto itself.

Consider any $g' \in G$. If g' meets an open set V_{g_i} for some $g_i \in H_\varepsilon$, then $g' \subset V_{g_i}$. For $j \neq i$, $h_{g_j} = \text{Id}$ on V_{g_i} because $h_{g_j} = \text{Id}$ on $E^3 \setminus V_{g_j}$ and $V_{g_i} \cap V_{g_j} = \phi$, so $h_\varepsilon(g') = h_{g_i}(g')$. Since h_{g_i} shrinks each g' in V_{g_i} into a set of diameter less than ε, we have $\text{diam}\, h_{g_i}(g') < \varepsilon$ and hence $\text{diam}\, h_\varepsilon(g') < \varepsilon$.

If $g' \cap V_{g_j} = \phi$, for all $g_j \in H_\varepsilon$ then since $h_{g_j} = \text{Id}$ on $E^3 \setminus V_{g_j}$, we have $h_\varepsilon(g') = (h_{g_1} \circ \ldots \circ h_{g_r})(g') = g'$. Hence, $\text{diam}\, h_\varepsilon(g') = \text{diam}\, g' < \varepsilon/2$, because each element of G with diameter $\geq \varepsilon/2$ lies in some V_{g_i}.

Since for each j, $V_{g_j} \subset U$, $h_{g_j} = Id$ on $E^3\backslash U$ for each j. Hence $h_\varepsilon = Id$ on $E^3\backslash U$. Consequently, h_ε is a homeomorphism from E^3 onto itself which shrinks each $g' \in G$ into a set $h_\varepsilon(g')$ of diameter less than ε, is Id on $E^3\backslash U$ and takes each component of U onto itself. This completes the proof of the basic lemma.

Now the proof of our Main Theorem is similar to the proof of Theorem 1 in [1, page 363] if we replace Lemma 1 in [1, page 363] by our Basic Lemma in Section 4.

Acknowledgement: The contents of this paper form a part of my thesis for the degree of Master of Science. I thank my advisor Dr. Arlo W. Schurle for his guidance and encouragement.

REFERENCES

[1] Bing, R.H., Upper semicontinuous decompositions of E^3, Annals of Mathematics (2) 65(1957), pp. 363-374.

[2] Bing, R.H., A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3, Annals of Mathematics (2) 65(1957), pp. 484-500.

[3] Bing, R.H., A simple closed curve that pierces no disk, Journal de Mathematiques Pures et Appliques, (9) 35(1956), pp. 337-343.

[4] Row, W.H. and Walsh, John, A non-shrinkable decomposition of S^3, whose non-degenerate elements are contained in a cellular arc, Transactions of American Mathematical Society 289(1985), pp. 227-252.

[5] Schurle, A.W., Topics in Topology, Elsevier, North Holland, New York, New York, 1979.

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