King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 117

March 1990

On the Structure of Orthodom Unions of Groups

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To PROFESSOR MIYUKI YAMADA on his sixtieth birthday

Structure theorems for orthodox unions of groups have been given by Fantham [5], Preston [1961], unpublished, see [2]], Yamada [17], Warne [14], Petrich [8], and Clifford [3]. Structure theorems for bands have been given by Petrich [1967, see [6, p. 102]] and Warne [13]. We give a global structure theorem for orthodox unions of groups (Theorem 31) and a global structure theorem for bands (corollary 32). Let $\mathcal{L}$ and $\mathcal{J}$ denote Green's relations, let "<" denote division, let "o" denote wreath product, let $\widehat{\ }$ denote the Rhodes expansion, and let "1" denote an appended identity. We first give our construction for bands. Let S be a band. Then,

$$S^1 < V \,o\, (\widehat{S/_{\mathcal{J}}})^1$$

where V is an $\mathcal{L}$-trivial and idempotent monoid and $(\widehat{S/_{\mathcal{J}}})$ is a semilattice $S/_{\mathcal{J}})$ of right zero semigroups. For each $t \in (\widehat{S/_{\mathcal{J}}})$, V covers a semigroup $(V_t)^1$ where V_t is a chain $1 > 2 > \ldots > |t|$ (length of t, a positive integer) of right zero semigroups. We next give our construction for orthodox unions of groups. Let S be an orthodox union of groups.
Then

$$S^1 < V \,o\, (\widehat{S/_{\rho}})^1$$

where ρ is the smallest inverse semigroup congruence on S and $\widehat{S/_{\rho}}$ is a semilattice $S/_{\mathcal{J}}$ of right groups.

Furthermore,

$$\widehat{(S/\rho)} \le ((\widehat{S/\rho})/\delta)^{1} \overset{r}{\circ} (E((\widehat{S/\rho})))^{1}$$

where "$\le$" denotes "embedding", δ denotes the smallest inverse semigroup congruence on $\widehat{(S/\rho)}$, $\widehat{(S/\rho)}/\delta$ is a semilattice $S/\mathcal{J}$ of maximal subgroups of S, "$\overset{r}{\circ}$" denotes the reverse wreath product, and $E((\widehat{S/\rho}))$, the set of idempotents of $\widehat{(S/\rho)}$, is a semillatice $S/\mathcal{J}$ of right zero semigroups.

A semigroup S is an orthodox union of groups if S is the union of its subgroups and $E(S)$, the set of idempotents of S, is a subsemigroup. A semigroup S is a band if $S = E(S)$.

AMS Subject Classification 20M10

A semigroup (monoid) S is said to divide a semigroup (monoid) T if there exists a homomorphism of subsemigroup (submonoid) of T onto S. We also say T covers S in this case. Let $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ denote Green's relation on a semigroup S.

Let us first describe the Rhodes expansion $\widehat{S}$ of an arbitrary semigroup S. The Rhodes expansion has been an important tool in our investigations of the structure theory of regular semigroups (see [15] and [16]). If $a, b \in S$, $a \le b$ means $a \cup Sa \le b \cup Sb$ and $a < b$ means $a \le b$ but $a \not\geq b$. Let $S_+ = \{(s_n, \ldots, s_1): s_i \in S$ for $1 \le i \le n$ and $s_1 \le s_2 \ldots \le s_n\}$. If $x = (s_n, \ldots, s_1)$, $y = (t_m, \ldots, t_1)$, define $xy = (s_n t_m, \ldots, s_1 t_m, t_m, \ldots, t_1)$. Then S_+ is a semigroup under this multiplication. If $a = (s_n, \ldots, s_1) \in S_+$ and $s_{k+1} \mathcal{L} s_k$ for some $1 \le k \le n - 1$ delete s_k to obtain $a_1 \in S_+$ and denote the deletion by $a \longrightarrow a_1$. Perform $a \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_k$ where $a_k = (s_n, s_{n_1}, \ldots, s_{n_r})$ with $s_n < s_{n_1} < \ldots < s_{n_r}$ (such an a_k is termed an irreducible element of S_+). Write $a_k = \text{red } a$ and $a \sim b$ if $\text{red } a = \text{red } b$. The equivalence relation $\sim$ is a congruence relation on S_+. Let $\widehat{S} = S_{+/\sim}$. $\widehat{S}$ is termed the Rhodes expansion of S after its inventor, John Rhodes. $\widehat{S}$ will be treated as the set of irreducible elements of S_+ under the multiplication $ab = \text{red}(ab)$. Let ζ be a homomorphism of a semigroup S onto a semigroup T. Then $(s_n, \ldots, s_1)\widehat{\zeta} = \text{red}(s_n \zeta, \ldots, s_1\zeta)$ defines a homomorphism of $\widehat{S}$ onto $\widehat{T}$ [Tilson, 10, Proposition 6.6].

We next give a specialization and slight modification of Tilson's derived category of a relational morphism [12]. Tilson's Derived Category Theorem [12] (see Theorem 2.6) will be crucial in our proof of Theorem 31. Let δ be a homomorphism of a monoid S onto a monoid T (a monoid is a semigroup with identity element). We define a category R_δ as follows: obj $R_\delta = T$. For $t_1, t_2 \in T$, $R_\delta(t_1,t_2) = \{(t_1,s,t_2): s \in S$ and $t_2 = t_1(s\delta)\}$. For $(t_1,s_1,t_2) \in R_\delta(t_1,t_2)$ and $(t_2,s_2,t_3) \in R_\delta(t_2,t_3)$, define the composition $(t_1,s_1,t_2)(t_2,s_2,t_3) = (t_1,s_1s_2,t_3)$. It is easily checked that $(t_1,s_1s_2,t_3) \in R_\delta(t_1,t_3)$ and our composition is associative where defined. The identity arrow of $R_\delta(t,t)$ is $(t,1,t)$ where 1 is the identity of S. So, R_δ is a category. Let λ be a congruence relation on S and for $(t_1,s_1,t_2), (t_1,s_2,t_2) \in R_\delta(t_1,t_2)$, define $(t_1,s_1,t_2) \sim (t_1,s_2,t_2)$ if and only if $ss_1 = ss_2$ for all $s \in t_1\delta^{-1}$ and $s_1\lambda s_2$. Then, $\sim$ is a congruence relation on the category R_δ (Lemma 25). Let $D_\delta^\lambda = R_\delta/_\sim$. Following Tilson [12], we term D_δ^λ the derived category of δ. By the Derived Category Theorem, if V is a monoid covering D_δ^λ (as categories) (see [12] and page 11-12), then $S < V \circ T$.

Let us sketch the proof of Theorem 31.

Let $U = S/_\rho$ and for $t \in U$, let $U_t = \{x \in U: tx = t\}$ (U_t is called the stabilizer of t in U ([11]). If $t =$

$(u_1,u_2,\ldots,u_n)$, let $|t| = n$, the length of t, and let $P_{|t|} = \{1,2,\ldots,n\}$ under the reverse of the usual order. We show $U_t\hat{\rho}^{-1}$ is a band on which $\mathcal{L}$ is a congruence relation and $U_t\hat{\rho}^{-1}/\mathcal{L}$ is a chain $P_{|t|}$ of right zero semigroups (Lemma 24). Let $\underset{\sim}{L}_1$ be the variety of $\mathcal{L}$-trivial (each $\mathcal{L}$-class is a singleton) and idempotent (each element is an idempotent) monoids. We show that $D_{\rho}\mathcal{G}(t,t) \cong (U_t\hat{\rho}^{-1}/\mathcal{L})^1$ for $t \in (\widehat{S/_{\rho}})$, and hence $D_{\rho}\mathcal{G} \in \underset{\sim}{L}_1$ (Lemma 30). We show $\underset{\sim}{L}_1$ is a local variety in the sense of Tilson [12] (a variety $\underset{\sim}{M}$ is local if for every category $S, S(c,c) \in M$ for every $c \in \mathrm{Obj}\ S$ implies $S < V$ for some $V \in M$) (Theorem 29). To prove Theorem 29, we use a specialization of an algorithm of Howie [6, p. 107, Theorem 4.7] for describing free idempotent monoids to free $\underset{\sim}{L}_1$ monoids (Theorem 27) and we use an extension of Simon's well known theorem on directed graphs [4, p. 224, Theorem 7.1] from the variety of commutative and idempotent monoids to $\underset{\sim}{L}_1$ (Theorem 28). Hence, $D_{\rho}\mathcal{G} < V$ for some $V \in \underset{\sim}{L}_1$. Thus, using the Derived Category Theorem, $(\hat{S})^1 < V\mathrm{o}(\widehat{S/_{\rho}})^1$. The proof of Theorem 31 may be completed by using [16, Theorem 2.6].

