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ON SOME STATISTICAL PROPERTIES OF THE VÓN–KÁRMAN CORRELATION FUNCTION

By

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ABSTRACT

Some statistical properties of the Vón-Kárman correlation function have been investigated. The function is fitted on a certain data on estimation of blood alcohol concentration distribution in driving people. It is found that the truncated form of the function is an appropriate model for the data.

1. INTRODUCTION

The probabilistic approach is more practical than the deterministic one in elastic wave scattering in complex media. The most commonly used correlation function for a complex media is the Vón-Kárman correlation function (see Wu 1989, Hudson et al 1967)

$$f_\alpha(x;a) = C_\alpha\, x^\alpha\, K_\alpha(ax), \qquad x > 0, a > 0, \alpha > 0, \tag{1}$$

where K_α is the modified Bessel function of the third kind and

$$C_\alpha = a^{\alpha+1}/2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+1/2). \tag{2}$$

In this paper we have investigated some statistical properties of the function. We have fitted the function to the data obtained from Lloyd (1990) of the estimation of the blood alcohol concentration distribution in driving population. It is shown that the truncated form of the Vón-Kárman correlation function is a more appropriate model for the data.

2. PROPERTIES

2.1 The Graph: The graphs of $f_\alpha(x;a)$ for some values of α and $a = 1$ are given at Fig (1). When $\alpha = 1/2$, f_α is equivalent to the exponential correlation function and for $\alpha = 1/3$, f_α is the Kolmogorov turbulence function, Lloyd (1990) and Wu (1989).

2.2 Recurrence Relation: We state and prove the following recurrence relation useful for constructing tables and computing frequencies.

Theorem

$$f_{\alpha+1}(x;a) = (\frac{2\alpha}{2\alpha+1})f_\alpha(x;a) + \frac{(ax)^2}{4\alpha^2-1}f_{\alpha-1}(x;a). \tag{3}$$

Proof

By definition

$$f_{\alpha+1}(x;a) = \frac{4}{\sqrt{\pi}}(\frac{a}{2})^{\alpha+2} K_\alpha(ax)/\Gamma(\alpha+3/2).$$

By using the recurrence relations (Erdelyi, et al 1954) for K_α we get

$$\begin{aligned}
f_{\alpha+1}(x;a) &= \frac{4}{\sqrt{\pi}}\frac{(\frac{a}{2})^{\alpha+2}x^{\alpha+1}}{\Gamma(\alpha+3/2)}\left\{\frac{2\alpha}{ax}K_\alpha(ax) + K_{\alpha-1}(ax)\right\} \\
&= \frac{4}{\sqrt{\pi}}\frac{a}{2\alpha+1}\left\{\frac{2\alpha}{a}\frac{(\frac{a}{2})^\alpha}{\Gamma(\alpha+1/2)}x^\alpha K_\alpha(ax) + \frac{ax^2}{2\alpha-1}\frac{(\frac{a}{2})^\alpha}{\Gamma(\alpha-1/2)}x^{\alpha-1}K_{\alpha-1}(ax)\right\} \\
&= \frac{a}{2\alpha+1}\left\{\frac{2\alpha}{a}f_\alpha(x;a) + \frac{ax^2}{2\alpha-1}f_{\alpha-1}(x;a)\right\} \\
&= \frac{2\alpha}{2\alpha+1}f_\alpha(x;a) + \frac{(ax)^2}{4\alpha^2-1}f_{\alpha-1}(x;a).
\end{aligned}$$

In particular when $\alpha = 1$, we get

$$3f_2(x;a) - 2f_1(x;a) = (ax)^2.$$

2.3 The Cumulative Distribution Function (c.d.f.): The series representation of the c.d.f. $F_\alpha(x;a)(\alpha > 0, \alpha \neq 1,2,3,\ldots)$ can be found by term-by-term integration of the series representation of the Bessel function K_α Watson (1916). Since

$$K_\alpha(x) = \frac{\pi}{2\sin\alpha\pi}\sum_{k=1}^{\infty}\frac{1}{k!}\{\frac{(\frac{x}{2})^{2k-\alpha}}{\Gamma(k-\alpha+1)} - \frac{(\frac{x}{2})^{2k+\alpha}}{\Gamma(k+\alpha+1)}\}, \tag{4}$$

and

$$\begin{aligned}
F_\alpha(x;a) &= \int_0^x f_\alpha(t;a)dt \\
&\Rightarrow F_\alpha(x;a) = \frac{a^{\alpha+1}}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^x t^\alpha K_\alpha(at)dt \\
&= \frac{2\sqrt{\pi}}{\sin\alpha\pi\Gamma(\alpha+1/2)}[\sum_{k=0}^{\infty}\frac{1}{k!}\{\frac{(\frac{x}{2})^{2k+1}}{(2k+1)\Gamma(k-\alpha+1)} - \frac{(\frac{x}{2})^{2k+2\alpha+1}}{(2k+2\alpha+1)\Gamma(k+\alpha+1)}\}]. \quad (5)
\end{aligned}$$

It follows from (5) that $\lim_{x\to\infty} F_\alpha(x;a) = 1$ ($\alpha > 0$, $\alpha \neq 1,2,3,\ldots$) the graphs of $F_\alpha(x;a)$ for some values of α when $a = 1$ are given at Fig (2).

2.4 The Moment Generating Function: Let $M_X(t)$ be the moment generating function of the Vón-Kárman random variable. Then,

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{-tx} f_\alpha(x;a)dx \\ &= C_\alpha \int_0^\infty x^\alpha e^{-tx} K_\alpha(ax)dx. \end{aligned}$$

By using the tables (Erdelyi et al, 1954) and Gradshteyn and Ryshik (1980) we get

$$M_X(t) = \frac{1}{2} C_\alpha Q_\alpha^\alpha (\frac{t}{\sqrt{t^2-a^2}}/cos(\alpha\pi)\sqrt{(t^2-a^2)\alpha+1}), \qquad |t| > a \tag{6}$$

where $Q_\beta^\alpha(t)$ is the Legendre function of the second kind. The rth noncentral moment can now be derived by using the rth derivative of $Q_\alpha^\alpha(t)$ with respect to t at $t = 0$.

2.5 Moments And Coefficient of Variation

The rth non-central moment ($r \geq 0$) of the Vón-Kárman random variable is given by

$$\begin{aligned} \mu_r' &= \int_0^\infty x^r f_\alpha(x;a)dx \\ &= \frac{a^{\alpha+1}}{2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^\infty x^{\alpha+r} K_\alpha(ax)dx \\ &= \frac{(\frac{2}{a})^r \Gamma(\frac{r+1}{2})\Gamma(\frac{r+1}{2}+\alpha)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}. \end{aligned} \tag{7}$$

The expectation of the random variable is

$$\mu_1' = \frac{(\frac{2}{a})\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}. \tag{8}$$

The variance of the random variable is

$$\sigma^2 = \frac{4}{a^2}\left[(\frac{2\alpha+1}{4}) - (\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)})^2\right]. \tag{9}$$

The higher moments are related to the lower moments via the relation given by

$$\mu_{r+2}' = (\frac{2}{a})^2(\frac{r+1}{2})(\frac{r+1}{2}+\alpha)\mu_r'. \tag{10}$$

The relation (10) can easily be proved by using the properties of the gamma function. In general we have the following

$$\begin{aligned} \mu'_{2m} &= \frac{1}{a^{2m}} 1.3.5 \cdots (2m-1)(1+2\alpha)(3+2\alpha)\cdots(2m-1+2\alpha) \\ &= \frac{1}{a^{2m}}(2m-1)!!\,\Gamma(2m+2\alpha)/\Gamma(2\alpha+1) \end{aligned} \qquad (11)$$

and

$$\mu'_{2m+1} = (\frac{2}{a})^{2m}\, m!\, \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \cdot \mu'_1 \qquad (12)$$

In particular, we have

$$\mu'_2 = \frac{1+2\alpha}{a^2}, \qquad \mu'_3 = \frac{4}{a^2}(1+\alpha)\mu'_1,$$

and

$$\mu'_4 = \frac{3}{a^4}(1+2\alpha)(3+2\alpha).$$

If $\alpha \to 0, \mu'_1 = \frac{2}{a\pi}, \mu'_2 = 1/a^2, \mu'_3 = \frac{8}{a^3\pi}$ and $\mu'_4 = \frac{9}{a^4}$ and thus

$$\mu_2 \to \frac{1}{a^2}(1-\frac{4}{\pi^2}) = \frac{0.5950}{a^2},$$

$$\mu_3 \to \frac{2}{\pi a^3}(1+\frac{8}{\pi^2}) = \frac{1.1518}{a^3},$$

$$\mu_4 \to \frac{a}{a^4}(1-\frac{40}{a\pi^2}-\frac{48}{a\pi^4}) = \frac{4.4584}{a^4},$$

$$\beta_1 \to \mu_3/\mu_1'^{3/2} = 2.5096,$$

$$\beta_2 \to \mu_4/\mu_2^2 = 12.5935.$$

If $\alpha \to \infty, \beta_1 \to 1.75$ and $\beta_2 \to 10.4$. It follows that β_1 and β_2 are monotonically decreasing functions of α. The coefficient of variation of the random variable is found to be

$$cv(\alpha) = \left[\frac{\pi}{4}(2\alpha+1)(\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)})^2 - 1\right]^{1/2}. \qquad (13)$$

The inequality $0.75 \le cv(\alpha) \le 1.225$ holds for $0 \le \alpha < \infty$. In particular for $\alpha = 1/2, cv(1/2) = 1$.

3. Estimation of Parameters

Let m'_1 and m'_2 be the two sample moments about the origin. By using the moment method (MM) of estimation we get the estimation equations

$$\frac{m_1'^2}{m_2'} = \frac{4}{(2\hat{\alpha}+1)\pi}\left(\frac{\Gamma(\hat{\alpha}+1)}{\Gamma(\hat{\alpha}+1/2)}\right)^2 \tag{14}$$

and

$$\hat{a} = \sqrt{\frac{2\hat{\alpha}+1}{m_2'}}. \tag{15}$$

If we let $m(\hat{\alpha}) = m_1'^2/m_2'$, then, $m(0) = 0.4053$ and $m(\hat{\alpha}) \to 0.63662$ as $\hat{\alpha} \to \infty$. The estimating equation $m(\hat{\alpha})$ has been tabulated for various values of $\hat{\alpha}$ (see Table I). For a given value of $m(\hat{\alpha})$ from the data, we can interpolate inversely the $\hat{\alpha}$–value from the table. Simple linear interpolation will suffice. We can substitute this $\hat{\alpha}$– value in the equation (15) in order to obtain $\hat{a}$.

The asymptotic variances and covariance of $\hat{a}$ and $\hat{\alpha}$ are given by

$$var(\hat{a}) = |\sum|^{-1}\frac{8}{a(2\alpha+1)}[RL(R-\frac{1}{2\alpha+1}) + \frac{2\alpha+1}{4}RS^2M_2 - S^2M_1]$$

$$var(\hat{\alpha}) = |\sum|^{-1}\frac{8L}{a^3}[R-\frac{1}{2\alpha+1}]$$

$$cov(\hat{\alpha},\hat{a}) = |\sum|^{-1}\frac{4}{a^2}[L(R^2-\frac{1}{2\alpha+1)^2}) + \frac{S^2M_2}{4} - S^2M_1],$$

where

$$|\sum| = \frac{8}{a^2}S^2(R-\frac{1}{2\alpha+1}),$$

$$R = \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} - \frac{\Gamma'(\alpha+1/2)}{\Gamma(\alpha+1/2)},$$

$$S = 2(\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} - \frac{\Gamma'(\alpha+1/2)}{\Gamma(\alpha+1/2)} - \frac{1}{2\alpha+1}),$$

$$M_1 = \frac{1}{\mu_1'^2}var(m_1'),$$

$$M_2 = \frac{1}{\mu_2'^2}var(m_2'),$$

and

$$L = \frac{4}{\mu_1'^2}var(m_1') + \frac{1}{\mu_2'^2}var(m_2') - \frac{4}{\mu_1'\mu_2'}cov(m_1', m_2').$$

4. APPLICATIONS

EXAMPLE Estimation of Blood Alcohol Concentration Distribution in Driving Population.

The 2003 records of BACs of randomly sampled drivers are obtained from Lloyd (1990). The following model

$$g(x) = \begin{cases} g_0 & x = 0 \\ (1-g)_0)g^+(x) & x > 0 \end{cases} \tag{16}$$

where

$$g^+(x) = \frac{\beta^\alpha x^{\alpha-1} exp(-\beta x)}{\Gamma(\alpha)} \tag{17}$$

was considered as an appropriate model for the data. The main BAC of those drivers giving a positive reading is 45.352 mg/100 ml with a standard deviation of 42.383 mg/100 ml, coefficient of variation 1.145. The geometric mean is 32.375 mg/100 ml so that the ratio T of the geometric to the arithmetic mean is $T = 0.7138$. This leads to ML estimates $\hat{\alpha}_{ML} = 1.632$, $\hat{\beta}_{ML} = 0.03598$ and MM estimates $\hat{\alpha}_{MM} = 1.145$, $\hat{\beta}_{MM} = 0.02525$. It was found that the ML fit ($\chi^2 = 121.0$ on 17 degree of freedom) was worse than the MM fit ($\chi^2 = 47.3$).

Table 2

Interval Or Midpoint	Observed Frequencies	Chi-squared Contribution (Gamma p.d.)	Chi-squared Contribution (Vón-Kárman p.d.)
10	397	3.03	5.17
20	388	3.75	44.95
30	298	1.55	9.36
40	215	0.00	0.03
50	162	0.18	2.52
60	120	0.68	6.16
70	86	1.88	9.76
80	64	2.09	9.59
90	63	0.29	1.14
100	48	0.18	0.72
110	33	0.07	1.28
120	29	0.24	0.00
130	16	0.90	1.26
140	15	0.02	0.00
150	17	2.23	11.18
160	12	0.95	2.81
170	13	5.32	12.02
[175 – 195]	8	0.19	23.88
[195 – ∞]	19	2.44	
Total	2003	25.99	141.83

We have fitted Vón-Kárman function (1) on the same data based on MM method and found that χ^2–value is 141.83 which shows that the fit is rather poor. However, if we truncate the Vón-Kárman function at $x = 5$, it is obviously clear from the table that the χ^2–contribution reduces considerably. The truncated form of the function is given by $f_T(x;a) = (1 - F(x_0))^{-1} x^\alpha \cdot K_\alpha(ax),\ \ x \geq x_0$. The mean and non-central moments for the non-truncated functions are $m_1' = 44.696, m_2' = 3295.9935$ giving $m(\hat{\alpha}) = 0.60611$. So that $\hat{\alpha} = 4.589$ and $a = 0.0557$. The asymptotic variances and covariance of $\hat{\alpha}$ and $\hat{a}$ are $var(\hat{\alpha}) = 0.1526$, $var(\hat{a}) = 0.00031$ and $cov(\hat{\alpha}, \hat{a}) = 0.2513$.

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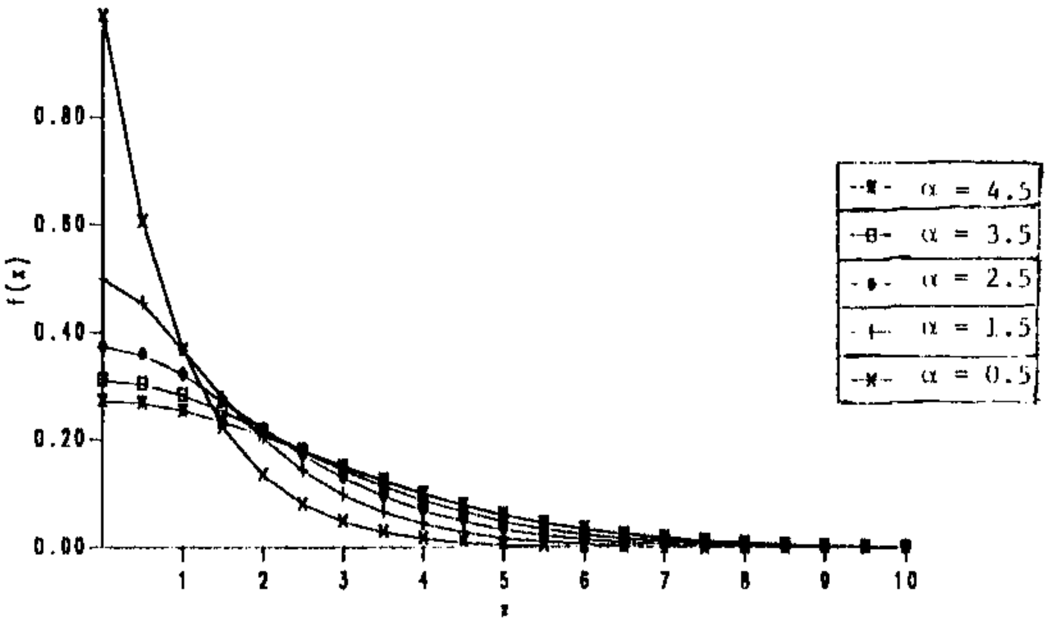

Fig (1): Graphs of f(x) for various values of α

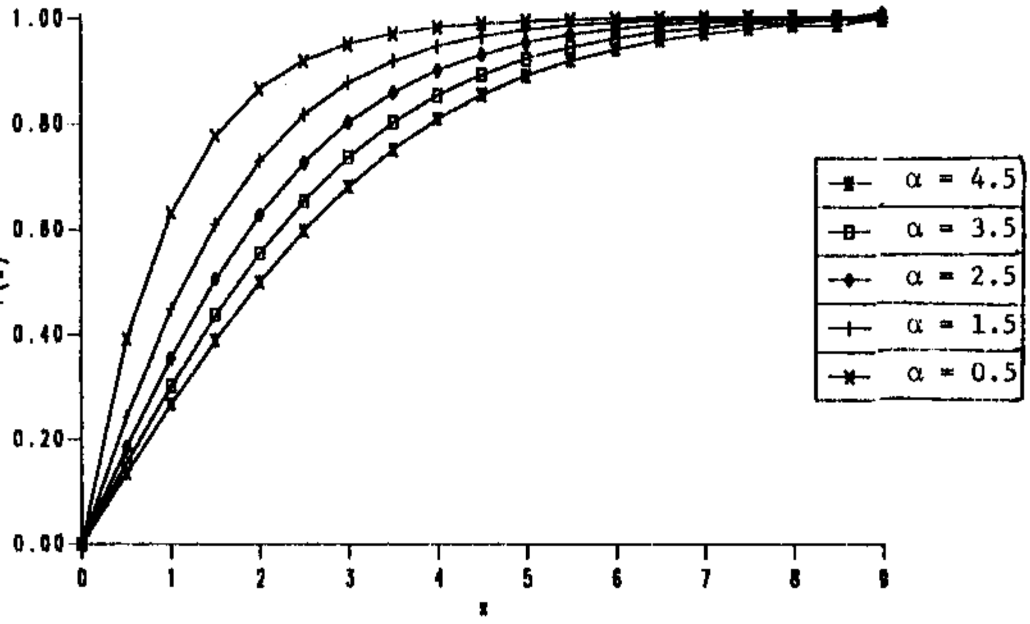

Fig. (2): Graphs of F(x) for various values of α

Table 1: $m(\hat{\alpha})$ function for various values of $\hat{\alpha}$.

$\hat{\alpha}$	$m(\hat{\alpha})$	$\hat{\alpha}$	$m(\hat{\alpha})$	$\hat{\alpha}$	$m(\hat{\alpha})$	$\hat{\alpha}$	$m(\hat{\alpha})$	$\hat{\alpha}$	$m(\hat{\alpha})$
0.00	0.40529	0.63	0.51337	1.26	0.55329	1.89	0.57381	2.52	0.58625
0.01	0.40838	0.64	0.51430	1.27	0.55372	1.90	0.57405	2.53	0.58641
0.02	0.41139	0.65	0.51521	1.28	0.55414	1.91	0.57430	2.54	0.58657
0.03	0.41433	0.66	0.51610	1.29	0.55456	1.92	0.57454	2.55	0.58672
0.04	0.41720	0.67	0.51699	1.30	0.55498	1.93	0.57478	2.56	0.58688
0.05	0.42000	0.68	0.51786	1.31	0.55539	1.94	0.57502	2.57	0.58704
0.06	0.42273	0.69	0.51872	1.32	0.55580	1.95	0.57525	2.58	0.58719
0.07	0.42539	0.70	0.51956	1.33	0.55620	1.96	0.57549	2.59	0.58734
0.08	0.42799	0.71	0.52040	1.34	0.55660	1.97	0.57572	2.60	0.58749
0.09	0.43054	0.72	0.52122	1.35	0.55700	1.98	0.57595	2.61	0.58764
0.10	0.43302	0.73	0.52203	1.36	0.55739	1.99	0.57618	2.62	0.58780
0.11	0.43545	0.74	0.52283	1.37	0.55778	2.00	0.57640	2.63	0.58794
0.12	0.43782	0.75	0.52362	1.38	0.55816	2.01	0.57663	2.64	0.58809
0.13	0.44014	0.76	0.52440	1.39	0.55854	2.02	0.57685	2.65	0.58824
0.14	0.44241	0.77	0.52517	1.40	0.55892	2.03	0.57708	2.66	0.58838
0.15	0.44463	0.78	0.52593	1.41	0.55929	2.04	0.57730	2.67	0.58853
0.16	0.44680	0.79	0.52667	1.42	0.55966	2.05	0.57752	2.68	0.58867
0.17	0.44892	0.80	0.52741	1.43	0.56003	2.06	0.57773	2.69	0.58882
0.18	0.45100	0.81	0.52814	1.44	0.56039	2.07	0.57795	2.70	0.58896
0.19	0.45304	0.82	0.52886	1.45	0.56075	2.08	0.57816	2.71	0.58910
0.20	0.45503	0.83	0.52957	1.46	0.56110	2.09	0.57837	2.72	0.58924
0.21	0.45698	0.84	0.53027	1.47	0.56146	2.10	0.57858	2.73	0.58938
0.22	0.45889	0.85	0.53097	1.48	0.56181	2.11	0.57879	2.74	0.58952
0.23	0.46077	0.86	0.53165	1.49	0.56215	2.12	0.57900	2.75	0.58966
0.24	0.46260	0.87	0.53232	1.50	0.56250	2.13	0.57920	2.76	0.58979
0.25	0.46440	0.88	0.53299	1.51	0.56284	2.14	0.57941	2.77	0.58993
0.26	0.46617	0.89	0.53365	1.52	0.56317	2.15	0.57961	2.78	0.59007
0.27	0.46790	0.90	0.53430	1.53	0.56351	2.16	0.57982	2.79	0.59020
0.28	0.46959	0.91	0.53494	1.54	0.56384	2.17	0.58002	2.80	0.59033
0.29	0.47125	0.92	0.53558	1.55	0.56417	2.18	0.58022	2.81	0.59047
0.30	0.47288	0.93	0.53620	1.56	0.56449	2.19	0.58041	2.82	0.59060
0.31	0.47449	0.94	0.53682	1.57	0.56482	2.20	0.58061	2.83	0.59073
0.32	0.47606	0.95	0.53743	1.58	0.56514	2.21	0.58081	2.84	0.59086
0.33	0.47760	0.96	0.53804	1.59	0.56545	2.22	0.58100	2.85	0.59100
0.34	0.47911	0.97	0.53863	1.60	0.56577	2.23	0.58120	2.86	0.59113
0.35	0.48059	0.98	0.53922	1.61	0.56608	2.24	0.58139	2.87	0.59126
0.36	0.48205	0.99	0.53981	1.62	0.56639	2.25	0.58158	2.88	0.59138
0.37	0.48348	1.00	0.54038	1.63	0.56669	2.26	0.58177	2.89	0.59151
0.38	0.48489	1.01	0.54095	1.64	0.56700	2.27	0.58195	2.90	0.59164
0.39	0.48627	1.02	0.54151	1.65	0.56730	2.28	0.58214	2.91	0.59177
0.40	0.48763	1.03	0.54207	1.66	0.56760	2.29	0.58232	2.92	0.59189
0.41	0.48896	1.04	0.54262	1.67	0.56789	2.30	0.58251	2.93	0.59202
0.42	0.49027	1.05	0.54316	1.68	0.56818	2.31	0.58269	2.94	0.59214
0.43	0.49156	1.06	0.54370	1.69	0.56847	2.32	0.58287	2.95	0.59227
0.44	0.49283	1.07	0.54423	1.70	0.56876	2.33	0.58305	2.96	0.59239
0.45	0.49407	1.08	0.54475	1.71	0.56905	2.34	0.58323	2.97	0.59251
0.46	0.49530	1.09	0.54527	1.72	0.56933	2.35	0.58341	2.98	0.59263
0.47	0.49650	1.10	0.54578	1.73	0.56961	2.36	0.58358	2.99	0.59275
0.48	0.49769	1.11	0.54629	1.74	0.56989	2.37	0.58376	3.00	0.59287
0.49	0.49885	1.12	0.54679	1.75	0.57016	2.38	0.58393	3.50	0.59814
0.50	0.50000	1.13	0.54729	1.76	0.57044	2.39	0.58411	4.00	0.60228
0.51	0.50113	1.14	0.54778	1.77	0.57071	2.40	0.58428	4.50	0.60562
0.52	0.50224	1.15	0.54827	1.78	0.57098	2.41	0.58445	5.00	0.60837
0.53	0.50333	1.16	0.54875	1.79	0.57125	2.42	0.58462	5.50	0.61066
0.54	0.50440	1.17	0.54922	1.80	0.57151	2.43	0.58478	6.00	0.61262
0.55	0.50546	1.18	0.54970	1.81	0.57177	2.44	0.58495	6.50	0.61430
0.56	0.50650	1.19	0.55016	1.82	0.57203	2.45	0.58512	7.00	0.61576
0.57	0.50753	1.20	0.55062	1.83	0.57229	2.46	0.58528	7.50	0.61704
0.58	0.50854	1.21	0.55108	1.84	0.57255	2.47	0.58545	8.00	0.61818
0.59	0.50954	1.22	0.55153	1.85	0.57281	2.48	0.58561	8.50	0.61919
0.60	0.51052	1.23	0.55198	1.86	0.57306	2.49	0.58577	9.00	0.62009
0.61	0.51148	1.24	0.55242	1.87	0.57331	2.50	0.58593	9.50	0.62090
0.62	0.51243	1.25	0.55286	1.88	0.57356	2.51	0.58609	10.00	0.62165
								∞	0.63662