King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 115

October 1989

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Super Quasi-adequate Semigroups

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Let S be a semigroup and let L denote Green's relation on S, For a, b ∈ S, let $(a,b) \in L^*$ if and only if $(a,b) \in L$ in some oversemigroup of S. R^* is defined dually and let $H^* = L^* \cap R^*$. From ([11] or [12]), $(a,b) \in L^*$ if and only if, for all x, $y \in S^1$ (S with an appended identity), ax = ay if and only if bx = by. So L^* is a right congruence relation and R^* is a left congruence relation. Fountain [9] terms a semigroup S abundant if each L^*-class of S and each R^*-class of S contains an idempotent, and Fountain [9] terms S superabundant if each H^*-class of S contains an idempotent. If S is a regular semigroup, $L^* = L$ and $R^* = R$. Hence, regular semigroups are abundant semigroups and unions of groups are superabundant semigroups.

In [9], Fountain gave superabundant analogues to the Rees Theorem and Clifford's well known theorem that a semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups. In [7], El-Qallali terms an abundant semigroup S to be L^*-unipotent if E(S), the set of idempotents of S, form a subsemigroup and each L^*-class of S contains precisely one idempotent. In [7], El-Qallali gives a structure theorem for super L^*-unipotent semigroups on which H^* is a congruence (L^*-unipotent bands of cancellative monoids [7]). A semigroup S is termed L-unipotent if each L-class of S contains precisely one idempotent (equivalently, S is orthodox and each J-class of E(S)

AMS Subject Classification 20 M 10

is a right zero semigroup [20]). El-Qallali's theorem is a superabundant analogue to Bailes' structure theorem for L-unipotent union of groups on which H is a congruence (L-unipotent bands of groups) [1].

Let S be an abundant semigroup. Fountain [8] terms S an adequate semigroup if E(S) is a semilattice. El-Qallali and Fountain [6] term S a quasi-adequate semigroup if E(S) is a subsemigroup. If, furthermore, L is a congruence relation on E(S), we term S a generalized L^*-unipotent semigroup. El-Qallali and Fountain [5], term a congruence ρ on S good if a L^*b implies $a\rho L^* b\rho$ and aR^*b implies $a\rho R^* b\rho$.

In section 1, we give a structure theorem for super quasi adequate semigroups S (Theorem 1.11). We first specialize the above mentioned results of Fountain to super quasi-adequate semigroups S. In particular, S is a semilattice Y of semigroups $(S_\gamma : \gamma \in Y)$ where $S_\gamma = T_\gamma \times E(S_\gamma)$ (algebraic direct product) where T_γ is a cancellative monoid and $E(S_\gamma)$ is a rectangular band (Lemma 1.1). For $(g;i,j)$, $(h;r,s) \in S$, define $(g;i,j)\delta(h;r,s)$ if $(g;i,j)$, $(h;r,s) \in S_\gamma$, say, and $g=h$. Then, δ is the minimum adequate good congruence on S (Proposition 1.3) and S/δ is a strong semilattice Y of the T_γ (Lemma 1.4). Then, S^1 divides $V \circ (\widehat{S/\delta})^1$ where V is an L-trivial and idempotent monoid, $\circ$ is wreath product, $\wedge$ is the Rhodes expansion, $(\widehat{S/\delta})$ is a semilattice Y of left cancellative semigroups $(X_\gamma : \gamma \in Y)$ with idempotents, and $E((\widehat{S/\delta}))$ is a semilattice Y of right zero semigroups $(E(X_\gamma) : \gamma \in Y)$ (Theorem 1.11). If S is an orthodox union of groups, δ becomes the smallest inverse semigroup congruence on S, T_y becomes a maximal subgroup of S, and $X_y = T_y \times E(X_y)$ (algebraic direct product) (see Lemma 1.12). Hence, Theorem 1.11 is a superabundant semigroup analogue to our structure theorem for orthodox unions of groups [26].

In section 2, we give a structure theorem for super generalized L^*-unipotent semigroups S(Theorem 2.4). We first show that $\delta \cap L$ is the smallest L^*-unipotent good congruence on S and $S/\delta \cap L$ is a semilattice Y of the semigroups $((T_y \times J_y): y \in Y)$ were J_y is an R-class of $E(S_y)$(Proposition 2.1). Then,

$S \leq W^1 \circ (E(S)/L)^1 \circ (S/\delta \cap L)^1$

($\leq$ means "is embedded in") and $S/\delta \cap L$

$\leq (S/\delta \cap L/\varrho)^1 \hat{\circ} (E(S)/L)^1$ where

W is a lower partial chain Y of left zero subsemigroups of E(S), ϱ is the smallest adequate good congruence on $S/\delta \cap L$, $S/\delta \cap L/\varrho$ is a strong semilattice Y of the T_y, and $\hat{\circ}$ is reverse wreath product (Theorem 2.4). An orthodox semigroup S is termed generalized L-unipotent if L is a congruence relation on E(S). If S is a generalized L-unipotent union of groups, $\delta \cap L$ becomes the smallest L-unipotent congruence on S. Hence, Theorem 2.4 is a superabundant analogue to our structure theorem for generalized L-unipotent unions of groups [24].

In section 3, we show that if S is a super R^*-unipotent semigroup, then $S \leq (E(S))^1 \circ (S/\delta)^1$ where E(S) is a semilattice Y of left zero semigroups (Theorem 3.1). Theorem 3.1 is a superabundant analogue to our structure theorem for R-unipotent unions of groups [24].

Abundant semigroup analogues to many theorems in regular semigroup theory have been given by Fountain ([8],[9]), El-Qallali and Fountain ([5],[6]), and El-Qallali [7].

We have studied the structure of generalized L-unipotent semigroups in ([21],[22],[23],[24]), R-unipotent semigroups have been studied extensively by many authors – most recently by Szendrei ([14],[15]).

A submonoid of a monoid S is a subsemigroup of S containing the identity of S.

A semigroup (monoid) S is said to divide a semigroup (monoid) T if there exists a homomorphism of a subsemigroup (submonoid) of T onto S. We also say T covers S in this case and write $S<T$. If there exists an isomorphism of S into T, we write $S\leq T$. R, L, H, D and J will denote Green's relations and E(S) will denote the set of idempotents of a semigroup S.

See [9] for the definition of $J^{\bullet}$. If S is a regular semigroup $J^{\bullet}=J$.

We adopt the following notation and definitions from [24, p.181-182]: S^1 (S with appended identity), $S^{\bullet}$, wreath product "o" of semigroups, reverse wreath product "$\hat{o}$" of semigroups, type A semigroup congruence (for example, inverse semigroup congruence), $a\rho$ ($a\in S$, a semigroup)(ρ, a congruence on S, will also denote the natural homomorphism of S onto S/ρ), and unions of groups.

For other definitions not given in this paper, see [2] or [10]. We also adopt the notation of [2] unless otherwise specified.

A monoid S is termed L-trivial and idempotent if each L-class of S is a singleton and S is a band.

Section 1 - The Structure of Super Quasi-Adequate Semigroups

In this section, we describe the minimum adequate good congruence δ on a super quasi-adequate semigroup (Proposition 1.3 and Lemma 1.4) and give a structure theorem for super quasi-adequate semigroups (Theorem 1.11)

Let S be a semigroup. For $_\bullet \in S$, $L^*_\bullet$ or $L^*_\bullet(S)$(in case of ambiguity) will denote the L^*-class of S containing $_\bullet$ (notation of [9]).

Let S be a semigroup and I and J be sets and let $P: J \times I \rightarrow S$ with $(_j,_i)P = p_{ji}$. Let M(S,I,J,P) denote $S \times I \times J$ under the multiplication $(a;_{i,j})(b;_{r,\bullet}) = (a\ p_{jr}\ b;_{i,\bullet})$. We term M(S,I,J, P) a Rees Matrix semigroup over S with entries in P.

The following lemma gives the "gross" structure of super quasi-adequate semigroups.

Lemma 1.1. *A semigroup S is super quasi-adequate if and only if S is a semilattice* $Y = S/J^*$ *of semigroups* $(S_y : {}_y \in Y)$ *where* $S_y = T_y \times E(S_y)$ *where* T_y *is a cancellative monoid and* $E(S_y)$ *is a rectangular band,* $L^*_\bullet(S) = L^*_\bullet(S_y)$ *and* $R^*_\bullet(S) = R^*_\bullet(S_y)$ *for* ${}_y \in Y$ *and* $_\bullet \in S_y$ *and* E(S) *is a semilattice Y of rectangular bands* $(E(S_y) : {}_y \in Y)$.

Proof. Utilizing [9, Theorem 6.8 and its proof and Corollary 5.2], we obtain the above theorem (except the statement about $E(S)$) with $S_y = M(T_y, I_y, J_y, P_y)$, a Rees matrix semigroup over a cancellative monoid T_y where the entries of P_y are units U of T_y. As is easily shown, [2, Lemma 3.6] is valid for the above matrix semigroups if we require the mappings to have range U. Using this Lemma, we may "normalize" P_y such that all the elements in a given row and a given column are the identity e of T_y. Then, using the assumption that $E(S)$ is a subsemigroup, we may show $p_{ji} = e$ for all $j \in J_y$ and $i \in I_y$. Hence, $M(T_y, I_y, J_y, P_y) = T_y \times E(S_y)$ where $E(S_y)$ is a rectangular band.

To show δ is a congruence relation (Proposition 1.3), we will need the following lemma.

Lemma 1.2. *Let* $S_y = T_y \times E_y$ *and* $S_z = T_z \times I_z \times J_z$ *where* T_y *and* T_z *are cancellative monoids,* E_y *is a rectangular band,* I_z *is a left zero semigroup, and* J_z *is a right zero semigroup. Assume these exists*

a) *a left répresentation* $\mathbf{a} \rightarrow \lambda_{\mathbf{a}}$ *of* S_y *by transformations of* I_z

b) *a right representation* $\mathbf{a} \longrightarrow \rho_{\mathbf{a}}$ *of* S_y *by transformations of* J_z

c) *a homomorphism* $\emptyset$ *of* T_y *into* T_z,

Define a binary operation on $S_y \cup S_z$ *extending the given ones on* S_y *and* S_z *by defining products of* $\mathbf{a} = (a, e) \in S_y$ *and* $(b; i, j) \in S_z$ *as follows:*

$$(a, e)(b; i, j) = (a\emptyset b;\ \lambda_{\mathbf{a}} i, j)$$

$$(b; i, j)(a, e) = (b(a\emptyset);\ i,\ j\rho_{\mathbf{a}}).$$

<u>Then,</u> $S_\gamma \cup S_\alpha$ <u>becomes a semigroup with</u> S_α <u>an ideal</u>.

<u>Conversely every possible binary associative operation on</u> $S_\gamma \cup S_\alpha$ <u>extending the given ones on</u> S_γ <u>and</u> S_α, <u>and such that</u> S_α <u>is an ideal, can be constructed in the above manner</u>.

Proof. Lemma 1.2 has been established by Clifford [3, Lemma 2.5] in the case T_γ and T_α are groups. Clifford's proof is easily seen to be valid when T_γ and T_α are just cancellative monoids.

Proposition 1.3 <u>Let</u> S <u>be a super quasi-adequate semigroup. Then,</u> δ <u>is the minimum adequate good congruence on</u> S.

Proof. We first show that δ is a congruence relation on S. Let $\bar{\delta}$ denote the smallest congruence on S containing δ. Suppose a $\bar{\delta}$ b. Then, there exists $a = a_1, a_2, \ldots\ldots, a_n = b \in S$ such that $a_i = x_i u_i y_i$, $a_{i+1} = x_i v_i y_i$ where $x_i, y_i \in S^1$ and $(u_i, v_i) \in \delta$ for $1 \le i \le n-1$. Let $x_i = (w;i,j)_\alpha \in S_\alpha$, $y_i = (h;r,s)_\beta \in S_\beta$, $u_i = (g;m,n)_\gamma$, and $v_i = (g;c,d)_\gamma$. Hence, $a_i = (A;p,q)_{\alpha\beta\gamma} \in S_{\alpha\beta\gamma}$ and $a_{i+1} = (B;k,l)_{\alpha\beta\gamma} \in S_{\alpha\beta\gamma}$ say. Let $\theta = \alpha\beta\gamma$. Thus,

$(A;p,q)_\theta = (w,i,j)_\alpha (g;m,n)_\gamma (h;r,s)_\beta$

$(B;k,l)_\theta = (w;i,j)_\alpha (g;c,d)_\gamma (h;r,s)_\beta$

Multiply both of the above equations on the left and right by $(e;p,q)_\theta$ where e is the identity of T_θ.

Hence,

$(A;p,q)_\theta = (\bar{W};\bar{i},\bar{j})_\theta (g;m,n)_\gamma (\bar{h};\bar{r},\bar{s})_\theta$

$(B;p,q)_\theta = (\bar{W};\bar{i},\bar{j})_\theta (g;c,d)_\gamma (\bar{h};\bar{r},\bar{s})_\theta$

say,

Using Lemma 1.2

$(A;p,q)_\theta = (\bar{W}(g\omega_{\gamma,\theta});\bar{i},\bar{j}\rho_{[illegible]})_\theta (\bar{h};r,s)_\theta = (\bar{W}(g\omega_{[illegible]})\bar{h};\bar{i},s)_\theta$.

where $\omega_{\gamma,\alpha}$ is the homomorphism of T_γ into T_α given by Lemma 1.2 and $(B;p,q)_\alpha=(\widetilde{W}(g\omega_{\gamma,\alpha});\bar{i},\bar{j}\rho_{[illegible]})_\alpha(\bar{h};r,s)_\alpha = (\widetilde{W}(g\omega_{\gamma,\alpha})\bar{h};\bar{i},s)_\alpha$. Hence, A=B. Thus $a_i \delta a_{i+1}$ for $1 \le i \le n-1$. Hence, $a\delta b$. Thus, $\bar{\delta}=\delta$, and, hence, δ is a congruence on S.

Let $a \in S$ and let $a^+, a^* \in E(S)$ such that $a^+ R^* a$ and $a^* L^* a$. Using [9, Corollary 6.2 and Proposition 6.5] and Lemma 1.1, a^+, a^*, $a \in S_\gamma$, say. Hence, using [6, Corollary 2.4 and Proposition 2.6], δ is the minimum adequate good congruence on S.

Lemma 1.4. <u>Let</u> S <u>be a super quasi-adequate semigroup. Then,</u> S/δ <u>is a strong semilattice</u> Y <u>of cancellative monoids</u> $(T_\gamma; \gamma \in Y)$.

Proof. Let $\overline{(g;i,j)}$ denote the δ - class of S containing $(g;i,j)$. Since $\overline{(g;i,j)}\tau=g$ defines a 1-1 map of S/δ onto $T=\cup(T_\gamma:\gamma\in Y)$, T becomes a groupoid under the multiplication $ab=(a\tau^{-1}b\tau^{-1})\tau$ and τ defines an isomorphism of S/δ onto T. If $g,h\in T_\gamma$, $g,h=(\overline{(g;i,j)}\,\overline{(h;k,e)})\tau = \overline{(gh;i,e)}\tau = gh$(the last product is multiplication in T_γ). Hence, T is a semilattice Y of cancellative monoids $(T_\gamma:\gamma\in Y)$. For $a\in T_\alpha$ and $\alpha\ge\gamma$, define $a\varphi_{\alpha,\gamma} = ae_\gamma$ where e_γ is the identity of T_γ. It is routine to verify that $\varphi_{\alpha,\gamma}$ is a homomorphism of T_α into T_γ, $\varphi_{\gamma,\gamma}$ is the identity map on T_γ, and, for $a\in T_\gamma$, $b\in T_\alpha$, $ab=a\varphi_{\gamma,\gamma\alpha}b\varphi_{\alpha,\gamma\alpha}$. Using the fact that the idempotents of T commute by Proposition 1.3, its easily seen that $\varphi_{\gamma,\alpha}\varphi_{\alpha,\beta}=\varphi_{\gamma,\beta}$ for $\gamma\ge\alpha\ge\beta$. Hence, T is a strong semilattice $\varphi(Y;T_\gamma;\varphi_{\gamma,\alpha})$ of cancellative monoids (notation of [10]). We identify S/δ and T.

We next describe the Rhodes expansion $\hat{S}$ of an arbitrary semigroup S (see [17] and [13]). The Rhodes expansion and certain of its properties will be crucial in developing our structure theory of super quasi-adequate semigroups. If $a, b \in S$, $a \leq b$ means $aUSa \leq bUSb$ and $a < b$ means $a \leq b$ but $a \not\leq b$. Let $S_+ = \{(s_n, \ldots, s_1) : s_i \in S$ for $1 \leq i \leq n$ and $s_1 \leq s_2 \leq \ldots \leq s_n\}$. If $x = (s_n, \ldots, s_1)$, $y = (t_m, \ldots, t_1)$ define $xy = (s_n t_m, \ldots, s_1 t_m, t_m, \ldots, t_1)$. Then, S_+ is a semigroup under this multiplication. If $a = (s_n, \ldots, s_1) \in S_+$ and $s_{k+1} L s_k$ for some $1 \leq k \leq n-1$ delete s_k to obtain $a_1 \in S_+$ and denote the deletion by $a \rightarrow a_1$. Perform $a \rightarrow a_1 \rightarrow \ldots \rightarrow a_k$ where $a_k = (s_n, s_{n-1}, \ldots, s_{n-r})$ with $s_n < s_{n-1} < \ldots < s_{n-r}$ (such an a_k is termed an irreducible element of S_+). Write a_k = red a and $a \sim b$ if red a = red b. The equivalence relation $\sim$ is a congruence relation on S_+. Let $\hat{S} = S_+/\sim$. $\hat{S}$ is termed the Rhodes expansion of S after its inventor John Rhodes. $\hat{S}$ will be treated as the set of irreducible elements of S_+ under the multiplication ab= red(ab).

Lemma 1.5. Let S be a super quasi-adequate semigroup. Then, $\hat{S}$ is a semilattice Y of subsemigroups $(F_y : y \in Y)$ where $F_y = \{(a_n, a_{n-1}, \ldots, a_1) : a_n \in S_y, a_j \in S\}$ and $E(\hat{S})$ is the semilattice Y of rectangular bands

$E(F_y) = \{((e_y; i, j), a_{n-1}, \ldots, a_1) : (e_y; i, j) \in E(S_y), a_j \in S\}$.

$U = (\widehat{S/\delta})$ is a semilattice Y of left cancellative semigroups with idempotent $(X_y : y \in Y)$ where $X_y = \{(a_n, a_{n-1}, \ldots, a_1) : a_n \in T_y, a_j \in S/\delta\}$. $E(U)$ is a semilattice Y of right zero semigroups $(E(X_y) : y \in Y)$ where $E(X_y) = \{(e_y, a_{n-1}, \ldots, a_1) : e_y$, the identity of T_y, $a_j \in S/\delta\}$. For $(a_n, a_{n-1}, \ldots, a_1) \in \hat{S}$, let $(a_n, a_{n-1}, \ldots, a_1)\hat{\delta} = \mathrm{red}(a_n\delta, a_{n-1}\delta, \ldots, a_1\delta)$. Then, $\hat{\delta}$ defines a homomorphism of $\hat{S}$ onto $(\widehat{S/\delta})$.

Proof. To establish the second sentence of the lemma, utilize Lemma 1.1 and [16, Lemma 6.7](see also [17, Lemma 11.4] and [24, Theorem 3.1(f)]). Utilizing Lemma 1.4 and [24, Theorem 3.1(f)], it is easily checked that U is a semilattice Y of the semigroups $(X_y : y \in Y)$ and that the fourth sentence of the lemma is valid. We next show X_y is left cancellative for $y \in Y$. Let $(x_r, x_{r-1}, \ldots, x_1), (a_n, a_{n-1}, \ldots, a_1)$, and $(b_s, b_{s-1}, \ldots, b_1)$ be elements of X_y and suppose that $(x_r, x_{r-1}, \ldots, x_1).(a_n, a_{n-1}, \ldots, a_1) = (x_r, x_{r-1}, \ldots, x_1).(b_s, b_{s-1}, \ldots, b_1)$. Hence, $red(x_r a_n, x_{r-1} a_n, \ldots, x_1 a_n, a_n, a_{n-1}, \ldots, a_1) = red(x_r b_s, x_{r-1} b_s, \ldots, x_1 b_s, b_s, b_{s-1}, \ldots, b_1)$. Thus, $x_r a_n = x_r b_s$. Hence, since T_y is a cancellative semigroup, $a_n = b_s$. Thus, $n=s$ and $a_i = b_i$ for $1 \leq i \leq n$. The last sentence of the lemma is a consequence of [16, Proposition 6.6](see also [17] and [24, Theorem 3.11(b)]).

In the remainder of this section, S will denote a super quasi-adequate semigroup.

If A is a semigroup and $a = (a_n, \ldots, a_1) \in \hat{A}$, let $|a| = n$. We term $|a|$ the length of a.

Lemma 1.6, If $z \in \hat{S}$, $|z| = |z\hat{\delta}|$

Proof. Let $z = (a_n, a_{n-1}, \ldots, a_1)$. Suppose $a_{k+1}\delta \mathcal{L} a_k \delta$ for some $1 \leq k \leq n-1$. Using Lemma 1.1, let $a_{k+1} = (g_{k+1}, i_{k+1}, j_{k+1}) \in S_y$, say, and $a_k = (g_k; i_k, j_k) \in S_z$, say. Thus, $a_{k+1}\delta = g_{k+1} \in T_y$ and $a_k \delta = g_k \in T_z$, and, hence, $g_{k+1} \mathcal{L} g_k$ (in S/δ). Using Lemma 1.4, it easily seen that $y=z$ and $g_{k+1} = \mu g_k$ where μ is a unit of T_y. Since $a_{k+1} < a_k$, $a_{k+1} = s a_k$ for some $s \in S$. We may take $s = (s'; m, n) \in S_y$. Hence, $(g;_{k+1} i_{k+1}, j_{k+1}) = (s'; m, n)(g_k; i_k, j_k)$. So, $j_{k+1} = j_k$. Thus,

$(g_k;i_k,j_k)=(\mu^{-1};i_k,j_k).(g_{k+1};i_{k+1},j_{k+1})$. Hence, $a_{k+1}La_k$, a contradiction. Thus, $red(a_n\delta,a_{n-1}\delta,...,a_1\delta)=(a_n\delta,a_{n-1}\delta,...,a_1\delta)$ and $|z|=|z\delta|$.

For $t\in U=(\widehat{S/\delta})$, let $U_t=\{x\in U: tx=t\}$

Lemma 1.7. For $t\in U$, $U_t\hat{\delta}^{-1}\le E(\hat{S})$. If $t\in X_y$, $U_t\hat{\delta}^{-1}\le \cup(E(F_x):x\ge y)$.

Proof. Let $s\in U_t\hat{\delta}^{-1}$. Hence, $s\hat{\delta}\in U_t$. Using an important theorem of Rhodes [13, Theorem A.1V.1], $(s\hat{\delta})^{|s|+1}=(s\hat{\delta})^{|s|}$. Let $s=(s_n,s_{n-1},...,s_1)$. Then, $s\hat{\delta}=(s_n\delta,s_{n-1}\delta,...,s_1\delta)$. If $s_n=(g;i,j)\in S_y$, $s_n\delta=g\in T_y$. Thus, $pr_1(s\hat{\delta})^{|s|+1}=g^{|s|+1}$ and $pr_1(s\hat{\delta})^{|s|}=g^{|s|}$. Let e denote the identity of T_y. Thus, since T_y is a cancellative monoid, $g^{|s|}e=g^{|s|}g$ implies $e=g$. Hence, $s_n\in E(S)$. Thus, using [24, Theorem 3.1(f)], $s\in E(\hat{S})$. Hence $U_t\hat{\delta}^{-1}\le E(\hat{S})$. The last sentence of the lemma is a consequence of the definitions of U_t and δ, Lemma 1.5, and the first sentence of the lemma.

If we replace "ϱ" by "δ", "X_y" by F_y","G_y" by "T_y", and "U_y"by "X_y" in [26, Lemma 5, Lemma 7, Lemma 8, Lemma 9, Lemma 11] (if $U_t\hat{\delta}^{-1}\ne 0$ and the last sentence is omitted), Lemma 12, Lemma 13, the first two sentences of Lemma 15, Lemma 16, Lemma 17, and Lemma 18 (with "and ...Y^*" omitted)], these lemmas are valid for quasi-adequate semigroups S. The proofs of these modified lemmas are the same as the proofs of the original lemmas in [26] except that we replace Lemma 1 of [26] by Lemmas 1.1, 1.4, and 1.5 and Proposition 1.3; Lemma 2 of [26] by Lemma 1.6; and Lemma 6 of [26] by Lemma 1.7 in the proofs of the original lemmas. Using

Lemmas 1.1, 1.4 and 1.5, Proposition 1.3, Lemma 1.6, [26, Lemma 3], Lemma 1.7, and the modified Lemmas, we obtain

Lemma 1.8. <u>If</u> $U_t\hat{\delta}^{-1}\neq\square$, <u>then</u> $U_t\hat{\delta}^{-1}$ <u>is a chain</u> $\tilde{P}_{(t)}$ <u>of rectangular bands</u> $(W_j : j\in\tilde{P}_{(t)})$ <u>where</u> $\tilde{P}_{(t)}$ <u>is a sub-chain of</u> $P_{(t)}=\{1,2,\ldots,|t|\}$ <u>under the reverse of the usual order. Furthermore, every element of</u> W_j <u>has length</u> j.

Let $t\in X$, and suppose that $|t|=n$. If $x,y\in U_t\hat{\delta}^{-1}$, define $x\sigma' y$ if and only if $ax=ay$ for all $a\in W_n$ where n is the least element of $\tilde{P}_n$.

If we make the usual modifications and furthermore replace "σ" by "σ'", [26, Lemma 21 and Lemma 23] are valid for super quasi-adequate semigroups S. The proofs also remain valid of we replace "σ" by σ', "ϱ" by "δ", k by $\bar{k}$, and Lemma 7 by modified Lemma 7 if we note that $e_j L g_j$ (notation of [26, Lemma 23]) by virtue of the modified Lemma 5.

Lemma 1.9. <u>If</u> $U_t\hat{\delta}^{-1}\neq\square$, <u>L is a congruence relation on</u> $U_t\hat{\delta}^{-1}$. <u>Hence,</u> $U_t\delta^{-1}/L$ <u>is a chain</u> $\tilde{P}_{(t)}$ <u>of right zero semigroups</u> $(W_j/L : j\in\tilde{P}_{(t)})$.

Proof. Replace "δ" for "ϱ", Lemmas 21 and 23 by their modifications, and Lemma 1.8 for Lemma 20 in the proof of [26, Lemma 24].

Let τ be a homomorphism of a monoid S onto a monoid T, we define a category R_τ as follows: obj $R_\tau=T$. For $t_1, t_2\in T$, $R_\tau(t_1,t_2)=\{(t_1,s,t_2) : s\in S \text{ and } t_2=t_1(s\tau)\}$. For $(t_1,s_1,t_2)\in R_\tau(t_1,t_2)$ and $(t_2,s_2,t_3)\in R_\tau(t_2,t_3)$, we define the composition $(t_1,s_1,t_2)(t_2,s_2,t_3)=(t_1,s_1s_2,t_3)$. It is easily checked that

$(t_1, s_1 s_2, t_3) \in R_\tau(t_1, t_3)$ and the composition is associative where defined. The identity arrow of $R_\tau(t, t)$ is $(t, 1, t)$ where 1 is the identity of S. So, R_τ is a category. Let α be a congruence on S and for (t_1, s_1, t_2), $(t_1, s_2, t_2) \in R_\tau(t_1, t_2)$ define $(t_1, s_1, t_2) r (t_1, s_2, t_2)$ if and only if $ss_1 = ss_2$ for all $s \in t_1\tau^{-1}$ and $s_1 \alpha s_2$. Then, by [26, Lemma 25], r is a congruence on the category R_τ. Let $D_\tau^\alpha = R_\tau / r$. Following Tilson [18], we term D_τ^α the derived category of τ. Let $[t_1, s_1, t_2] \in D_\tau^\alpha(t_1, t_2)$ denote the r-class of R_τ containing $(t_1, s_1, t_2) \in R_\tau(t_1, t_2)$. We define $x \lambda y$ (in $\hat{S}$) if $x, y \in F_v$ for some v. Clearly, λ is a congruence relation on $\hat{S}$.

Lemma 1.10. <u>For</u> $e \in (\widehat{S/\delta})$, $[e, s, e]\tau = sL$ <u>defines an isomorphism of</u> $D_\tau^\lambda(e, e)$ <u>onto</u> $(U_e\hat{\delta}^{-1}/L)^1$.

Proof. Suppose $sLz (s, z \in U_e\hat{\delta}^{-1})$ Hence, using Lemma 1.8, $s, z \in W_j$ for some $j \in \tilde{P}_{|e|}$. Thus, using modified [26, Lemma 23], $s\sigma' z$. Hence, $xs = xz$ for all $x \in W_{\bar{k}}$ where $k = |e|$. Since $e(x\hat{\delta}) = e$, $e \le x\hat{\delta}$. Let $e = (g_k, g_{k-1}, \ldots, g_1)$. If $\bar{k} = k$, using [16, Proposition 7.1] (valid for arbitrary semigroups) (see also [17, Proposition 12.1]), Lemmas 1.6-1.8, and [24, Lemma 3.1[f], $x\hat{\delta} = (e_k, g_{k-1}, \ldots, g_1)$ where $e_k^2 = e_k L g_k$. Using Lemmas 1.5, 1.6 and 1.8 if $u \in e\hat{\delta}^{-1}$, then $u = ((g_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_1; i_1, j_1))$, say. Since $W_{\bar{k}} = W_k = E(F_{k'}) \cap U_e\hat{\delta}^{-1}$ (where $k \to k'$ defines isomorphism of $\tilde{P}_k$ into Y) (see [26]), let $x = ((e_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_1; i_1, j_1))$. Since $(g_k; i_k, j_k) L (e_k; i_k, j_k)$, it is easily checked that $ux = u$. Hence, $us = uxs = uxz = uz$. Since $s, z \in W_j$, $s\lambda z$. Thus, $[e, s, e] = [e, z, e]$. Next, assume $k > \bar{k}$. Then, using [17, Proposition 12.1]), Lemma 1.7 and [24, Theorem 3.1(f)], $t = (g_k, g_{k-1}, \ldots, g_{\bar{k}}, g_{\bar{k}-1}, \ldots, g_1)$ and

$x\hat{\delta}=(e_{\bar{k}}, g_{\bar{k}-1}, \ldots, g_1)$ where $g_{\bar{k}} L e_{\bar{k}} = e_{\bar{k}}^2$. Hence, $u=(g_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}}), (g_{\bar{k}-1}; i_{\bar{k}-1}, j_{\bar{k}-1}), \ldots, (g_1; i_1, j_1))$ and $x=((e_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}}), (g_{\bar{k}-1}; i_{\bar{k}-1}, j_{\bar{k}-1}), \ldots, (g_1; i_1, j_1))$. Since $(g_a; i_a, j_a) < (g_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}})$ for $\bar{k} < a \le k$, $(g_a; i_a, j_a)(e_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}}) = (g_a; i_a, j_a)$. Furthermore $(g_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}}) L (e_{\bar{k}}; i_{\bar{k}}, j_{\bar{k}})$. Hence, by a routine calculation, $ux=u$. Thus, as above, $[_i, s, _i] = [_i, z, _i]$. Conversely, assume $[_i, s, _i] = [_i, z, _i]$. Hence, $s, z \in F_i$, say and $xs=xz$ for all $x \in {}_i\hat{\delta}^{-1}$. Using [26, Lemma 22], $s \le z$ or $z \le s$. Using Lemma 1.7, $sz=s$ or $zs=z$. Since $s, z \in W_j$ for some j, sLz in either case. Thus, $[_i, s, _i]\tau = sL\,(s \in U_i\hat{\delta}^{-1})$ defines a 1-1 map of $D_i^{\lambda}\,(i,i)$ into $(U_i\hat{\delta}^{-1}/L)^1$. Clearly, τ is a surjection. Using Lemma 1.9, τ is an isomorphism.

Theorem 1.11: <u>Let</u> S <u>be a super quasi-adequate semigroup. Then,</u>

(1) $S^1 < V \circ (\widehat{S/\delta})^1$

<u>where</u> V <u>is an L-trivial and idempotent monoid,</u> δ <u>is the minimum adequate good congruence on</u> S, $(\widehat{S/\delta})$ <u>is a semilattice</u> $Y=S/J^*$ <u>of left cancellative semigroups</u> $(X_y : y \in Y)$ <u>with idempotents, and</u> $E((\widehat{S/\delta}))$ <u>is a semilattice Y of right zero semigroups</u> $(E(X_y) : y \in Y)$.

Proof. Utilize Lemma 1.5 (define $1\delta=1$), Lemma 1.10, [26, Lemma 29], and [26, Theorem 26] to establish (1). To complete the proof utilize Proposition 1.3 and Lemma 1.5.

Remark 1.12 If E is the edge set of the graph obtained from D_i^{λ} by removing the identity arrows, then V is the free monoid over E relative to the equation $xyx=yx\,(x, y \in E^1)$ (see [26]-especially the proof of [26, Lemma 29]). V is a semilattice A (set of all finite subsets of E under union) of right zero semigroups $(U_P : P \in A)$ where U_P denotes the set of all elements of V with content P(see[2],[10] and [26, especially Theorem 27])

Lemma 1.12. $X_y = C_y \times E_y$ where C_y is a cancellative monoid and E_y is a right zero semigroup if and only if T_y is a group. In the case, $X_y = T_y \times E(X_y)$.

Proof. Suppose $X_y = C_y \times E_y$. Then, Using [19, Theorem 2], $a \in aX_y$ for all $a \in X_y$. Thus, $(a_n) = (a_n)e$ for some $e \in X_y$. Hence, $(a_n)e = (a_n)e^2$. Thus, using Lemma 1.5, $e = e^2$. Hence, using Lemma 1.5, $(a_n) = (a_n)(e_y, x_{k-1}, \ldots, x_1)$ where e_y is the identity of T_y. Thus, $(a_n) = red(a_n, e_y, x_{k-1}, \ldots, x_1)$. So, $a_n L e_y$. Hence, using Lemma 1.4, $e_y = sa_n$ where s may be taken as an element of T_y. Thus, $a_n s a_n s = a_n e_y s = a_n s = a_n s e_y$. So, $a_n s = e_y$ and, hence, T_y is a group. Conversely, suppose T_y is a group. Let $(a_n, a_{n-1}, \ldots, a_1) \in X_y$. Then, $(a_n, a_{n-1}, \ldots, a_1) = (a_n)(e_y, a_{n-1}, \ldots, a_1)$. Since $(a_n)(b_n) = (a_n b_n)$ for $a_n, b_n \in T_y$, $T_y \cong \{(a_n : a_n \in T_y\}$. Thus, it is easily checked that every element of X_y may be uniquely expressed in the form $(a)e$ where $a \in T_y$ and $e \in E(X_y)$ and $(a, e) \rightarrow (a)e$ defines an isomorphism of $T_y \times E(X_y)$ onto X_y.

Remark 1.13. In the case S is an orthodox union of groups in Theorem 1.11, δ becomes the minimum inverse semigroup congruence on S, $J^* = J$ and $X_y = T_y \times E(X_y)$ where T_y is a maximal subgroup of S (hence, X_y is a right group). These facts are a consequence of Proposition 1.3, Lemma 1.1, and Lemma 1.12. In this case, the structure of $(\widehat{S/\delta})$ is further refined by [25, Theorem 2.6](see also [26, Theorem 31]).

Section 2. <u>The Structure of Super Generalized L^*-unipotent Semigroups</u>.

In this section, we describe the smallest L^*-unipotent good congruence on a super generalized L^*-unipotent semigroup (Proposition 2.1) and give a structure theorem for super generalized L^*-unipotent semigroups (Theorem 2.4).

Proposition 2.1. <u>Let</u> S <u>be a super generalized L^*-unipotent semigroup. Then,</u> $\delta\cap L$ <u>is the smallest L^*-unipotent good congruence on</u> S. $S/\delta\cap L$ <u>is a semilattice</u> $Y=S/J^*$ <u>of semigroups</u> $(M_\gamma : \gamma\in Y)$ <u>where</u> $M_\gamma=T_\gamma\times J_\gamma$ <u>where</u> T_γ <u>is the cancellative monoid of Lemma 1.1 and</u> J_γ <u>is an R-class of</u> $E(S_\gamma)$. $E(S/\delta\cap L)$ <u>is a semilattice</u> Y <u>of the right zero semigroups</u> $(J_\gamma : \gamma\in Y)$.

Proof. We first show that $\delta\cap L$ is a congruence relation on S. Utilizing Proposition 1.3, $\delta\cap L$ is a right congruence relation on S. Let $\overline{\delta\cap L}$ be the smallest congruence relation on S containing $\delta\cap L$. We will show that $\overline{\delta\cap L}=\delta\cap L$. Suppose a $(\overline{\delta\cap L})$b. Then, there exists $a=a_1, a_2, \ldots a_n=b\in S$ such that $a_i=x_i u_i$, $a_{i+1}=x_i v_i$ where x_i $y_i\in S^1$ and $(u_i, v_i)\in\delta\cap L$ for $1\le i\le n-1$. Let $x_i=(g;i,k)_\gamma\in S_\gamma$, $u_i=(w;s,j)_\lambda$ $\in S_\lambda$, and $v_i=(w;t,j)_\lambda\in S_\lambda$. Since δ is a congruence relation, $a_i=(m;p,q)_{\gamma\lambda}$ and $a_{i+1}=(m;i,d)_\gamma$, say. Let $\alpha=\gamma\lambda$. Then, $\alpha\gamma=\alpha\lambda=\alpha$. Hence, $(m;p,q)_\alpha=(g;i,k)_\gamma(e_\gamma;i,k)_\gamma(e_\lambda;s,j)_\lambda(w;s,j)_\lambda$ and $(m;c,d)_\alpha=(g;i,k)_\gamma(e_\gamma;i,k)_\gamma(e_\lambda;t,j)_\lambda$ $(w;s,j)_\lambda$ where e_γ is the identity of T_γ.

Since L is a congruence relation on E(S), $(e_\gamma;i,k)_\gamma(e_\lambda;s,j)_\lambda$ $L(e_\gamma;i,k)_\gamma(e_\lambda;t,j)_\lambda$. Hence, $(e_\lambda;i,k)_\gamma(e_\lambda;s,j)_\lambda=(e_\alpha;s',j')_\alpha$ and $(e_\gamma;i,k)_\gamma(e_\lambda;t,j)_\lambda=(e_\alpha;t',j')_\alpha$, say. Hence, (m;p,q).=(g;i,k),(e.;s′,j′).$(w;s,j)_{\lambda}$

(m;c,d).=(g;i,k),(e.;t′,j′).$(w;s,j)_{\lambda}$

Since L is a right congruence relation on S, (e.;s′,j′).$(w;s,j)_{\lambda}$L(e.;t′,j′).$(w;s,j)_{\lambda}$. Hence, (e.;s′,j′).$(w;s,j)_{\lambda}$ =(w*;s*,j*). and (e.;t′,j′).$(w;s,j)_{\lambda}$ =$(\bar{w};\bar{s},j^{*})$.., say. Thus,

(m;p,q).=(g;i,k),(w*;s*,j*).

(m;c,d).=(g;i,k),(w;s,j*)..

Hence,

(e.;p,q).(m;p,q).=(e.;p,q).(g;i,k),(w*;s*,j*).

(e.;p,q).(m;c,d).=(e.;p,q).(g;i,k),$(\bar{w};\bar{s},j^{*})$..

Suppose that (e.;p,q).(g;i,k),=$(\bar{g};\bar{i},\bar{k})$.. Then,

(m;p,q).=$(\bar{g};\bar{i},\bar{k})$.(w*;s*,j*).

(m;p,q).=$(\bar{g};\bar{i},\bar{k})$.$(\bar{w};\bar{s},j^{*})$.

Hence, q=d=j*. Thus, $a_i(\delta\cap L)a_{i+1}$ for $1\le i\le n-1$. Hence, $a(\delta\cap L)b$ and, thus $\delta\cap L=\overline{\delta\cap L}$.

We will need to show that $\delta\cap L^{*}=\delta\cap L$. Suppose $a(\delta\cap L^{*})b$. Since $a\delta b$, a=(g;i,j).∈S. and b=(g;r,s).∈S.., say. There exists an over-semigroup S* of S such that s(g;i,j).=(g;r,s). where s∈S*. Hence, (g;r,s).(e.;i,j).=(g;r,s)..

Thus, j=s. Hence, $a(\delta\cap L)b$. Thus, $\delta\cap L^{*}\le\delta\cap L$. Since $L\le L^{*}$, $\delta\cap L^{*}=\delta\cap L$.

We next show that $\delta\cap L$ is a good congruence. We will use [5, Corollary 1.5]. Suppose $aL^{*}e$ where $e\in E(S)$. Let $ax(\delta\cap L)ay$ where $x,y\in S^{1}$. Thus, $ax(\delta\cap L^{*})ay$. Since $aL^{*}e$, $axL^{*}ex$ and $ayL^{*}ey$. Thus, $exL^{*}ey$. Using [5, Corollary 1.5] and Proposition 1.3, $ex\delta ey$ for some $e^{*}=e\in L^{*}_{a}$. Thus, $ex(\delta\cap L^{*})ey$. Hence, $ex(\delta\cap L)ey$. Next, let $aR^{*}e$

where $e \in E(S)$. Assume $xa(\delta \cap L)ya$ where $x,y \in S^1$. Thus, $xa=(h;m,n)_e$ and $ya=(h;p,n)_e$, say. Let $f=(e_y;m,n)_e$. Then, $xa=fya$. Hence, $fxa=fya$. Thus, using [11, Lemma 1.7], $fxe=fye$. Since xaR^*xe and yaR^*ye, it is easily seen that xe, ye, and $f \in S_e$. Hence $fxe=fye$ implies $xe(\delta \cap L)ye$. Thus, $\delta \cap L$ is a good congruence on S by [5, Corollary 1.5].

We next show that $S/\delta \cap L$ is an L^*-unipotent semigroup. Using [6, Proposition 1.6], $S/\delta \cap L$ is a quasi-adequate semigroup. Using [6, Lemma 1.5], $E(S/\delta \cap L)=\{e(\delta \cap L):e \in E(S)\}$. Suppose $e(\delta \cap L)Lf(\delta \cap L)$ (in $E(S/\delta \cap L)$). Thus, $(ef,e) \in \delta \cap L$ and $(fe,f) \in \delta \cap L$. Hence, $e,f \in S_r$, say. Thus, $e=efe=ef$. Hence, $e(\delta \cap L)f$. Thus, $S/\delta \cap L$ is an L^*-unipotent semigroup.

Let ρ be an L^*-unipotent congruence on S. Suppose $a(\delta \cap L)b$. Then, $a=(g;m,n)_e$ and $b=(g;p,n)_e$, say. Thus $a=(e_e;m,n)_e b$. Since $(e_e;m,n)_e L (e_e;p,n)_e$, $(e_e;m,n)_e\rho=(e_e;p,n)_e\rho$. Hence, $a\rho=(e_e;m,n)_e\rho b\rho=(e_e;p,n)_e b\rho=b\rho$. Thus, $\delta \cap L \leq \rho$. Thus, $\delta \cap L$ is the smallest L^*-unipotent congruence on S.

Using Lemma 1.1, $S_y=T_y \times I_y \times J_y$ (algebraic direct product) where I_y is a left zero semigroup and J_y is a right zero semigroup. Let $M_y=T_y \times J_y$ (algebraic direct product). Let $\overline{(g;i,j)}$ denote the $\delta \cap L$-class of S containing $(g;i,j)$. Then, $\overline{(g;i,j)}\lambda=(g,j)$ defines a 1-1 mapping of $S/\delta \cap L$ onto $M=\cup(M_y:y \in Y)$. In a similar manner to the proof of Lemma 1.4, we may define a multiplication on M such that M is a semilattice Y of the semigroups $(M_y:y \in Y)$ and $M \cong S/\delta \cap L$. The last sentence follows since E(M) is a semigroup.

Remark 2.2 will be used in the proof of Theorem 2.4.

Remark 2.2. Let θ be a homomorphism of a semigroup S onto a semigroup T. Define $D(\theta)=\{(t,s,t(s\theta)):t\in T^{*};s\in S\}\cup\{0\}$ under the multiplication $(t_1,s_1,t_1(s_1\theta))(t_2,s_2,t_2(s_2\theta))=(t_1,s_1s_2,t_1(s_1s_2)\theta)$ if $t_1(s_1\theta)=t_2$; 0 if $t_1(s_1\theta)\neq t_2$ and $0(t,s,t(s\theta))=(t,s,t(s\theta))0=0.0=0$. $D(\theta)$ was termed the derived semigroup of θ by its inventor Bret Tilson (see[16] and [17]). Let $\emptyset$ be a mapping of $D(\theta)-\{0\}$ into a semigroup P. Following Rhodes [13, Definition A.I.2.1, p.94], we term $\emptyset:D(\theta)-\{0\}\rightarrow P$ a parametrization of $D(\theta)$ if 1) $\emptyset$ is a partial homomorphism of $D(\theta)-\{0\}$ into P(i.e. if $x,y\in D(\theta)-\{0\}$ and $xy\neq 0$, then $x\theta y\theta=(xy)\theta$) 2) $\emptyset$ satisfies the embedding condition: $s_1\theta=s_2\theta$ and $(t,s_1,t(s_1\theta))\emptyset=(t,s_2,t(s_2\theta))\emptyset$ for all $t\in T^{*}$ implies $s_1=s_2$. For brevity, we also term P a parametrization of $D(\theta)$. Using [13, Proposition AI.2.3], $S\leq P\circ T$ where $p|S=\theta$ (p is the projection if $P\circ T$ onto T). Following Rhodes [13], we define $D^{*}(\theta)$(dual derived semigroup) as follows: $D^{*}(\theta)=\{((s\theta)t,s,t):s\in S,t\in T^{*}\}\cup\{o\}$ under the multiplication $((s_1\theta)t_1,s_1,t_1)((s_2\theta)t_2,s_2,t_2)=((s_1\theta)t_1,s_1s_2,t_2)$ if $t_1=(s_2\theta)t_2$; o if $t_1\neq(s_2\theta)t_2$; $o((s\theta)t,s,t)=((s\theta)t,s,t)o=oo=o$. A parametrization P^{*} of $D^{*}(\theta)$ is defined as above and $S\leq T\circ P^{*}$ with $P|S=\theta$.

Remark 2.3 will be needed for the statement of Theorem 2.4

Remark 2.3. Let W be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups $(T_y:y\in Y)$ where Y is a semilattice. If $a\in T_y$, $b\in T_z$ and $y\geq z$ (in Y) imply ab is defined (in W) and $ab\in T_z$ and $z\geq w$ and $c\in T_w$ imply $(ab)c=a(bc)$, we term Wa lower partial chain Y of the semigroups $(T_\gamma : \gamma \in Y)$. Let X be a semilattice Y of semigroups $(X_\gamma : \gamma \in Y)$ and let R and S be semigroups. For the definition of $W \circ X \circ R$ and $S \leq W \circ X \circ R$, see [24, p. 188 and p. 189].

Theorem 2.4. <u>Let</u> S <u>be a super generalized L^*-unipotent semigroup</u>. <u>Then,</u>

(1) $S \leq W^1 \circ (E(S)/L)^1 \circ (S/\delta \cap L)^1$ <u>where</u> W <u>is a lower partial chain</u> $Y = S/J^*$ <u>of left zero subsemigroups of</u> $E(S)$, $E(S)/L$ <u>is a semilattice</u> Y <u>of right zero semigroups, and</u> $\delta \cap L$ <u>is the smallest L^*-unipotent good congruence on</u> S. <u>Furthermore,</u>

(2) $S/\delta \cap L \leq (S/\delta \cap L/\rho)^1 \circ (E(S)/L)^1$ <u>where ρ is the smallest adequate good congruence on</u> $S/\delta \cap L$ <u>and</u> $S/\delta \cap L/\rho$ <u>is a strong semilattice</u> Y <u>of cancellative monoids</u> $(T_\gamma : \gamma \in Y)$ (T_γ <u>is a cancellative subsemigroup of</u> S).

Proof. We will first establish that $S \leq (E(S))^1 \circ (S/\delta \cap L)^1$. For each $(g,j)_\gamma \in M_\gamma$ $(\gamma \in Y)$ (Notation of Proposition 2.1), select a representative element $u_{(g,j)_\gamma}$ in S_γ. We first show that every element of S may be uniquely expressed in the form $w_{(e_\gamma,j)_\gamma} u_{(g,j)_\gamma}$ where $w_{(e_\gamma,j)_\gamma} \in (e_\gamma,j)_\gamma (\delta \cap L)^{-1}$. Let $(g;i,j)_\gamma \in S_\gamma$ and suppose $u_{(g,j)_\gamma} = (g;i_0,j)_\gamma$. Then, $(g;i,j)_\gamma = (e_\gamma;i,j)_\gamma (g;i_0,j)_\gamma$ where $(e_\gamma;i,j)_\gamma \in (e_\gamma,j)_\gamma (\delta \cap L)^{-1}$. It is easily checked that the above expression is unique. If $\alpha = (g,j)_\gamma$, let $\alpha^+ = (e_\gamma,j)_\gamma$. Thus every element of S may be uniquely expressed in the form $w_{\alpha^+} u_\alpha$ where $w_{\alpha^+} \in \alpha^+ (\delta \cap L)^{-1}$. Let $\alpha \in S/\delta \cap L$ and $\beta \in_x (\delta \cap L)^{-1}$. Hence, we may write $u_\alpha s = f(\alpha,\beta) u_{\alpha x}$ where $f(\alpha,\beta) \in (\alpha x)^+ (\delta \cap L)^{-1}$. First assume S has an identity. For $(\alpha,\beta,\alpha(\beta(\delta \cap L)) \in D(\delta \cap L) - \{0\}$, define

$(t, s, t(s(\delta\cap L)))\theta = f(t,s)$. We will show that $\theta: D(\delta\cap L)-\{0\} \to E(S)$ is a parametrization of $D(\delta\cap L)$. It is easily checked that θ defines a mapping of $D(\delta\cap L)-\{0\}$ into $E(S)$. Next, we show that θ defines a partial homomorphism. Let $(t_1, s_1, t_1(s_1(\delta\cap L)))$, $(t_2, s_2, t_2(s_2(\delta\cap L))) \in D(\delta\cap L)$ with $t_1(s_1(\delta\cap L)) = t_2$. We must show $f(t_1,s_1)f(t_2,s_2) = f(t_1, s_1 s_2)$. Suppose $s_1 \in s_1(\delta\cap L)^{-1}$ and $s_2 \in s_2(\delta\cap L)^{-1}$. Then, $u_{t_1}(s_1 s_2) = f(t_1, s_1 s_2)u_{t_1 s_1 s_2} = f(t_1, s_1 s_2)u_{t_2 s_2}$ where $f(t_1, s_1 s_2) \in (t_2 s_2)^{+}(\delta\cap L)^{-1}$. However, $(u_{t_1} s_1)s_2 = f(t_1, s_1)(u_{t_2} s_2) = f(t_1, s_1)f(t_2, s_2)u_{t_2 s_2}$. Let $t_2 \in M_y$ and $s_2 \in M_z$, say. Hence, $t_2 s_2 \in M_{yz}$. Furthermore, $t_2^{+} \in E(M_y)$ and $(t_2 s_2)^{+} \in E(M_{yz})$. Using the last sentence of Proposition 2.1, $t_2^{+}(t_2 s_2)^{+} = (t_2^{+}(t_2 s_2)^{+})(t_2 s_2)^{+} = (t_2 s_2)^{+}$. Hence, $f(t_1, s_1)f(t_2, s_2) \in (t_2 s_2)^{+}(\delta\cap L)^{-1}$. Thus, $f(t_1, s_1)f(t_2, s_2) = f(t_1, s_1 s_2)$, and, hence, θ is a partial homomorphism. We next show the embedding condition is valid. Let e denote the identity of $S/\delta\cap L$ and let $u_e = 1$, the identity of S. Thus, if $s_1(\delta\cap L) = s_2(\delta\cap L) = s$ and $f(e, s_1) = f(e, s_2)$, then $s_1 = u_e s_1 = f(e, s_1)u_s = f(e, s_2)u_s = u_e s_2 = s_2$. Hence, $E(S)$ is a parametrization of $D(\delta\cap L)$. Thus, using Remark 2.2, $S \le E(S) \circ S/\delta\cap L$. If S has no identity consider S^1. Note that $a(\delta\cap L)1$ (in S^1) implies $a = 1$. Hence, $S^1/\delta\cap L \cong (S/\delta\cap L)^1$. Furthermore, $E(S^1) \cong (E(S))^1$. Hence, $S \le S^1 \le (E(S))^1 \circ (S/\delta\cap L)^1$. Thus utilizing [24, Theorem 1.24, Remark(1.24)', Lemma 1.23, and Lemma 1.25], we obtain(1). We next establish (2). Let $M = S/\delta\cap L$. Utilizing [9, Corollary 6.2 and Proposition 6.5], Proposition 2.1 and Lemma 1.4, M/ϱ is the strong semilattice Y of cancellative monoids $(T_y : y \in Y)$. If $t \in T_y$, let $t^{+} = e_y$, the identity of T_y. For each $t \in M/\varrho$, select a representative element $u_{\cdot} \in {}_{\cdot}\rho^{-1}$. We show that every element of M may be uniquely expressed in the form $u_{\cdot}\, v_{\cdot}^{*}$ where $v_{\cdot}^{*} \in {}_{\cdot}{}^{*}\rho^{-1}$. Let $(g,j)_y \in M_y$ and suppose $u_{\cdot} = (g, j_{\cdot})_y \in M_y$. Hence, $(g,j)_y = (g, j_{\circ})_y (e_y, j)_y$ where $(e_y, j)_y \in e_y\rho^{-1}$ and $g^{\circ} = e_y$. Suppose $u_{\cdot} g_{\cdot}^{*} = u_{\cdot} h_{\cdot}^{*}$. Then, since $M_y\,(y \in Y)$ is left cancellative, $g_{\cdot}^{*} = h_{\cdot}^{*}$. Let $\cdot \in M/\rho$ and $\cdot \in {}_{\cdot}\rho^{-1}$. Hence, we may write ${}_{\cdot}u_{\cdot} = u_{\cdot\cdot} f(\cdot,\cdot)$ where $f(\cdot,\cdot) \in (\cdot\cdot)^{*}\rho^{-1}$. First, assume that M has an identity. For $((\cdot\rho)\cdot,\cdot,\cdot) \in D^{*}(\rho) - \{0\}$, define $((\cdot\rho)\cdot,\cdot,\cdot)\theta = f(\cdot,\cdot)$. Using the fact that M/ρ is a strong semilattice Y of cancellative monoids $(T_y : y \in Y)$, we proceed as above to show that $\theta : D^{*}(\rho) - \{0\} \to E(M)$ is a parametrization of $D^{*}(\rho)$. Thus, using Remark 2.2, $M \le M/\rho\, \hat{\circ}\, E(M)$. Again, proceeding as above, $M \le M^{1} \le (M/\rho)^{1}\, \hat{\circ}\, (E(M))^{1}$. Using Proposition 2.1, $E(M) \cong E(S)/L$. Hence (2) is valid. To complete the proof, utilize Proposition 2.1.

Remark 2.5. W is a lower partial chain Y of L-classes of E(S). Eacn J-class of E(S) contains precisely one of these L-classes (see[24, Theorem 1.24]).

Remark 2.6. Let S be a generalized L-unipotent union of groups. Then, $\delta \cap L$ is the smallest L-unipotent congruence on S (δ is the smallest inverse semigroup congruence on S), ρ is the smallest inverse semigroup congruence on $S/\delta \cap L$, T_y is a maximal subgroup of S, and $J^{*} = J$ in the statement of Theorem 2.4. Thus, Theorem 2.4 generalizes [24, Theorem 1.27, Theorem 1.28, and Theorem 1.26] in the case S is also a union of groups (our structure theorem for generalized L-unipotent unions of groups).

A different type structure theorem for generalized R-unipotent unions of groups is given in [22, Theorem 4.7].

Section 3 Super R^*-unipotent Semigroups

In this section, we give a structure theorem for super R^*-unipotent semigroups (Theorem 3.1)

Theorem 3.1. Let S be a super R^*-unipotent semigroup. Thus,

* $S \leq (E(S))^1 \circ (S/\delta)^1$ where E(S) is a semilattice $Y = S/J^*$ of left zero semigroups, δ is the smallest adequate good congruence on S, and S/δ is a strong semilattice Y of cancellative monoids $(T_y : y \in Y)(T_y$ is a subsemigroup of S).

Proof. Using Lemma 1.1, $S_y = T_y \times E(S_y)$ where $E(S_y)$ is a left zero semigroup. Hence, by a routine calculation, $\delta \cap L = \delta$. Thus, utilizing the proof of Theorem 2.4, * is valid. Use Proposition 1.3 and Lemma 1.4 to complete the proof.

Remark 3.2. Let S be an R-unipotent union of groups. Then, δ is smallest inverse semigroup congruence on S, T_y is a maximal subgroup of S, and $J = J^*$ in the statement of Theorem 3.1. Hence, Theorem 3.1 generalizes [24, Remark 1.14, Theorem 1.12, and Theorem 1.8](our structure theorem for R-unipotent unions of groups). A different type structure theorem for L-unipotent unions of groups is given in [22, Theorem 7.2].

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