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Abstract

We characterize semigroups obeying the term condition and give a more detailed characterization of periodic semigroups obeying the term condition.

McKenzie [5] characterized semigroups of fixed exponent obeying the term condition and, in [5], he posed the problem of characterizing all semigroups obeying the term condition. The term condition for algebras was introduced by McKenzie [4] and algebras obeying the term condition (also called abelian algebras) have been studied extensively (see, for example, [2] and [3]). An algebra A satisfies the term condition if $p(a,\tilde{x}) = p(a,\tilde{y})$ if and only if $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a, b \in A$, $\tilde{x}, \tilde{y} \in A^n$ and any $n+1-ary$ term p. We determine the structure of all semigroups obeying the term condition (Theorem 1.14) and we give a more detailed structure theorem for periodic semigroups obeying the term condition (Theorem 2.11). We build these semigroups from subsemigroups of $GXIXJ$ (algebraic direct product) where G is an abelian group, I is a left zero semigroup ($xy = y$ for all $x, y \in I$), and J is a right zero semigroup ($xy = y$ for all $x, y \in J$).

For definitions not given here, see [1] or [6].

1 Semigroups Obeying the term Condition

Let R be any semigroup obeying the term condition.

Lemma 1.1 *Let $x, y \in R$. Then, $xy = x^2 = y^2$ iff $zx = zy$ and $xz = yz$ for all $z \in R$.*

Proof. Suppose $xy = x^2 = y^2$. Then $xy = xx$ implies $zx = zy$ for all $z \in R$ (use the term $p(x, y) = xy$). Furthermore, $xy = yy$ implies $xz = yz$ for all $z \in R$.

Conversely, suppose $xz = yz$ and $zx = zy$ for all $z \in R$. Thus $xy = y^2$ and $x^2 = xy$. So, $x^2 = y^2 = xy$.   q.e.d.

Define $x \equiv y (x, y \in R)$ iff $x^2 = y^2 = xy$ iff $xz = yz$ and $zx = zy$ for all $\in R$.

Lemma 1.2 *$\equiv$ is a congruence relation on R.*

Proof. Reflexivity and symmetry are clear. First, we show transitivity. Suppose $x \equiv y$ and $y \equiv z$. Thus, $mx = my = mz$ and $xm = ym = zm$ for all $m \in R$. Hence, $x \equiv z$. Next, we show $\equiv$ is a compatible relation. Suppose $x \equiv y$ and $t \in R$. Thus, $xtz = ytz$ and $zxt = zyt$ for all $z \in R$. Hence, $xt \equiv yt$. Similarly, $tx \equiv ty$.

The following lemma will be used in the proof of Lemma 1.4, Lemma 1.5, Lemma 1.7, Lemma 2.5 and Remark 2.9.

Lemma 1.3 *(Taylor [7, Lemma 4]). A semigroup obeying the term condition satisfies the identity $xyzw = xzyw$.*

Let $T = R/\equiv$. If $x \in R, \bar{x}$ will denote the $\equiv$ class containing x. For $x, y \in T$, define $x \rho y$ if $xT \cap yT \neq \Box$.

Lemma 1.4 *ρ is a congruence relation on T and T/ρ is a left zero semigroup.*

Proof. Clearly, ρ is a reflexive and symmetric. We first show ρ is a transitive. Suppose $\bar{x}\rho\bar{y}$ and $\bar{y}\rho\bar{z}$. So, $\bar{x}T \cap \bar{y}T \neq \Box$ and $\bar{y}T \cap \bar{z}T \neq \Box$. Thus, there exist elements r_1, r_2, r_3, and r_4 of R such that $\bar{x}\bar{r}_1 = \bar{y}\bar{r}_2$ and $\bar{y}\bar{r}_3 = \bar{z}\bar{r}_4$. Hence, $xr_1 \equiv yr_2$ and $yr_3 \equiv zr_4$. Thus, for $a \in R$, $xr_1r_3a = yr_2r_3a$ and $yr_3r_2a = zr_4r_2a$. However, by Lemma 1.3, $yr_2r_3a = yr_3r_2a$. Thus, $xr_1r_3a = zr_4r_2a$, and, hence, $\bar{x}\bar{r}_1\bar{r}_3\bar{a} = \bar{z}\bar{r}_4\bar{r}_2\bar{a}$. Thus, $\bar{x}\rho\bar{z}$. Next, we show ρ is compatible. Suppose $\bar{x}\rho\bar{y}$. Thus, there exist $t_1, t_2 \in R$ such that $\bar{x}\bar{t}_1 = \bar{y}\bar{t}_2$. Let $z \in T$. Thus, $\bar{x}\bar{t}_1\bar{z} = \bar{y}\bar{t}_2\bar{z}$. Hence, $xt_1zm = yt_2zm$ for all $m \in R$. So, again using Lemma 1.3, $xzt_1m = yzt_2m$. Hence, $\bar{x}\bar{z}\overline{t_1m} = \bar{y}\bar{z}\overline{t_2m}$ and, thus, $\bar{x}\bar{z}\rho\bar{y}\bar{z}$. Since $\bar{z}\bar{x}\bar{t}_1 = \bar{z}\bar{y}\bar{t}_2$, it follows that $\bar{z}\bar{x}\rho\bar{z}\bar{y}$. Thus, ρ is a congruence relation. Finally, we show T/ρ is a left zero semigroup. Let $a, b \in T$. Thus, $abT \subseteq aT$. Hence, $abT \cap aT \neq \Box$ and, thus, $ab\rho$ or $a\rho b\rho = a\rho$. q.e.d.

For $x, y\ inT$, define $x\lambda y$ if $Tx \cap Ty \neq \Box$.

Lemma 1.5 *λ is a congruence relation on T and T/λ is a right zero semigroup.*

Proof. The proof of Lemma 1.5 is the right left dual of the proof of Lemma 1.4.

Lemma 1.6 *T obeys the term condition.*

Proof. Let p be a term. Suppose $p(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, \bar{u}) = p(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_u, \bar{u})$. Then, $(p(a_1, a_2 \ldots, a_n, u))^2 = (p(b_1, b_2, \ldots, b_n, u))^2 = p(a_1, a_2, \ldots, a_n, u)$.

$p(b_1, b_2 \ldots, b_n, u)$. So, since R obeys the term condition, $(p(a_1, a_2, \ldots, a_n w))^2 = (p(b_1, b_2, \ldots, b_n w))^2 = p(a_1, a_2, \ldots, a_n, w)p(b_1, b_2, \ldots, b_n, w)$ for all $w \in R$. Thus, $p(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, \bar{w}) = p(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n, \bar{w})$. Let $x_0 \in T$ and let $M = x_0 T x_0$. Define an operation $*$ on M as follows $x_0 a x_0 * x_0 b x_0 = x_0 a b x_0$.

Lemma 1.7 *$(M, *)$ is an abelian cancellative semigroup.*

Proof. Suppose $u = u_1$ and $v = v_1$ where u, u_1, v, and v_1 are elements of M. Then, $u = x_0 a x_0, v = x_0 b x_0, u_1 = x_0 a_1 x_0$, and $v_1 = x_0 b_1 x_0$ where a, a_1, b, and b_1 are elements of T. Thus, $u * v = x_0 a b x_0$ and $u_1 * v_1 = x_0 a_1 b_1 x_0$. Since $x_0 a x_0 x_0 b x_0 = x_0 a_1 x_0 x_0 b_1 x_0$, using Lemma 3, $x_0^3 a b x_0 = x_0^3 a_1 b_1 x_0$. Let $z = x_0^3$. Hence $z a b x_0 = z a_1 b_1 x_0$. Thus, using Lemma 1.6, $x_0 a b x_0 = x_0 a_1 b_1 x_0$. Hence, $u * v = u_1 * v_1$. Thus, $(M, *)$ is a groupoid. Let $u = x_0 a x_0$, $v = x_0 b x_0$, and $w = x_0 c x_0$ where $a, b, c \in T$. Hence, $(u * v) * w = (x_0 a b x_0) * (x_0 c x_0) = x_0(ab)c x_0 = x_0 a(bc) x_0 = x_0 a x_0 * x_0 b c x_0 = x_0 a x_0 * (x_0 b x_0 * x_0 c x_0) = u * (v * w)$. Hence, $(M, *)$ is a semigroup. Furthermore, using Lemma 1.3, $u * v = x_0 a b x_0 = x_0 b a x_0 = v * u$. Finally, suppose $u * v = u * w$. Thus, $x_0 a b x_0 = x_0 a c x_0$. Using Lemma 1.6, $x_0 b x_0 = x_0 c x_0$. Thus, $v = w$. Hence, $(M, *)$ is an abelian cancellative semigroup.

Lemma 1.8 *Let $V = \{x_0 a x_0, a/\rho, a/\lambda : a \in T\}$. Then, V is a subsemigroup of $(M, *) X T/\rho X T/\lambda$ (algebraic direct product). In fact, V is a subdirect product of $(M, *)$, T/ρ, and T/λ. $a \to (x_0 a x_0, a/\rho, a/\lambda)$ defines an isomorphism of T onto V.*

Proof. Let $(x_0 a x_0, a/\rho, a/\lambda), (x_0 b x_0, b/\rho, b/\lambda) \in V$. Then, $(x_0 a x_0, a/\rho, a/\lambda)$ $(x_0 b x_0, b/\rho, b/\lambda) = (x_0 a b x_0, ab/\rho, ab/\lambda) \in V$. So, V is a subsemigroup of

$(M,*)XT/\rho XT/\lambda$. Clearly, V is a subdirect product of $(M,*)$, T/ρ, and T/λ. Next, we show $a \to (x_0ax_0, a/\rho, a/\lambda)$ defines a one–to–one mapping of T onto V. Suppose, $(x_0ax_0, a/\rho, a/\lambda) = (x_0bx_0, b/\rho, b/\lambda)$. Let, $x_0 = \bar{y}_0$, $a = \bar{a}_1$ and $b = \bar{b}_1$ for $y_0, a_1, b_1 \in R$. Thus, $\bar{y}_0\bar{a}_1\bar{y}_0 = \bar{y}_0\bar{b}_1\bar{y}_0$, $\bar{a}_1\rho\bar{b}_1$ and $\bar{a}_1\lambda\bar{b}_1$. Hence, $\bar{t}_0\bar{a}_1 = \bar{t}_1\bar{b}_1$ and $\bar{a}_1\bar{t}_2 = \bar{b}_1\bar{t}_3$ for some $t_0, t_1, t_2, t_3 \in R$. Using Lemma 1.6, $\bar{t}_0\bar{a}_1\bar{t}_0 = \bar{t}_0\bar{b}_1\bar{t}_0$. Hence, $\bar{t}_1\bar{b}_1\bar{t}_0 = \bar{t}_0\bar{b}_1\bar{t}_0$. Thus, again using Lemma 1.6, $\bar{t}_1\bar{a}_1 = \bar{t}_0\bar{a}_1$. Hence, $\bar{t}_1\bar{b}_1 = \bar{t}_1\bar{a}_1$. Furthermore, $\bar{t}_2\bar{a}_1\bar{t}_2 = \bar{t}_2\bar{b}_1\bar{t}_2$ by virtue of Lemma 1.6. Hence, $\bar{t}_2\bar{b}_1\bar{t}_3 = \bar{t}_2\bar{b}_1\bar{t}_2$. Thus, using Lemma 1.6, $\bar{b}_1\bar{t}_3 = \bar{b}_1\bar{t}_2$. Hence, $\bar{a}_1\bar{t}_2 = \bar{b}_1\bar{t}_2$. Thus, since $\bar{t}_1\bar{a}_1 = \bar{t}_1\bar{b}_1$ and $\bar{a}_1\bar{t}_2 = \bar{b}_1\bar{t}_2$, $mt_1\,a_1 = mt_1\,b_1$ for all $m \in R$ and $a_1t_2m = b_1t_2m$ for all $m \in R$. Thus, since R obeys the term condition, $a_1^2 = a_1b_1$ and $a_1b_1 = b_1^2$. Hence, $a_1^2 = b_1^2 = a_1b_1$. Thus, $\bar{a}_1 = \bar{b}_1$. So, $a = b$. Hence, $a \to (x_0ax_0, a/\rho, a/\lambda)$ defines an one–to–one mapping of T onto V. Thus, using the first sentence of this proof, $a \to (x_0ax_0, a/\rho, a/\lambda)$ defines an isomorphism of T onto V.

We identify T and V. Let $R_v = v \equiv -1$..

Lemma 1.9 *R_uR_v is a single element of R_{uv}.*

Proof. Let $a, a_1 \in R_u$ and $b, b_1 \in R_v$. Then, $a \equiv a_1$ implies $ab = a_1b$ and $b \equiv b_1$ implies $a_1b = a_1b_1$. So, $ab = a_1b_1$.

Write $R_uR_v = \phi(u,v)$. Thus, ϕ defines a mapping of VXV into R. Let $S = \{(x,u) : x \in R_u,\, u \in V\}$ under the multiplication $(x,u)(y,v) = (\phi(u,v), uv)$. Clearly, S is a groupoid and $x\alpha = (x,u)$ defines an isomorphism of R onto S. Hence, S is a semigroup obeying the term condition.

If p is a term in V, let $\bar{p}$ denote the term obtained by deleting the rightmost letter of p. For $p(\vec{a},u)(\vec{a} \in V^n, u \in V)$, let y_u denote the rightmost letter of $p(\vec{a}, u)$.

Lemma 1.10 *$\phi(\bar{p}(\vec{a}, u), y_u) = \phi(\bar{p}(\vec{c}, u), y_u)$ and $p(\vec{a}, u) = p(\vec{c}, u)$ implies $\phi(\bar{p}(\vec{a}, w), y_w) = \phi(\bar{p}(\vec{c}, w), y_w)$ for all $w \in V$.*

Proof. Let $q((x_1, a_1), \ldots, (x_n, a_n), (x, u))$ be a term in S. Thus, $q((x_1, a_1), \ldots, (x_n, a_n), (x, u)) = (\phi(\bar{p}(a_1, a_2, \ldots, a_n, u), y_u), p(a_1, a_2, \ldots, a_n, u))$. Hence, $q((x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n), (x, u)) = q((y_1, c_1), (y_2, c_2), \ldots, (y_n, c_n), (x, u)))$. Thus, since S obeys the term condition, $q((x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n), (y, w)) = q((y_1, c_1), (y_2, c_2), \ldots, (y_n, c_n), (y, w))$ for all $(y, w) \in S$. Thus, $\phi(\bar{p}(\vec{a}, w), y_w) = \phi(\bar{p}(\vec{c}, w), y_w)$ for all $w \in V$.

Lemma 1.11 *$\phi(uv, w) = \phi(u, vw)$ for all $u, v, w \in V$.*

Proof. Let $x \in X_u, y \in X_v$, and $z \in X_w$. Then, $(xy)z = \phi(uv, w)$ and $x(yz) = \phi(u, vw)$. Hence, $\phi(uv, w) = \phi(u, vw)$.

We will next give our construction of semigroups obeying the term condition.

Let G be an abelian group, I be a left zero semigroup, and J be a right zero semigroup. Let V be a subsemigroup of $GXIXJ$ (algebraic direct product). Let $\{X_v : v \in V\}$ be a collection of pairwise disjoint subsets and let $X = U(X_v : v \in V)$.

Let (X, V, ϕ) denote $\{(x, v) : x \in X_v,\ v \in V\}$ under the multiplication

$$(x, u)(y, v) = (\phi(u, v), uv)$$

where ϕ is a function from $V^2 \to X$ such that

1. $\phi(u,v) \in X_{uv}$

2. $\phi(uv,w) = \phi(u,vw)$ for all $u,v,w \in V$

3. $\phi(\bar{p}(\vec{a},u),y_u) = \phi(\bar{p}(\vec{b},u),y_u)$ and $p(\vec{a},u) = p(\vec{b},u)$

imply $\phi(\bar{p}(\vec{a},w),y_w) = \phi(\bar{p},(\vec{b},w),y_w)$ for all $w \in V$.

Lemma 1.12 *Let R be a semigroup obeying the term condition. Then $R \cong (X,V,\phi)$ for some X,V, and ϕ.*

Proof. Let $X_u = R_u$ for $u \in V$ and let $X = U(X_u : u \in V)$. Note that $(M,*)$ may be embedded in an abelian group G. Apply Lemma 8, