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A NOTE ON CERTAIN NEXT-TO-INTERPOLATORY RATIONAL FUNCTIONS

M.A. Bokhari

(Dedicated to Prof. R.S. Varga on his 60th Birthday)

1. Introduction.

This note owes its motivation to an extension of a simple and elegant theorem of J.L. Walsh ([6], p. 153). Let $f \in A_\rho$ (the set of functions analytic in $|z| < \rho$ but not in $|z| \leq \rho$, $\rho > 1$). Let Π_s denote the class of all polynomials of degree $\leq s$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then we put

(1.1) $$S_{n-2}(z,f) = \sum_{j=0}^{n-2} a_j z^j.$$

Let $p^*_{n-2}(z,f)$ denote the polynomial of degree $\leq n-2$ which minimizes

$$\max_{0 \leq k \leq n-1} |p_{n-2}(\lambda^k) - f(\lambda^k)|, \quad \lambda = \exp(2\pi i/n),$$

over all polynomials $p_{n-2} \in \Pi_{n-2}$. Then Cavaretta et al established:

Theorem A. ([2], Theorem 7) Let $f \in A_\rho$, $1 < \rho < \infty$. Then

(1.2) $$\lim_{n\to\infty} \{S_{n-2}(z,f) - p^*_{n-2}(z,f)\} = 0, \quad |z| < \rho^2,$$

the convergence being uniform and geometric for all $|z| \leq \tau < \rho^2$. Moreover, the result is sharp in the sense that (1.2) is not valid at each point of $|z| = \rho^2$ for all $f \in A_\rho$.

It may be noted that both the sequences $\{S_{n-2}(z,f)\}_{n=1}^{\infty}$ and $\{p^{*}_{n-2}(z,f\}_{n=1}^{\infty}$ in Theorem A do not converge beyond the region $|z| < \rho$ whereas their difference converges in $|z| < \rho^2$. This phenomenon introduced by Walsh ([6], p. 153) is known as overconvergence or equiconvergence in the literature. For further details on this topic we refer the interested reader to [1] and [5].

Recently, E.B. Saff and A. Sharma [4] discussed the equiconvergence of certain sequences of rational interpolants. The classic theorem of Walsh ([6], p. 153) is a special case of their result ([4], Theorem 2.3). Our object in the present paper is to generalize Theorem A in the spirit of Saff-Sharma result. Building on results of Motzkin and Sharma [3], we solve a min-max problem in §3 for which the solution is the so-called next-to-interpolatory rational function. We prove our main result, i.e., Theorem 2.1, in §4.

2. Preliminaries and Statement of Main Result.

Let $m \geq -n+1$ be a fixed integer and let $r_{n+m-1}(z,f)$ be a rational function of the form

$$(2.1) \qquad p(z)/(z^n - \sigma^n), \quad p(z) \in \Pi_{n+m-1}, \ \sigma > 1,$$

which minimizes (see [4])

$$\int_{|z|=1} |f(z) - r(z)|^2 \, |dz|$$

over all rational functions $r(z)$ of the form (2.1).

If $f \in A_\rho$, then it is known [4] that the minimizing function $r_{n+m-1,n}(z,f)$ is given by

$$
(2.2)\quad \begin{cases} r_{n+m-1,n}(z,f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \left\{\dfrac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)} \quad \dfrac{t^m(t^n-\sigma^{-n})-z^m(z^n-\sigma^{-n})}{t^m(t^n-\sigma^{-n})}\right\}dt, \\ \text{and} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } m \geq 0, \\ r_{n+m-1,n}(z,f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \left\{\dfrac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)} \quad \dfrac{t^{n-m}-z^{n-m}}{t^m(t^n-\sigma^{-n})}\right\} dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } m < 0, \end{cases}
$$

where Γ is a circle $|t| = \rho'$, $1 < \rho' < \rho$, and $z \in \mathbb{C}$ with $|z| \neq \sigma$.

Next, consider the following problem:

(PI) Let m be a fixed integer with $m \geq -n+1$, and let $\omega = \exp(2\pi i/(n+m+1))$. For $f \in A_\rho$, minimize

$$\max_{0 \leq k \leq n+m} |f(\omega^k) - R(\omega^k,f)|$$

over all rational functions $R(.,f)$ of the form (2.1).

The existence and the uniqueness of the solution for (PI) is based on the results of Motzkin and Sharma ([3]), Theorems 1 & 2). If the solution (see §3) is denoted by

$$(2.3)\qquad R^*_{n+m-1,n}(z,f) := P^*_{n+m-1}(z,f)/(z^n-\sigma^n), \quad P^*_{n+m-1} \in \Pi_{n+m-1},$$

then we formulate our main result as follows:

Theorem 2.1. *Let* m *be a fixed integer and let* $\sigma > 1$. *If* $f \in A_\rho$, $1 < \rho < \infty$, *then*

(2.4) $$\lim_{n\to\infty} \{R^*_{n+m-1,n}(z,f) - r_{n+m-1,n}(z,f)\} = 0 \quad \begin{cases} |z| < \rho^2 & \text{if } \sigma \geq \rho^2, \\ |z| \neq \sigma & \text{if } \sigma < \rho^2, \end{cases}$$

the convergence being uniform and geometric in any compact subset of the regions described above. Moreover, the result is sharp in the sense of Theorem A.

3. Solution of Minimization Problem (PI).

Indeed, we want to solve the problem

$$\min_{p\in\Pi_{n+m-1}} \max_{0\leq k\leq n+m} \left|f(\omega^k) - \frac{p(\omega^k)}{\omega^{kn}-\sigma^n}\right|, \quad (\omega \text{ being as in (PI)}),$$

which is equivalent to

(3.1) $$\min_{p\in\Pi_{n+m-1}} \max_{0\leq k\leq n+m} b_k|F_n(\omega^k) - p(\omega^k)|$$

where

$$b_k = |\omega^{kn} - \sigma^n|^{-1},$$

and

$$F_n(z) = (z^n - \sigma^n)\, f(z).$$

Based on a result of Motzkin and Sharma ([3], Theorem 2), it can be verified that the solution $P^*_{n+m-1}(z,f)$ of the problem (3.1) is given by

$$(3.2)\qquad P^*_{n+m-1}(z,f) = \sum_{k=1}^{n+m+1} b_k^{-1}\, g_k(z) \Big/ \sum_{k=1}^{n+m+1} b_k^{-1}$$

where

$$(3.3)\qquad g_k(z) := \sum_{\substack{j=1\\ j\neq k}}^{n+m+1} \frac{W_k(z)}{(z-\omega^j)\, W_k'(\omega^j)} F_n(\omega^j)$$

with

$$W_k(z) := (z^{n+m+1} - 1) / (z - \omega^k).$$

In order to prove our main result, we shall require an integral representation of $P^*_{n+m-1}(z,f)$. For this we prove

Lemma 3.1. The polynomial $g_k(z)$, $1 \leq k \leq n+m+1$, given by (3.3) has the following representation:

$$(3.4)\qquad g_k(z) = \beta_{n+m,n}(z,f) - C_n(z^{n+m+1} - 1)/(z - \omega^k)$$

where

$$(3.5)\qquad \beta_{n+m,n}(z,f) := \frac{1}{2\pi i} \int_\Gamma \frac{F_n(t)}{t - z} \left\{\frac{t^{n+m+1} - z^{n+m+1}}{t^{n+m+1} - 1}\right\} dt,$$

and

$$(3.6)\qquad C_n := \frac{1}{2\pi i} \int_\Gamma \frac{F_n(t)}{t^{n+m+1} - 1}\, dt.$$

Here Γ is a circle $|t| = \rho'$, $1 < \rho' < \rho$.

Proof. It is easy to see that

$$g_k(z) = \sum_{\substack{j=1 \\ j\neq k}}^{n+m+1} \frac{(z^{n+m+1} - 1)\ (\omega^j - \omega^k)\omega^j}{(n+m+1)(z-\omega^k)(z-\omega^j)} F_n(\omega^j).$$

Since

$$\frac{1}{(z-\omega^k)(z-\omega^j)} = \left\{\frac{1}{z - \omega^j} - \frac{1}{z - \omega^k}\right\}(\omega^j - \omega^k)^{-1}, \quad j \neq k,$$

we have

$$g_k(z) = \sum_{j=1}^{n+m+1} \frac{z^{n+m+1}-1}{z - \omega^j}\cdot\frac{\omega^j}{n+m+1} F_n(\omega^j) - \sum_{j=1}^{n+m+1} \frac{z^{n+m+1} - 1}{z - \omega^k}\ \frac{\omega^j}{n+m+1} F_n(\omega^j).$$

The first summation on the right side of the above equation represents the Lagrange interpolation polynomial of degree $n+m$ to $F_n(z)$ on the $n+m+1$ roots of unity and therefore equals $\beta_{n+m,n}(z,f)$. The second summation upon using the Cauchy integral formula gives us $C_n(z^{n+m+1} - 1) / (z - \omega^k)$.

Remark 3.1. From (3.2) and (3.4) we have

$$(3.7) \qquad P^*_{n+m-1}(z,f) = \beta_{n+m,n}(z,f) - C_n \sum_{k=1}^{n+m+1} b_k^{-1}\left(\frac{z^{n+m+1}-1}{z - \omega^k}\right) \Big/ \sum_{k=1}^{n+m+1} b_k^{-1}.$$

An alternate representation of $P^*_{n+m-1}(z,f)$ is therefore given by

$$(3.8) \qquad P^*_{n+m-1}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)f(t)}{t^{n+m+1} - 1}\left\{\frac{t(t^{n+m} - z^{n+m})}{t - z}\right\}dt - \gamma_n(z,f)$$

where

$$(3.9) \qquad \gamma_n(z,f) := C_n \sum_{k=1}^{n+m+1} b_k^{-1}\ \omega^k\left(\frac{z^{n+m} - \omega^{k(n+m)}}{z - \omega^k}\right) \Big/ \sum_{k=1}^{n+m+1} b_k^{-1}.$$

This can easily be seen upon writing the integral representation of $\beta_{n+m,n}(z,f)$ (cf. (3.5)) and considering the relation:

$$\frac{z^{n+m+1}-1}{z-\omega^k} = z^{n+m} + \omega^k \left(\frac{z^{n+m}-\omega^{k(n+m)}}{z-\omega^k}\right)$$

in (3.7).

4. Proof of the Main Result.

When $|z| \le 1$, the direct estimates from (2.2), (3.2) and (3.3) show that

$$\lim_{n\to\infty} \{R^*_{n+m-1,n}(z,f) - r_{n+m-1}(z,f)\} = 0.$$

However, the proof of (2.4) for $|z| > 1$ requires a detailed analysis of $\gamma_n(z,f)$. For this, first we note that $\omega = \exp(2\pi i/(n+m+1))$. Therefore,

$$b_k^{-1} := |\omega^{kn} - \sigma^n| = \sigma^n\{1 + (\frac{1}{\sigma^{2n}} - \frac{2}{\sigma^n}\cos\frac{2kn\pi}{n+m+1})\}^{\frac{1}{2}}$$

which upon using binomial expansion simplifies to

$$b_k^{-1} = \sigma^n + O(1). \tag{4.1}$$

Also

$$\sum_{k=1}^{n+m+1} \frac{\omega^k(z^{n+m}-\omega^{k(n+m)})}{z-\omega^k} = 0, \quad \forall\ z. \tag{4.2}$$

This follows from the properties of the roots of unity. From (4.1) and (4.2) we observe that

$$\left|\sum_{k=1}^{n+m+1} b_k^{-1} \frac{\omega^k(z^{n+m}-\omega^{k(n+m)})}{z-\omega^k}\right| \le D_1(n+m+1)^2|z|^{n+m}\frac{|z|^{n+m}+1}{|z^{n+m+1}-1|} \tag{4.3}$$

when $|z| > 1$. Here D_1 is a constant independent of n. Since

$$\sum_{k=1}^{n+m+1} b_k^{-1} \ge (n+m+1)(\sigma^n - 1) \tag{4.5}$$

and

$$C_n := \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)f(t)}{t^{n+m+1}-1}\,dt = O(\frac{\rho^n+\sigma^n}{\rho^n}), \tag{4.6}$$

we obtain the following inequality from (3.9) and (4.3)-(4.6):

$$|\gamma_n(z,f)| \le D_2(n+m+1)|z|^{n+m} \cdot \frac{|z|^{n+m}+1}{|z^{n+m+1}-1|} \cdot \frac{\rho^n+\sigma^n}{\rho^n\sigma^n}, \quad |z| > 1,$$

where D_2 is a constant independent of n. This shows that

$$\lim_{n\to\infty} \frac{\gamma_n(z,f)}{z^n-\sigma^n} = 0, \ \forall\ |z| > 1 \text{ and } |z| \ne \sigma. \tag{4.7}$$

Proof of Theorem 2.1. We shall prove this result for $m \ge 0$. The proof for the case $m < 0$ will be similar. From (2.2), (2.3) and (3.8) we have

$$R^*_{n+m-1,n}(z,f) - r_{n+m-1,n}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)} K_n(t,z)dt - \frac{\gamma_n(z,f)}{z^n-\sigma^n} \tag{4.8}$$

where

$$K_n(t,z) := \frac{(t^{n+m}-z^{n+m})(1-t^{m+1}\sigma^{-n})}{(t^{n+m+1}-1)(t^n-\sigma^{-n})t^m} - \frac{(z^m-t^m)\sigma^{-n}}{t^m(t^n-\sigma^{-n})}. \tag{4.9}$$

Notice that Γ is a circle $|t| = \rho'$, $1 < \rho' < \rho$. Since $\sigma > 1$, for all sufficiently large n we obtain

$$|R^*_{n+m-1}(z,f) - r_{n+m-1,n}(z,f)|$$

$$\le D_3 \ \frac{(\rho^n+\sigma^n)}{|z^n-\sigma^n|}\left\{\frac{\rho^n+|z|^n}{\rho^{2n}} + \frac{\sigma^{-n}}{\rho^n}\right\} + \frac{|\gamma_n(z,f)|}{|z^n-\sigma^n|}$$

where D_3 is a constant independent of n. Taking into account (4.7) and considering different cases for $\sigma \ge \rho^2$, and $\sigma < \rho^2$, a straightforward analysis now yields (2.4).

Next, we show that the result is sharp in the sense that the first region of convergence in (2.4) cannot be improved. For this we follow the usual technique (cf. [4]) and select the function $\hat{f}(z) = (\rho-z)^{-1}$ in A_ρ along with the point $z = \rho^2$ on the boundary of the region $|z| < \rho^2$. It is easy to see that

$$\frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)(t^{n+m}-z^{n+m})\,\hat{f}(t)\,dt}{(z^n-\sigma^n)(t-z)(t^{n+m+1}-1)(t^n-\sigma^{-n})t^m}$$

$$= \frac{(\rho^n-\sigma^n)\;(\rho^{n+m}-z^{n+m})}{(z^n-\sigma^n)(\rho-z)(\rho^{n+m+1}-1)(\rho^n-\sigma^{-n})\rho^m},$$

whereas for $z = \rho^2$ and $\sigma > \rho^2$ we have

$$\lim_{n\to\infty} \frac{\gamma_n(z,\hat{f})}{z^n-\sigma^n} = 0$$

and

$$\lim_{n\to\infty}\int_\Gamma \frac{(t^n-\sigma^n)\;\sigma^{-n}f(t)}{(z^n-\sigma^n)(t-z)(t^n-\sigma^{-n})t^m}\;\left\{\frac{(t^{n+m}-z^{n+m})t}{t^{n+m+1}-1}-(z^m-t^m)\right\}dt = 0.$$

This along with (4.8) and (4.9) gives us

$$\text{(4.10)}\qquad \lim_{n\to\infty}\{R^*_{n+m-1,n}(\rho^2,\hat{f}) - r_{n+m-1,n}(\rho^2,\hat{f}) = \frac{\rho^{2m}}{(\rho^2-\rho)\rho^{m+1}\rho^m},\quad (\sigma > \rho^2).$$

When $z = \rho^2$ and $\sigma = \rho^2$, the limit on the left side of (2.4) does not exist because of the factor $(z^n - \sigma^n)$ in the denominator. This completes the proof.

Remark 4.3. If we choose $m = -1$ and let $\sigma \to \infty$ in (2.4), we obtain Theorem A.

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