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ON A CHARACTERIZATION OF INNER PRODUCT SPACES

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ABSTRACT

In this paper we present some new characterizations of inner product spaces. It is also proved that there are certain simple inequalities which are not satisfied by the normed spaces.

INTRODUCTION

It is well known that if a certain non-trivial inequality holds in a normed linear space, then the norm is determined by an innerproduct [1], [2], [3] and [4]. For some norm identities, M.M. Day [2] has shown that if the identity holds only for unit vectors, the space must still be an innerproduct space. Day's result was extended by Schoenberg [5]. He showed that the sign of equality in Day's result could be replaced by either $\geq$ or $\leq$. In this paper we have used Schoenberg's result to prove some new characterizations of the innerproduct spaces. To the best of the author's knowledge the results proved in this paper are not known before.

Theorem 1

A normed space $(X, \|\cdot\|)$ is an innerproduct space iff for all $x, y \in X$, $\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \leq \max(\|x\|^2, \|y\|^2)$. However, there does not exist any normed space X for which $\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 > \max(\|x\|^2, \|y\|^2)$ for all $x, y \in X$ is satisfied.

Proof:

($\Rightarrow$) If X is an innerproduct space then from the parallelogram identity [2] the result follows. Since

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in X$$

which implies

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \le \max(\|x\|^2, \|y\|^2) \qquad \text{for all } x, y \in X \tag{1}$$

($\Leftarrow$)

For $x, y \in X$ with $\|x\| = \|y\| = 1$ in (1), we get

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \le 1 \tag{2}$$

But then according to Schoenberg [5] the normed space X must be an innerproduct space.

Lastly suppose that there exists a normed space $(X, \|\cdot\|)$ for which the inequality

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 > \max(\|x\|^2, \|y\|^2) \qquad \text{for all } x, y \in X \tag{3}$$

holds. Then, in particular for $x, y \in X$ with $\|x\| = \|y\| = 1$

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 > 1 \tag{4}$$

and hence according to [5] the space X must be an inner product space. But, for any inner-product space X,

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 = 1 \qquad \text{for all } x, y \in X \text{ with } \|x\| = \|y\| = 1$$

which contradicts (4). Hence the inequality (3) does not hold in any normed space.

Example:

Consider the normed space $X = l_\infty(R^2)$. For $x = (1,1), y = (1,-1)$ we have

$$\|\frac{x+y}{2}\|_\infty^2 + \|\frac{x-y}{2}\|_\infty^2 = 2 > \max(\|x\|_\infty^2, \|y\|_\infty^2) = 1.$$

However, for $x = (2,0)$ and $y = (0,2)$

$$\|\frac{x+y}{2}\|_\infty^2 + \|\frac{x-y}{2}\|_\infty^2 = 2 < \max(\|x\|_\infty^2, \|y\|_\infty^2) = 4.$$

It follows that the inequality (3) may be satisfied by some of the elements in a normed space X but it is not satisfied by all the elements in the normed space.

Now we state another characterization of an innerproduct space. The proof of Theorem 2 is similar to that of Theorem 1 and is therefore omitted.

Theorem 2

A normed space $(X, \|\cdot\|)$ is an innerproduct space iff for all $x, y \in X$, $\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \geq \min(\|x\|^2, \|y\|^2)$. However, there does not exist any normed space X for which $\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 < \min(\|x\|^2, \|y\|^2)$ for all $x, y \in X$ is satisfied.

Theorem 3

A normed space $(X, \|\cdot\|)$ is an innerproduct space if there exists some $0 \leq \theta \leq 1$ such that

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \sim \theta\|x\|^2 + (1-\theta)\|y\|^2 \quad \text{for all } x, y \in X \tag{5}$$

where $\sim$ is one of the relations $=, \geq, \leq$.

Proof:

Suppose, there exists some $0 \leq \theta \leq 1$ such that, (5) is satisfied. If we take $x, y \in X$ with $\|x\| = \|y\| = 1$ in (5) we get

$$\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \sim 1 \qquad \text{for } 0 \leq \theta \leq 1.$$

But then, according to Schoenberg [5], X must be an innerproduct space.

The following two corollaries are immediate consequences of Theorem 3.

Corollary 1: A normed space X is an innerproduct space if $\|x+y\|^2 + \|x-y\|^2 \sim 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in X$.

Corollary 2: A normed space X is an innerproduct space if $\|\frac{x+y}{2}\|^2 + \|\frac{x-y}{2}\|^2 \sim \|x\|^2$ for all $x, y \in X$.

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