We adopt the following notation and definitions from [15, p. 181-182], S^1 (S with appended identity), wreath product "o" of semigroups, reverse wreath product "$\underset{\mathrm{o}}{r}$" of semigroups, type A semigroup congruence (for example, inverse semigroup congruence), $a\rho$ ($a \in S$, a semigroup) (ρ, a congruence on S, will also note the natural homomorphism of S onto $S/_{\rho}$), union of groups, $S \xrightarrow{\psi} T$, and $S \cong T$ where S and T are semigroups and ψ is a homomorphism of S onto T.

A submonoid of a monoid S is a subsemigroup of S containing the identity of S.

If S and T are semigroups and there exists an isomorphism of S into T, we write $S \leq T$.

We will use the following definitions and notation from Clifford and Preston [1] and/or Howie [6] and/or Krohn, Rhodes, and Tilson [7]. Idempotent, dentity, Green's relations ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$), $\mathcal{J}$-class, semilattice, semilattice of semigroups, completely simple semigroup, natural ordering of idempotents, inverse, regular semigroup, maximal subgroup, inverse semigroup, right zero semigroup, right group, rectangular band, orthodox semigroup, and equivalence between semilattice and commutative idempotent semigroup.

A chain is a linearly ordered set. N will denote the set of natural numbers $1,2,\ldots$.

If S is a semigroup and $s \in S$, let $S_s = \{x \in S\colon sx = s\}$. S_s is a subsemigroup of S and is called the stabilizer of s in [11]. Let $\mathcal{I}(s)$ denote the collection of inverses of s.

Lemma 1 will be used in the proof of Lemmas 2,6, 9, 10 ,12 ,14, 15, 16, and 19 and Theorem 31.

Lemma 1. Let S be an orthodox union of groups (i.e. S is a semilattice Y of orthodox completely simple semigroups $(S_y\colon y \in Y)$). Then S is a semilattice Y of orthodox completely simple semigroups $(X_y\colon y \in Y)$ where $X_y = \{(s_n,s_{n-1},\ldots,s_1)\colon n \in$

$N\backslash\{0\}$, $s_n \in S_y$ *and* $s_j \in S$ *for* $1 \le j \le n-1$). *Let* ρ *be the smallest inverse semigroup congruence on* S. *Then,* $S/_\rho$ *is a semilattice* Y *of groups* $(G_y\colon\ y \in Y)$ *where* G_y *is a maximal subgroup of* S_y *and* $S_y\rho \le G_y$ *for all* $y \in Y$. *We may write* $S_y = G_y \times B_y$ *(algebraic direct product) where* B_y *is a rectangular brand and* $(g,b)\rho = g$ *for* $(g,b) \in G_y \times B_y$, $\widehat{S/_\rho}$ *is a semilattice* Y *of right groups* $(U_y\colon\ y \in Y)$ *where* $U_y = \{(g_n,\ldots,g_1)\colon g_n \in G_y$ *and* $g_j \in S/_\rho$ *for* $1 \le j \le n-1\}$. *For* $(s_n,s_{n-1},\ldots,s_1) \in \hat{S}$, *define* $(s_n,s_{n-1},\ldots,s_1)\hat{\rho} = \mathrm{red}(s_n\rho,s_{n-1}\rho,\ldots,s_1\rho)$. *Then,* $\hat{\rho}$ *defies a homomorphism of* $\hat{S}$ *onto* $(\widehat{S/_\rho})$.

Proof. Combine [16, lemma 2.2] and [16, lemma 2.3]. Utilize the proof of [16, lemma 2.3] and [15, Theorem 3.3(a)].

We will utilize the notation of Lemma 1 without explicit mention. For brevity, we let $U = (\widehat{S/\rho})$

If $z = (x_n, x_{n-1},\ldots,x_1) \in \hat{X}$, where X is an arbitrary semigroup, we write $|z| = n$, the length of z.

Lemma 2 will be used in the proof of Lemmas 5, 13, 15, 19, and 30.

Lemma 2. *If* $z \in \hat{S}$, $|z| = |z\hat{\rho}|$.

Proof. Let $z = (a_n, a_{n-1},\ldots,a_1)$. Suppose $a_{k+1}\rho \not\mathcal{L} a_k\rho$ for some $1 \le k \le n-1$. Using Lemma 1, let $a_{k+1} = (g_{k+1},b_{k+1}) \in S_y$, say, and $a_k = (g_k,b_k) \in S_z$, say. Thus, $a_{k+1}\rho = g_{k+1} \in G_y$ and $a_k\rho = g_k \in G_z$ and, hence, $g_{k+1}\not\mathcal{L} g_k$. Since S/ρ is the semilattice Y of groups $(G_y\colon y \in Y)$ by lemma 1, $y = z$ and $a_{k+1}, a_k \in S_y$. Since $a_{k+1} \le a_k$, $a_{k+1} = sa_k$ for some $s \in S$. We may take $s \in S_y$. Hence, $a_{k+1} \mathcal{L} a_k$, a contradiction. So, $a_{k+1}\rho < a_k\rho$ for $1 \le k \le n - 1$. Hence, $\mathrm{red}(a_n\rho, a_{n-1}\rho, \ldots, a_1\rho) = (a_n\rho, a_{n-1}\rho, \ldots, a_1\rho)$ and $|z| = |z\rho|$. Lemma 3 will be used in the proof of Lemmas 5, 10, 14, 15, 19, 22, 23, and 30.

Lemma 3. Let X be an arbitrary semigroup. Then

3(a) (Tilson, [10, Lemma 6.7]; see also [11, Lemma 11.4]). $E(\widehat{X}) = \{(e, x_{n-1}, \ldots, x_1)\colon e \in E(X)\}$

3(b) (Tilson [10, Proposition 7.1]; see also [11, Proposition 12.1]). Let $a = (s_n, \ldots, s_1)$ and $b = (t_m, \ldots, t_1) \in \widehat{X}$. Then, $a \le b$ if and only if $m \le n$. $s_1 = t_1, \ldots, s_{m-1} = t_{m-1}$, and $s_m \mathcal{L} t_m$ in X.

3(c) (Tilson, [10, Corollary 7.2]; see also [11, Corollary 12.2]).

$a \mathcal{L} b$ in X if and only if $a = (s_n, \ldots, s_1)$, $b = (t_n, s_{n-1}, \ldots, s_1)$

and $s_n \mathcal{L} t_n$ in X.

3(d) (Tilson, [10, Corollary 7.3]; see also [11, Corollary 12.3]).

Let $a, b, c \in \widehat{X}$ and let $a \le b$ and $a \le c$. If $|b| \ge |c|$,

then $b \le c$. In fact if $|b| > |c|$, then $b < c$, and if $|b| = |c|$, then $b \mathcal{L} c$.

Lemma 4 will be used in the proof of Lemma 6.

Lemma 4. _For_ $t \in U$, _let_ $U_t = \{x \in U: tx = t\}$. _Then,_ $(t,s,t) \rightarrow s$ _defines an isomorphism of_ $R_\rho(t,t)$ _onto_ $U_t\rho^{-1}$.

Proof. Utilize [11, page 358].

Lemma 5 will be used in the proof of Lemmas 7, 16, and 18.

Lemma 5. _Let_ $s_1, s_2 \in U_t\rho^{-1}$. _If_ $|s_1| \geq |s_2|$, $s_1\rho \leq s_2\rho$. _If_ $|s_1| > |s_2|$, $s_1\rho < s_2\rho$. _If_ $|s_1| = |s_2|$, $s_1\rho \not< s_2\rho$.

Proof. Let $s_1, s_2 \in U_t\rho^{-1}$. Hence, $t(s_1\rho) = t$ and $t(s_2\rho) = t$. Thus, $t \leq s_1\rho$ and $t \leq s_2\rho$. Apply lemma 2 and Lemma 3(d).

Lemma 6 will be used in the proof of Lemmas 7, 11, 15, 18, 19, and 30.

Lemma 6. _For_ $t \in U$, $U_t\rho^{-1} \subseteq ES$. If $t \in U_y$, $U_t\rho^{-1} \subseteq U(E(X_z): z \geq y)$.

Proof. Using Lemma 1, $e\rho^{-1} \subseteq E(S)$ for all $e \in E(S/\rho)$. Thus, using [16, lemma 2.4(b)], $R_\rho(t,t)$ consists just of idempotents. Hence, using Lemma 4, $U_t\rho^{-1} \subseteq E(S)$. The last sentence of the lemma is a consequence of the definition of U_t, Lemma 1, and the first sentence of the lemma.

Lemma 7 or its proof will be used in the proof of Lemmas 9, 12, 18, and 23.

Lemma 7. _Let_ $s_1, s_2 \in U_t\rho^{-1}$ _where_ $t \in U$. _Then,_ $s_1\rho s_2\rho = s_1\rho$ _or_ $s_2\rho s_1\rho = s_2\rho$.

Proof. Using Lemma 5, $s_1\rho \leq s_2\rho$ or $s_2\rho \leq s_1\rho$. If $s_1\rho \leq s_2\rho$

there exists $x \in U$ such that $s_1\hat{\rho} = xs_2\hat{\rho}$. Using Lemma 6, $s_2\hat{\rho} \in E(U)$. Hence, $s_1\hat{\rho}s_2\hat{\rho} = xs_2\hat{\rho}s_2\hat{\rho} = xs_2\hat{\rho} = s_1\hat{\rho}$. Similarly, $s_2\hat{\rho} \leq s_1\hat{\rho}$ implies $s_2\hat{\rho}s_1\hat{\rho} = s_2\hat{\rho}$.

Lemma 8 will be used in the proof of Lemma 11.

Lemma 8. <u>For</u> $t \in U$ <u>and</u> $z \in Y$, $U_t\hat{\rho}^{-1} \cap E(X_z) = \square$ <u>or a rectangular band.</u>

Proof. Suppose $U_t\hat{\rho}^{-1} \cap E(X_z) \neq \square$. Let $a, b \in U_t\hat{\rho}^{-1} \cap E(X_z)$. Then, $t(ab)\hat{\rho} = (t(a\hat{\rho}))b\hat{\rho} = t(b\hat{\rho}) = t$. Hence $ab \in U_t\hat{\rho}^{-1} \cap E(X_z)$. Since $E(X_z)$ is a rectangular band, $aba = a$ and $bab = b$. Thus, $U_t\hat{\rho}^{-1} \cap E(X_z)$ is a rectangular band.

Lemma 9 will be used in the proof of Lemma 13.

Lemma 9. <u>For</u> $t \in U$ <u>and</u> $z \in Y$, $(U_t\hat{\rho}^{-1} \cap E(X_z))\hat{\rho} = \square$ <u>or a singleton.</u>

Proof. Suppose $(U_t\hat{\rho}^{-1} \cap E(X_z))\hat{\rho} \neq \square$. Let $s_1, s_2 \in U_t\hat{\rho}^{-1} \cap E(X_z)$. Thus, using Lemma 7, $s_1\hat{\rho}s_2\hat{\rho} = s_1\hat{\rho}$ or $s_2\hat{\rho}s_1\hat{\rho} = s_2\hat{\rho}$. However, using Lemma 1, $(U_t\hat{\rho}^{-1} \cap E(X_z))\hat{\rho}$ is contained in the right zero semigroup $E(U_z)$. Hence, $s_1\hat{\rho} = s_2\hat{\rho}$.

Lemma 10 will be used in the proof of Lemma 11.

Lemma 10. <u>For</u> $t \in U_y$, $U_t\hat{\rho}^{-1} \cap E(X_y) \neq \square$.

Proof. Let $t^{-1} \in \mathcal{I}(t)$ (hence, $t^{-1} \in U_y$ by Lemma 1). Using Lemma 3(a) and Lemma 1, there exists $s \in E(X_y)$ such that $s\hat{\rho} =$

$t^{-1}t$. Hence, $t(s\hat{\rho}) = t$, and, thus, $s \in U_t\hat{\rho}^{-1} \cap E(X_y)$. Select and fix $t \in U_y$. For $z \in Y$, let $W_z = U_t\hat{\rho}^{-1} \cap E(X_z)$ if $U_t\rho^{-1} \cap E(X_z) \neq \square$.

Lemma 11 will be used in the proof of Lemmas 18 and 20.

Lemma 11. $U_t\hat{\rho}^{-1}$ _is a semilattice_ $Y^* \subseteq Y$ _of rectangular bands_ $(W_z: z \in Y^*)$. Y^* _has a least element_ y.

Proof. Let $Y^* = \{z \in Y: U_t\hat{\rho}^{-1} \cap E(X_z) \neq \square\}$. Using Lemma 10, $y^* \neq \square$. Let $z_1, z_2 \in Y^*$ and let $a \in W_{z_1}$ and $b \in W_{z_2}$. Thus, $ab \in E(X_{z_1})E(X_{z_2}) \subseteq E(X_{z_1z_2})$. Since $ab \in U_t\hat{\rho}^{-1}$, $ab \in U_t\hat{\rho}^{-1} \cap E(X_{z_1z_2}) = W_{z_1z_2}$ and $z_1z_2 \in Y^*$. Using Lemma 8, $W_z (z \in Y^*)$ is a rectangular band. Using Lemma 10 and Lemma 6, y is the least element of Y^*.

Lemma 12 will be used in the proof of Lemma 17.

Lemma 12. Y^* _is a sub-chain of_ Y.

Proof. Let $z_1, z_2 \in Y^*$ and let $s_1 \in W_{z_1}$ and $s_2 \in W_{z_2}$. Using Lemma 7, $s_1\hat{\rho}s_2\hat{\rho} = s_1\hat{\rho}$ or $s_2\hat{\rho}s_1\hat{\rho} = s_2\hat{\rho}$. Since $s_1\hat{\rho} \in E(U_{z_1})$ and $s_2\hat{\rho} \in E(U_{z_2})$ by Lemma 1, $z_1z_2 = z_1$ or $z_2z_1 = z_2$. Thus, $z_1 \leq z_2$ or $z_2 \leq z_1$. Hence Y^* is a chain.

Lemma 13 will be used in the proof of Lemma 17.

Lemma 13. <u>For</u> $z \in Y^*$, <u>every element of</u> W_z <u>has the same length.</u>

Proof. Using Lemma 9 and the definition of W_z, $W_z\hat{\rho}$ is single element of length k, say. Let $a \in W_z$. Using Lemma 2, $|a| = |a\hat{\rho}| = k$.

Lemma 14 will be used in the proof of Lemmas 15 and 30.

Lemma 14. <u>For</u> $t^{-1} \in \mathcal{I}(t)$, $W_y = (t^{-1}t)\hat{\rho}^{-1}$.

Proof. Let $s \in (t^{-1}t)\hat{\rho}^{-1}$. Thus, using Lemma 1 and Lemma 3(a), $s \in E(X_y)$. However, $t(s\hat{\rho}) = t(t^{-1}t) = t$. Thus, $s \in U_t\hat{\rho}^{-1} \cap E(X_y) = W_y$. If $s \in W_y$, then $t(s\hat{\rho}) = t$. Hence, $t^{-1}t(s\hat{\rho}) = t^{-1}t$. However, $t^{-1}t$, $(s\hat{\rho} \in E(U_y)$, a right zero semigroup. Thus, $s\hat{\rho} = t^{-1}t$ and $s \in (t^{-1}t)\hat{\rho}^{-1}$.

Lemma 15 will be used in the proof of Lemmas 17 and 18.

Lemma 15. <u>Let</u> $|t| = k$. <u>Then,</u> $a \in U_t\hat{\rho}^{-1}$ <u>implies</u> $|a| \leq k$. <u>If</u> $a \in U_t\hat{\rho}^{-1}$, $|a| = k$ <u>if and only if</u> $a \in W_y$.

aProof. Let $a \in U_t\hat{\rho}^{-1}$. Hence, $t = t(a\hat{\rho})$ and, thus, $t \leq a\hat{\rho}$. Hence, using Lemma 3(b), $|t| \geq |a\hat{\rho}|$. Thus, since $|a\hat{\rho}| = |a|$ by Lemma 2, $|a| \leq |t| = k$. Suppose $|a| = k$. Then, $|a\hat{\rho}| = k = |t|$. Thus, using Lemma 3(b), $a\hat{\rho} = (s_k, t_{k-1}, \ldots, t_1)$ and $t = (t_k, t_{k-1}, \ldots, t_1)$ where $\{s_k, t_k, t_{k-1}, \ldots, t_1\} \subseteq S/\rho$ and $s_k \not\leq t_k$. Hence, using Lemma 3(c), $a\hat{\rho} \not\leq t$. Thus, since $t \in U_y$, $a\hat{\rho} \in$

U_y. Hence, using Lemma 1 and Lemma 6, $a \in E(X_y)$, and, thus, $a \in W_y$. Conversely, if $a \in W_y$, $a\hat{\rho} = t^{-1}t(t^{-1} \in \mathcal{F}(t))$ by Lemma 14. Using Lemma 3(c) $|t^{-1}t| = |t|$. Hence, using Lemma 2, $|a| = |a\hat{\rho}| = |t^{-1}t| = |t| = k$.

Lemma 16 will be used in the proof of Lemma 17.

Lemma 16. _Let_ $a \in W_u$ _and_ $b \in W_v$ _and suppose_ $u \neq v$. _Then,_ $|a| \neq |b|$.

Proof. Suppose $|a| = |b|$. Then, using Lemma 5, $a\hat{\rho} \mathcal{L} b\hat{\rho}$. Thus, since $a\hat{\rho} \in U_u$ and $b\hat{\rho}$ U_v, by Lemma 1, $u = v$. Hence, we have a contradiction. Thus, $|a| \neq |b|$. For $k \in N\backslash\{0\}$, let $P_k = \{n: n \in N\backslash\{0\}$ and $1 \leq n \leq k\}$ under the reverse of the usual order.

Lemma 17 and its proof will be used in proof of Lemma 18.

Lemma 17. Y^* _is a finite chain with card_ $Y^* \leq k$.

Proof. Let $z \in Y^*$. Then, every element of W_z has the same length by Lemma 13. Let us call this length $|z|$. By Lemma 15, $|z| \leq k$. For $z \in Y^*$, let $z\gamma = |z|$. We will show that γ defines a one-to-one mapping of Y^* into P_k. By the comments above, γ is a well defined mapping of Y^* into P_k. Suppose $z_1 \neq z_2$. Then, $z_1\gamma \neq z_2\gamma$ by Lemma 16 and, hence, γ is a one-to-one map. Hence, Y^* is finite and card $Y^* \leq k$. Apply Lemma 12.

Lemma 18 and its proof will be used in the proof of Lemma 19.

Lemma 18. Y^* _is a sub-chain of the chain_ P_k _and_ k _is the least element fo_ Y^*.

Proof. We will use the notation of the proof of Lemma 17. Let $z_1, z_2 \in Y^*$ with $z_1 \geq z_2$ (in Y^*). We will first show $|z_1| \leq |z_2|$. Let $|z_1| = \ell$ and $|z_2| = s$. Let $a = (x_\ell, x_{\ell-1}, \ldots, x_1) \in W_{z_1}$ and $b = (y_s, y_{s-1}, \ldots, y_1) \in W_{z_2}$. Hence, $ba \in W_{z_2}$ and, thus, $|ba| = s$. However, $ba = (y_s, y_{s-1}, \ldots, y_1)(x_\ell, x_{\ell-1}, \ldots, x_1) = (\text{red } (y_s x_\ell, y_{s-1} x_\ell, \ldots, y_1 x_\ell, x_\ell), x_{\ell-1}, \ldots, x_1)$. Thus, since $|ba| = s, s \geq \ell$ or $|z_2| \geq |z_1|$. Conversely, suppose $|z_1| \leq |z_2|$. Thus, using Lemma 5, Lemma 6, and the proof of Lemma 7, $b\hat{\rho} a\hat{\rho} = b\hat{\rho}$. Hence, $z_2 \leq z_1$. Thus, using the proof of Lemma 17, γ defines an order preserving, one-to-one map of Y^* into P_k. Thus, using Lemma 17, we may identify Y^* with a subchain of P_k. Using Lemma 11, y is the least element of Y^*. Using Lemma 15, $y\gamma = |y| = k$. Hence, k is the least element of Y^*.

Lemma 19 will be used in the proof of Lemma 20.

Lemma 19. Y^* <u>is the chain</u> P_k.

Proof. Let $t^{-1} \in \mathcal{I}(t)$. Then, using Theorem 3(a) and 3(c) and Theorem 1, $t^{-1}t = (e_k, g_{k-1}, \ldots, g_1)$ where $e_k = g_k \in E(G_y)$ and $g_j \in S/\rho$ for $1 \leq j \leq k-1$. Let $j \in P_k$ and let $a = (e_j, g_{j-1}, \ldots, g_1) \in U$ where $e_j \in E(S/\rho)$ and $e_j \mathcal{H} g_j$ (in S/ρ). If $1 \leq j \leq s \leq k$, $g_s \leq g_j \leq e_j$. Thus $g_s e_j = g_s$. Hence, $t^{-1}ta = (e_k, g_{k-1}, \ldots, g_1) \cdot (e_j, g_{j-1}, \ldots, g_1) = (\text{red}(e_k, e_j, g_{k-1}$

$e_j, \ldots, g_j e_j, g_{j-1} e_j, \ldots, g_1 e_j, e_j)$, $g_{j-1}, \ldots, g_1)$ = $(e_k, g_{k-1}, \ldots, g_j, g_{j-1}, \ldots, g_1) = t^{-1}t$. Thus, $ta = t$. Let $b = ((e_j, c_j), (g_{j-1}, c_{j-1}), \ldots, (g_1, c_1)) \in S$. Thus, using Lemma 1 and Lemma 2, $b\hat{\rho} = a$ and, hence, $t(b\hat{\rho}) = t$. Thus, using Lemma 6, $b \in U_t\hat{\rho}^{-1} \cap E(X_z)$ for some $y \leq z$. Hence, $b \in W_z$. Since $|b| = j$, $z\gamma = j$ (notation of proof of Lemmas 17 and 18). Thus, $P_k \subseteq Y^*\gamma$. Hence, using Lemma 18 and its proof, we may identify Y^* and P_k.

Lemma 20 will be use in the proof of Lemmas 21, 24, 25, and 30.

Lemma 20. $U_t\hat{\rho}^{-1}$ <u>is a chain</u> $P_{|t|}$ <u>of rectangular bands</u> $(W_j : j \in P_{|t|})$.

Proof. Utilize Lemma 11 and Lemma 19.

Lemma 21, Lemma 22 and Lemma 23 will be used to show $\mathcal{L}$ is a congruence relation on $U_t\hat{\rho}^{-1}$ (Lemma 24).

Let $t \in U_y$ and suppose $|t| = k$. If $x, y \in U_t\hat{\rho}^{-1}$, define $x\sigma y$ if and only if $ax = ay$ for all $a \in W_k$.

Lemma 21 will be used in the proof of Lemma 24.

Lemma 21. σ <u>is a congruence relation on</u> $U_t\hat{\rho}^{-1}$.

Proof. It is easily checked that σ is a right congruence relation. Suppose $x\sigma y$ and $z \in U_t\hat{\rho}^{-1}$ and $a \in W_k$. Since W_k is an ideal of $U_t\hat{\rho}^{-1}$ by Lemma 20, $a(zx) = (az)x = (az)y =$

$a(zy)$. Thus, σ is a congruence relation.

Lemma 22 will be used in the proof of Lemmas 23 and 30.

Lemma 22. Let S be an arbitrary semigroup and let $a,b,c \in \hat{S}$. If $ab = ac$, then $b \le c$ or $c \le b$.

Proof. Let $a = (a_n,\ldots,a_1)$, $b = (b_r,\ldots,b_1)$, and $c = (c_k,\ldots,c_1)$. Thus,

$$ab = (a_n,\ldots,a_1)(b_r,\ldots,b_1) = (\text{red}(a_nb_r,\ldots,a_1b_r,b_r),b_{r-1},\ldots,b_1)$$

and

$$ac = (a_n,\ldots,a_1)(c_k,\ldots,c_1) = (\text{red}(a_nc_k,\ldots,a_1c_k,c_k), c_{k-1},\ldots,c_1).$$

If $r \ge k$, $c_1 = b_1,\ldots,c_{k-1} = b_{k-1}$ and $c_k \mathcal{L} b_k$. Hence, $b \le c$ by Lemma 3(b). Similarly, if $k \ge r$, $c \le b$.

Lemma 23 will be used in the proof of Lemmas 24 and 30.

Lemma 23. Let $x, y \in W_j$. Then, $x\sigma y$ if and only if $x\mathcal{L}y$.

Proof. First, let $x,y \in W_j$ and assume that $x\sigma y$. Using Lemma 22, $x \le y$ or $y \le x$. Thus, as in the proof of Lemma 7, $xy = x$ or $yx = y$. In either case, $x\mathcal{L}y$. Conversely, assume that $x\mathcal{L}y$. Hence, using Lemma 3(a) and Lemma 3(c),

$$x = ((e_j,(v_j,b_j)), (g_{j-1},(a_{j-1},b_{j-1})),\ldots,(g_1,(a_1,b_1)))$$

$$y = ((e_j,(z_j,b_j)),(g_{j-1},(a_{j-1},b_{j-1})),\ldots,(g_1,(a_1,b_1))).$$

Let $a = ((e_k,(c_k,d_k)),(g_{k-1},(c_{k-1},d_{k-1})),\ldots,(g_1,(c_1,d_1))) \in W_k$

We note that

$$(g_j,(c_j,d_j))(e_j,(v_j,b_j)) = (g_j,(c_j,b_j)) = (g_j,(c_j,d_j))(e_j,(z_j,b_j)).$$

Suppose $j \le r < k$. Then, $(g_r,(c_r,d_r)) \le (g_j\cdot(c_j,d_j))$. Hence, there exists $p \in S$ such that $(g_r,(c_r,d_r)) = p(g_j,(c_j,d_j))$.

Thus,

$$(g_r,(c_r,d_r))(e_j,(v_j,b_j)) = (g_r,(c_r,d_r))(e_j,(z_j,b_j)).$$

Furthermore,

$$(e_k,(c_k,d_k)) \le (g_j,(c_j,d_j)).$$

Hence, there exists $q \in S$ suchs that

$$(e_k,(c_k,d_k)) = q(g_j,(c_j,d_j)).$$

Thus,

$$(e_k,(c_k,d_k))(e_j,(v_j,b_j)) = (e_k,(c_k,d_k))(e_j,(z_j,b_j))$$

Hence,

$$\begin{aligned} ax &= \\ &(\mathrm{red}((e_k,(c_k,d_k))(e_j,(v_j,b_j)),(g_{k-1},(c_{k-1},d_{k-1}))(e_j,(v_j,b_j)), \\ &\ldots,(g_j,(c_j,d_j))(e_j,(v_j,b_j)),\ldots,(g_1,(c_1,d_1))(e_j,(v_j,b_j)),(e_j,(v_j,b_j)), \\ &(g_{j-1},(a_{j-1},b_{j-1})),\ldots,(g_1,(a_1,b_1))) \\ &= \\ &((e_k,(c_k,d_k))(e_j,(v_j,b_j)),(g_{k-1},(c_{k-1},d_{k-1}))(e_j,(v_j,b_j)),\ldots, \\ &\quad (g_j,(c_j,b_j)),(g_{j-1},(a_{j-1},b_{j-1})),\ldots,(g_1,(a_1,b_1))) \\ &= ay. \end{aligned}$$

Hence, $x\sigma y$.

Lemma 24 will be used in the proof of Lemmas 25 and 30.

Lemma 24. $\mathfrak{L}$ <u>is a congruence relation on</u> $U_t\hat{\rho}^{-1}$. <u>Hence,</u> $U_t\hat{\rho}^{-1}/\mathfrak{L}$ <u>is a chain</u> $P_{|t|}$ <u>of right zero semigroups</u> $(W_j/\mathfrak{L}\colon j \in P_{|t|})$.

Proof. Let $x, y \in U_t\rho^{-1}$ and suppose $x \not\ell y$. Thus, $x, y \in W_j$, say. Hence, using Lemma 23, $x \sigma y$. Let $z \in W_\ell$ say. Thus, using Lemma 21, $zx \sigma zy$. However, using Lemma 20, $zx, zy \in W_{j\ell}$. Hence, using Lemma 23, $zx \not\ell zy$. The last sentence of the lemma is a consequence of the first sentence and Lemma 20.

Lemma 25 will justify our definition of D_δ^λ.

Lemma 25. $\sim$ <u>is a congruence relation on the category</u> R_δ.

Proof. It is straightforward to show that $\sim$ is a right congruence relation on R_δ. We show $\sim$ is also a left congruence relation on R_δ. Let (t_1, s_1, t_2), $(t_1, s_2, t_2) \in R_\delta(t_1, t_2)$ with $(t_1, s_1, t_2) \sim (t_1, s_2, t_2)$ and suppose $y = (t_3, x, t_1) \in R_\delta(t_3, t_1)$. Then, $y(t_1, s_1, t_2) = (t_3, xs_1, t_2)$ and $y(t_1, s_2, t_2) = (t_3, xs_2, t_2)$. Let $v \in t_3\delta^{-1}$. Thus, $(vx)\delta = v\delta x\delta = t_3(x\delta) = t_1$. Hence $vx \in t_1\delta^{-1}$. So, $(vx)s_1 = (vx)s_2$. Hence, $v(xs_1) = v(xs_2)$ for all $v \in t_3\delta^{-1}$. Furthermore, $xs_1 \lambda xs_2$. So, $y(t_1, s_1, t_2) \sim y(t_1, s_2, t_2)$. Thus $\sim$ is a congruence relation R_δ.

Let $[t_1, s_1, t_2] \in D_\delta^\lambda(t_1, t_2)$, denote the $\sim$-class of R_δ containing $(t_1, s_1, t_2) \in R_\delta(t_1, t_2)$.

We need the definition of division for categories.

Let S and T be categories. We say S divides T (or T covers S) and write $S < T$ if there exists

(1) An object function $f:\text{obj}(S) \longrightarrow \text{obj}(T)$.

(2) For every pair of hom-sets $S(c,c')$ and $T(cf, c'f)$, there exists a hom-set relation

$$f:S(c,c') \longrightarrow T(cf,c'f)$$

such that

2(a) f is fully defined (i.e., $xf \neq \square$ for all $x \in S(c,c')$).

2(b) f is injective (i.e., $x \neq x'$ implies $xf \cap x'f = \square$).

2(c) For every pair of consecutive arrows s,s' of S, $sfs'f \subseteq (ss')f$.

2(d) $1_{cf} \in 1_c f$ for each object $c \in \text{obj}(S)$.

Note that if S and T are monoids (one object categories) the two definitions agree.

We will need the direct part of a special case of a slight modificatioon of Tilson's Derived Category Theorem [12].

Theorem 26 (Tilson, [12]). <u>Let</u> ζ <u>be a homomorphism of a monoid</u> S <u>onto a monoid</u> T <u>and let</u> V <u>be a monoid satisfying</u> $D_\zeta^\lambda < V$. <u>Then, there exists a division of monoids</u>

$$\theta: S < V\circ T$$

<u>Satisfying</u> $\theta\P = \zeta$ ($\P$ <u>denotes the projection of</u> $V\circ T$ <u>onto</u> T).

Proof (Sketch). Let V^T denote the set of functions of T into V and let $\psi:D^\lambda\zeta<V$ be the given division. For $s \in S$, define $F_s = \{f \in V^T: t_1 f \in [t_1,s,t_1(s\zeta)]\psi,\ t_1 \in \text{obj } D_\zeta^\lambda\}$. Define the relation $s\theta = \{(f,s\zeta):f \in F_s\}$ and show θ is a division of monoids (Thus just specializes Tilson's proof to the case ζ is a monoid epimorhism.)

It will be convenient to have the notion of local variety of monoids due to Tilson [12].

Let $\underset{\sim}{M}$ be a variety of monoids. Let Cat denote the collection of all categories and let $\ell\underset{\sim}{M} = \{S \in \text{Cat}: S(c,c) \in \underset{\sim}{M}$ for all $c \in \text{obj } S\}$. A category A belongs to $\underset{\sim}{M}_{\underset{\sim}{C}}$ if and only if $A < B$ for some $B \in \underset{\sim}{M}$. $\underset{\sim}{M}$ is termed a local variety of monoids if $\ell\underset{\sim}{M} = \underset{\sim}{M}_{\underset{\sim}{C}}$ (it is easily seen that $\underset{\sim}{M}_{\underset{\sim}{C}} \subseteq \ell\underset{\sim}{M}$). Cat, $\ell\underset{\sim}{M}$, and $\underset{\sim}{M}_{\underset{\sim}{C}}$ (notation of [12]) are varieties of categories in the sense of Tilson [12]. We show $\underset{\sim}{L_1}$ is a local variety of monoids (Theorem 29). We then show $D_{\hat{\rho}}^{g} \in \ell\underset{\sim}{L_1}$ (Lemma 30) (note, by defining $1\hat{\rho} = 1$, we may extend $\hat{\rho}$ to a homomorphism of $(\hat{S})^1$ onto $(\widehat{S/\rho})^1$). Hence, $D_{\hat{\rho}}^{g}\ell < V$ for some $V \in \underset{\sim}{L_1}$. Thus, $(\hat{S})^1 < Vo(\widehat{S/\rho})^1$ by Theorem 26.

To show that $\underset{\sim}{L_1}$ is a local variety, we need an algorithm describing the free $\underset{\sim}{L_1}$ monoid (Theorem 27) and a generalization of Simon's Theorem [4, p. 224, Theorem 7.1] from the variety of commutative and idempotent monoids to $\underset{\sim}{L_1}$ (Theorem 28). We will give the proof of Theorem 27 and Theorem 28 in the appendix.

We need the following terminology to state Theorem 27. If S is a set, $|S|$ denotes the cardinality of S. Let A^* denote the free monoid over an alphabet A. Following Howie [6, p. 106], if $\omega \in A^*$, the content $C(\omega)$ of ω is the set of elements of A appearing in ω. If $x \in A^*$ and $|C(x)| = n \geq 1$, $x(1)$ denotes the letter in x that is first to make its last appearance and $x(1)$ (if $n > 1$) is the subwrod of x

following the last appearance of $x(1)$. If $|C(x)| = 1$, define $x(1) = x(1) = 1$.
If $|C(x)| = 0$ (i.e., $x = 1$), define $1 = \bar{x}(1) = x(1)$. Let us define $x(1)^2 = \overline{x(1)}(1)$ and $x(1^2) = x(1)(1), \ldots, x(1^j) = \overline{x(1^{j-1})}(1)$ and $x(1^j) = x(1^{j-1})(1)$. If $|C(x)| = k$, then $x(1^j) = 1$ for all $j > k$ and $x(1^j) = 1$ for $j \geq k$. Let β denote the smallest congruence relation on A^* containing $L_1 = \{(xyx, yx) : x, y \in A^*\}$.

Theorem 27 (Cf. Howie [6, Theorem 4.7, p. 107]. *Suppose* x *and* y *are elements of* A^*. *Then,* $(x,y) \in \beta$ *if and only if* $C(x) = C(y) = \{a_1, a_2, \ldots, a_k\}$, *say, and* $x(1^j) = y(1^j)$ *for* $1 \leq j \leq k$.

To state Theorem 28, we will need the following terminology. Let X be a graph and let E denote the set of edges of X. Let X^* denote the free category over X and let E^* denote the free monoid on E. Every path in X may be regarded as a word in E^*. A faithful morphism ρ_X of X^* into E^* may be defined as follows. Let $p\rho_X = x_1x_2\ldots x_n$ if $p = x_1x_2\ldots x_n$ and $p\rho_X = 1$ if p is empty.

Theorem 28. *Let* $\sim$ *be the smallest congruence relatioon on* X^* *satisfying*

$$* \quad xyx \sim yx$$

for any two loops about the same vertex. Then, for any two coterminal paths

$$\xrightarrow{x}$$

$$\xrightarrow[y]{u \quad v}$$

<u>the condition</u> $x \sim y$ <u>and</u> $C(x\rho_X) = C(y\rho_X) = \{a_1, a_2, \ldots, a_k\}$, <u>say, and</u> $x\pi_X(1^j) = y\pi_X(1^j)$ <u>for</u> $1 \le j \le k$ <u>are equivalent.</u>

We are now in a position to prove Theorem 29.

Theorem 29. $\underset{\sim}{L}_1$ <u>is a local variety.</u>

Proof. Let $U \in \ell\underset{\sim}{L}_1$ and let X denote the subgraph obtained from U by removing identity arrows. Let X^* denote the free category over X. Let θ denote the identity mapping from obj (X^*) onto obj (U) and, for $p \in X^*(c,c')$, define $p\theta = x_1x_2\ldots x_n$ where $p = x_1x_2\ldots x_n$ and $e_c\theta = 1_c$ (where e_c is the empty path about $c \in \text{obj}(X^*)$ and 1_c is the identity arrow from $c \in \text{obj}(U)$). Then, θ is a quotient morphism of X^* onto U. So, $U \le X/\theta$ (i.e., there exists a faithful morphism of U into X^*/θ) (in fact, $U \cong X^*/\theta$). Next, we specialize a construction of Tilson [12] to $\underset{\sim}{L}_1$. Let E denote the set of edges of X and let E^* denote the free monoid on E. Let $E^*/\underset{\sim}{L}_1$ be the free monoid over E relative to $\underset{\sim}{L}_1$ ($\underset{\sim}{L}_1$ is defined by the equation $xyx = yx$) and let $\beta: E^* \longrightarrow E^*/\underset{\sim}{L}_1$ be the associated quotient morphism. So, $E^*/\underset{\sim}{L}_1 = E^*/\beta$. Thus, $\rho_X\beta$ is a morphism of X^* into $E^*/\underset{\sim}{L}_1$. Let $[\underset{\sim}{L}_1]$ (notation of Tilson [12]) denote the congruence on X^* induced by $\rho_X\beta$. Hence,

$$X^*/[\underset{\sim}{L_1}] \leq E^*/\underset{\sim}{L_1} \in \underset{\sim}{L_1}$$

Using Theorem 27 and Theorem 28, $[\underset{\sim}{L_1}]$ is the smallest congruence relation on X^* such that $xyx\ [\underset{\sim}{L_1}]\ yx$ for any two loops about the same vertex. Thus, $[\underset{\sim}{L_1}] \leq \theta$. Hence,

$$U \leq X^*/\theta < X^*/[\underset{\sim}{L_1}] \leq E^*/\underset{\sim}{L_1} \in \underset{\sim}{L_1}.$$

Thus, $U \in (\underset{\sim}{L_1})C$. Hence, $\underset{\sim}{L_1}$ is a local variety. q.e.d.

We next show $D_{\hat{\rho}}^{\mathcal{J}} \in \underset{\sim}{L_1}$.

LEMMA 30. <u>For</u> $t \in \widehat{(S/\rho)}$, $[t,s,t]\gamma = s\mathcal{L}$ <u>defines an isomorphism of</u>

$D_{\hat{\rho}}^{\mathcal{J}}(t,t)$ <u>onto</u> $(U_t\hat{\rho}^{-1}/\mathcal{L})^1$. <u>Thus</u> $D_{\hat{\rho}}^{\mathcal{J}} \in \ell\underset{\sim}{L_1}$.

Proof. First assume $[t,s,t] = [t,z,t]$. Hence, $s\mathcal{J}z$ and $xs = xz$ for all $x \in t\hat{\rho}^{-1}$. Using Lemma 22, $s \leq z$ or $z \leq s$. Using Lemma 6, $sz = s$ or $zs = z$. Since $s, z \in W_j$ for some j, $s\mathcal{L}z$ in either case. Conversely, suppose $s\mathcal{L}z$ $(s,z \in U_t\hat{\rho}^{-1})$. Hence, using Lemma 20, $s,z \in W_j$ for some $j \in P_{|t|}$. Thus, using Lemma 23, $s\sigma z$. Hence, $xs = xz$ for all $x \in W_{|t|}$. Using Lemma 14, $W_{|t|} = (t^{-1}t)\hat{\rho}^{-1}$ for $t^{-1} \in \mathcal{J}(t)$. Suppose $|t| = k$. Thus, using Lemma 3(c) and Lemma 3(a), $t = (g_k,g_{k-1},\ldots,g_1)$ and $t^{-1}t = (e_k,g_{k-1},\ldots,g_1)$ where $e_k^2 = e_k\mathcal{R}g_k$. Let $u \in t\hat{\rho}^{-1}$. Hence, using Lemma 2,

$$u = ((g_k,(i_k,j_k)),\ (g_{k-1},(i_{k-1},j_{k-1})),\ldots,(g_1,(i_1,j_1)),\ \text{say}$$

and

$x = ((e_k,(i_k,j_k)),\ (g_{k-1},(i_{k-1},j_{k-1})),\ldots,(g_1(i_1,j_1))) \in W_{|t|}$.

Since

$(g_k,(i_k,j_k))x = u, \quad us = uz.$

Hence, $[t,s,t] = [t,z,t]$. Thus, $[t,s,t]\gamma = s\mathcal{L} \quad (s \in U_t\hat{\rho}^{-1})$ defines a 1-1 map of $\mathcal{D}_{\hat{\rho}}^{\mathcal{Y}}(t,t)$ into $(U_t\hat{\rho}^{-1}/\mathcal{L})^1$. Clearly, γ is a surjection. Using Lemma 24, γ is an isomorphism.

We are now in a position to establish the main result of the paper.

Theorem 31. Let S be an orthodox union of groups. Then,

$$(1) \quad (S)^1 < Vo(\widehat{S/\rho})^1$$

where V is an $\mathcal{L}$-trivial and idempotent monoid, ρ is the smallest inverse semigroup congruence on S, and $(\widehat{S/\rho})$ is a semilattice $S/\mathcal{Y}$ of right groups. For each $t \in \widehat{S/\rho}$, $(V_t)^1 < V$ where V_t is a chain $1 > 2 > \ldots > |t|$ of right zero semigroups. Furthermore,

$$(2) \quad (\widehat{S/\rho}) \le ((\widehat{S/\rho})/\delta)^1 \overset{r}{o} (E((\widehat{S/\rho})))^1$$

where δ is the smallest inverse semigroup congruence on $(\widehat{S/\rho})$. $(\widehat{S/\rho})/\delta$ is a semilattice $S/\mathcal{Y}$ of groups $(G_y\colon\ y \in S/\mathcal{Y})$ where G_y is a maximal subgroup of S. $E((\widehat{S/\rho}))$ is a semilattice $S/\mathcal{Y}$ of right zero semigroups.

Proof. Utilizing Lemma 30 and Theorem 29, $D_{\hat{\rho}}^{\mathcal{Y}} < V$ for some $V \in \underset{\sim}{L_1}$. Thus, utilizing [16, Theorem 2.6], Lemma 1, and Theorem 26, (1) is valid. The existence of $(V_t)^1$ is a consequence of Lemma

24, Lemma 30, and the fact $D_{\mathscr{P}}^{\mathscr{J}} < V$. Utilizing [16, Theorem 2.6], (2) is valid.

REMARK. Let $S \xleftarrow{\eta} \hat{S} \overset{\theta}{<} Vo(\widehat{S/\rho})$ where $(a_n, a_{n-1}, \ldots, a_1)\eta = a_n$. Then $x, y \in (e^2 = e)\,\eta^{-1}\,\theta^{-1}$ implies $x^3 = x^2$ and $xy\,xy = x^2y$.

Next, we give our structure theorem for bands.

Corollary 32. <u>Let</u> S <u>be a band. Then,</u>

$$S^1 < Vo(\widehat{S/\mathscr{J}})^1$$

<u>where</u> V <u>is an</u> $\mathscr{L}$-<u>trivial and idempotent monoid and</u> $\widehat{S/\mathscr{J}}$ <u>is a semilattice</u> $S/\mathscr{J}$ <u>of right zero semigorups. For each</u> $t \in \widehat{S/\mathscr{J}}$, $(V_t)^1 < V$ <u>where</u> V_t <u>is a chain</u> $1 > 2 \ldots > |t|$ <u>of right zero semigroups</u>.

Appendix

In this appendix, we will prove Theorems 27 and 28. To prove Theorem 27, we will need the following Lemma.

Lemma 27′ (Cf. Howie [6, Lemma 4.6P. 106]. <u>If</u> $x, y \in A^*$, <u>then</u> $(x, y) \in \beta$ <u>if and only if</u> (1)$\bar{x}(1) = \bar{y}(1)$ <u>and</u> (2) $(x(1), y(1)) \in \beta$.

Proof. Suppose (1) and (2) are valid. If $|C(x)| = 0$, then, trivially $x\beta = \beta y$. So, we assume $|C(x)| > 0$. Since $x = u\bar{x}(1)x(1)$ for some $u \in A^*$, $x\bar{x}(1)x(1) = u\ \bar{x}(1)x(1)\bar{x}(1)x(1)\ \beta u\ \bar{x}(1)x(1) = x$. Since $\mathcal{L}$ is a congruence relation on A^*/β and $A^*/\beta/\mathcal{L}$ is a semillatice, $C(x) = C(\bar{x}(1)x(1))$ implies $(\bar{x}(1)x(1))\beta\mathcal{L}x\beta$ (cf Howie [6, p. 105]). Thus, $x\bar{x}(1)x(1)\beta\bar{x}(1)x(1)$. Hence, $x\beta\bar{x}(1)x(1)$. Similarly, $y\beta\bar{y}(1)y(1)$. Thus, $(x,y) \in \beta$. Conversely, if $(x,y) \in \beta$, then $x = y$ or x and y are connected by a finite sequence of elementary L_1-transitions (terminology of Howie [6]) of the types $a = p\ uvuq \longrightarrow b = pvuq$ or $a = pvuq \longrightarrow b = puvuq$ $(p,q \in A^*)$. Clearly, $\bar{a}(1) = \bar{b}(1)$. Hence, $\bar{x}(1) = \bar{y}(1)$. If the last appearance of $\bar{a}(1)$ is in vuq, then $a(1) = b(1)$. If the last appearance of $\bar{a}(1)$ is in p, then $a(1) \longrightarrow b(1)$ is an elementary L_1-transition.

Thus, in both cases, $(a(1), b(1)) \in \beta$. Hence, $(x(1), y(1)) \in \beta$. q.e.d.

Remark: (1) and (2) of Lemma 27′ imply $C(x) = C(y)$.

<u>Proof of Theorem</u> 27. First, we assume $\bar{x}(1^j) = \bar{y}(1^j)$ for $1 \le j \le k$ and $C(x) = C(y) = \{a_1, a_2, \ldots, a_k\}$. We shall then show that

$(x(1),y(1))) \in \beta$. Then, $(x,y) \in \beta$ by Lemma 27′. We first note that $x(1^k) = y(1^k) = 1$ and $\bar{x}(1^k) = \bar{y}(1^k)$. Using the proof of Lemma 27′, $x(1^{k-1})\beta\overline{x(1^{k-1})}(1)x(1^{k-1})(1) = \bar{x}(1^k)x(1^k) = \bar{y}(1^k)v(1^k)\beta y(1^{k-1})$. Proceeding inductively, if $\bar{x}(1^j)$ $x(1^j)\beta y(1^j)$. then $x(1^{j-1})\beta\bar{x}(1^j)\beta\bar{y}(1^j)y(1^j)$ $\beta y(1^{j-1})$. Finally, $x(1^2)\beta y(1^2)$ implies $x(1)\beta y(1)$. Thus, $(x,y) \in \beta$ by Lemma 27′.
Conversely, assume $(x,y) \in \beta$ and $C(x) = C(y) = (a_1,a_2,\ldots,a_k)$. By Lemma 27′, $\bar{x}(1) = \bar{y}(1)$ and $(x(1),y(1)) \in \beta$. Assume $\bar{x}(1^{j-1}) = \bar{y}(1^{j-1})$ and $(x(1^{j-1}), y(1^{j-1})) \in \beta$. Hence, using Lemma 27′, $\bar{x}(1^j) = \overline{x(1^{j-1})}(1) = \overline{y(1^{j-1})}(1) = \bar{y}(1^j)$ and $(x(1^j),y(1^j)) \in \beta$. Finally, $\bar{x}(1^{k-1}) = \bar{y}(1^{k-1})$ and $(x(1^{k-1}), y(1^{k-1})) \in \beta$. Thus, $\bar{x}(1^k)= \bar{y}(1^k)$. Hence, $\bar{x}(1^j) = \bar{y}(1^j)$ for $1 \leq j \leq k$.
Remark $1\beta = 1$, i.e., 1 is the only element in the β-class containing 1.

<u>Proof of Theorem</u> 28. We first assume that $u \underset{y}{\overset{x}{\rightrightarrows}} v$ are two coterminal paths such that $C(x\rho_X) = C(y\rho_X)$ and $\overline{x\rho_X}(1^j) = \overline{y\rho_X}(1^j)$ for $1 \leq j \leq k$. We show $x \sim y$. We first show that the left-right dual of Eilenberg [4, Lemma 7.2, p. 225] and the left-right dual of Eilenberg [4, Lemma 7.3, p. 226] are valid with the assumption * (one does not need the stronger assumptioon $xx \sim x$ and $xy \sim yx$). We show that if we are given paths

$$v \xrightarrow{x} w \xrightarrow{y} t$$

such that $C(x) \subseteq C(y)$ (the set of elements of X appearing in y), there exists a factorization

$$w \xrightarrow{y_0} v \xrightarrow{y_1} t$$

of y such that

$$y \sim y_0 x y$$

(Cf. Eilenberg [4, Lemma 7.2, p. 224]).

We call this lemma Result A. To prove Result A, we will use induction on $\|x\|$, the length of x. Suppose $\|x\| = 0$. Then, $v = w$ and $x = 1_w$ (the identity path about w). So

$$w \xrightarrow{1w} w \xrightarrow{y} t$$

$$w \xrightarrow{1w} w \xrightarrow{y} t, \quad y = 1_w y$$

$$y_0 = 1_w \quad y_1 = y$$

and $y \sim 1_w 1_w y$.

Next, suppose $x = ez$ where e is a path of length 1.

Thus, $v \xrightarrow{e} u \xrightarrow{z} w$, say.

Hence,

$$u \xrightarrow{z} w \xrightarrow{y} t$$

Thus, by the inductive hypothesis, there exists a factorization

$$w \xrightarrow{y_0} u \xrightarrow{y_1} t$$

of $y(y = y_0 y_1)$ such that

$$y \sim y_0 z y$$

Since $e \in C(x) \leq C(y)$, $y = y_2\ e\ y_3$, say. Recall

$$v \xrightarrow{e} u \xrightarrow{z} w \xrightarrow{y} t$$

Furthermore,

$$w \xrightarrow{y_2} v \xrightarrow{e} u \xrightarrow{y_3} t$$

with $y = y_2(ey_3)$.

We complete the proof of Result A by showing

$y \sim y_2xy = y_2ezy.$

We have

$y \sim y_0zy = y_0zy_2ey_3$ ($u \xrightarrow{z} w \xrightarrow{y_2} v \xrightarrow{e} u$ and hence zy_2e is a loop about the vertex u. So we may apply *.)

$\sim y_0zy_2ezy_2ey_3$

$= (y_0z)(y_2ez)y$

$\sim (y_0z)(y_2ez)(y_0zy)$ (since $y \sim y_0zy$)

$= (y_0z)(y_2ez)(y_0z)y$

$\sim (y_2ez)(y_0z)y$ ($w \xrightarrow{y_2} v \xrightarrow{e} u \xrightarrow{z} w$, $w \xrightarrow{y_0} u \xrightarrow{z} w$ and hence y_2ez and y_0z are loops about w. So we may apply *.)

$\sim y_2ezy$ (since $y \sim y_0zy$)

$= y_2xy.$

We next show that if we are given paths

$$x \;{}_{\sigma}y \xrightarrow{y} w$$

such that $C(x) \subseteq C(y)$, we have

$$y \sim xy$$

(Cf. Eilenberg, [4, Lemma 7.3, p. 226]).
We will call this lemma Result B. To prove Result B, we use Result A. Since

$$v \xrightarrow{x} v \xrightarrow{y} w$$

with $C(x) \subseteq C(y)$, using Result A, there exists a factorization

$$v \xrightarrow{y_0} v \xrightarrow{y_1} w$$

of y such that

$$y \sim y_0xy$$

Thus,

$$\begin{aligned} y &\sim y_0xy = y_0xy_0y_1 \\ &\sim xy_0y_1 \quad \text{(use *)} \\ &= xy \end{aligned}$$

Let

$$u \underset{y}{\overset{x}{\rightrightarrows}} w$$

be paths such that $Cx = Cy = \{a_1,a_2,\ldots,a_k\}$, say, and $\overline{x\rho_X}(1^j) = \overline{y\rho_X}(1^j)$ for all $1 \leq j \leq k$. For brevity, let $\overline{x}(1^j) = \overline{x\rho}_X(1^j)$. We will show $x \sim y$. Replacing E by Cx we may assume $E = Cx = Cy$. Let V_1 be the set of vertices accessible from w by a path, and let $V_0 = V - V_1$. The existence of trivial paths implies that $w \in V_1$.

Case I. $u \in V_1$. Hence, there is a path $z: w \longrightarrow u$. We will prove $x \sim y$ by mathematical induction on $|C(x)|$. If $|C(x)| = 0$, $x = 1u = y$. If $|C(x)| = 1$, $C(x) = C(y) = \{a_1\}$, say. Thus, $x = a_1^n$ and $y = a_1^s$ for positive integers n and s. If $n = s = 1$, $x = a_1 = y$. If $n > 1$, say, then a_1 is a loop about the vertex u. Thus, $x = a_1^n \sim a_1 \sim a_1^s = y$. Suppose $|C(x)| > 1$. Then, $|C(x(1))| < |C(x)|$. Hence $C(x(1)) = C(y(1))$ and $|C(x(1))(x))| = |C(y(1))| = |C(x)| -1$. We note $x(1^2) = x(1)(1)$. Suppose $x(1^j) = x(1)(1^{j-1})$. Thus, $x(1^{j+1}) = x(1^j)(1) = (x(1)(1^{j-1}))(1) = x(1)(1^j)$. So, $x(1^j) = x(1)(1^{j-1})$ for $2 \le j \le k$. Thus, if $2 \le j \le k-1$, $\bar{x}(1^{j+1}) = \overline{x(1^j)}(1) = \overline{x(1)(1^{j-1})}(1) = \overline{x(1)}(1^j)$. Hence, if $1 \le j \le k-1$, $\cancel{x(1)}\ (1^j) = x(1^{j+1}) = y(1^{j+1}) = \cancel{y(1)}(1^j)$. Thus, by the inductive hypothesis, $x(1) \sim y(1)$. Hence, since $\bar{x}(1) = \bar{y}(1)$, $\bar{x}(1)x(1) \sim \bar{y}(1)y(1)$. Let $x = a\bar{x}(1)x(1)$. and $y = b\bar{y}(1)y(1)$. Thus

$$\begin{tikzcd} u \arrow[r, "a"] & s \arrow[r, "\bar{x}(1)x(1)"] & w \arrow[ll, bend left, "z"] \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, "b"] & s \arrow[r, "\bar{y}(1)y(1)"] & w \arrow[ll, bend left, "z"] \end{tikzcd}$$

We note that $\bar{x}(1)x(1)za$ and $\bar{y}(1)y(1)zb$ are loops about the vertex s. So,

$$_{\sigma}s \xrightarrow{\bar{x}(1)x(1)} w$$

$\bar{x}(1)x(1)za$

$\bar{y}(1)y(1)zb$

Using Result B,

$$\begin{aligned} \bar{x}(1)x(1) &\sim \bar{y}(1)y(1)zb\bar{x}(1)x(1) \\ &\sim \bar{y}(1)y(1)zb\bar{y}(1)y(1) \\ &\sim \bar{y}(1)y(1)zy \\ &\sim \bar{x}(1)x(1)zy\,. \end{aligned}$$

Hence,

$a\bar{x}(1)x(1) \sim a\bar{x}(1)x(1)zy$

Thus, $x \sim xzy$

Again, using Result B,

$$xz\ {}_{\sigma}u \xrightarrow{y} w$$

implies

$y \sim xzy$

Thus,

$x \sim y$.

Case 2. $u \in V_0$. The proof of this case is exactly the same as the proof given in Eilenberg [4, p 237-238].

Conversely, we will show that for any two coterminal paths

$$u \underset{y}{\overset{x}{\rightrightarrows}} w$$

that $x \sim y$ implies $Cx = Cy = \{a_1, a_2, \ldots, a_k\}$, say, and $\bar{x}(1^j) = \bar{y}(1^j)$ for $1 \leq j \leq k$. Let p and p' be coterminal paths of X^*. We define $p \equiv p'$ if $C(p) = C(p') = \{a_1, a_2, \ldots, a_k\}$, say, and $\bar{p}(1^j) = \bar{p'}(1^j)$ for $1 \leq j \leq k$. Thus, $(p, p') \in \beta$ by Theorem 27. Let

$$r \underset{p'}{\overset{p}{\rightrightarrows}} s \quad \text{and} \quad s \xrightarrow{q} t$$

Thus,

$(pq, p'q) \in \beta$.

Hence, using Theorem 2.7, $C(pq) = C(p'q)$ and $\overline{pq}(1^j) = \overline{p'q}(1^j)$ for $1 \leq j \leq |C(pq)|$. So, $pq \equiv p'q$. Similarly, if $v \xrightarrow{q} r$, $qp \equiv qp'$. Hence, it easily follows that $\equiv$ is a congruence relation on X^*. Let x and y be loops about the same vertex of X^*. We show that $C(xyx) = C(yx)$ and $\overline{xyx}(1^j) = \overline{yx}(1^j)$ for $1 \leq j \leq |C(yx)|$. Clearly, $C(xyx) = C(yx)$. Since all letters in xyx will make their last appearance in yx, $\overline{xyx}(1^j) = \overline{(yx)}(1^j)$ for $1 \leq j \leq |C(yx)|$. Thus, $\equiv$ satisfies $*$ (i.e., for any two loops x and y about the same vertex, $xyx \equiv yx$). Hence, $\sim \subseteq \equiv$. Thus, if x and y are coterminal paths and $x \sim y$, then $C(x) = C(y) = \{a_1, a_2, \ldots, a_k\}$, say, and $\bar{x}(1^j) = \bar{y}(1^j)$ for $1 \leq j \leq k$.

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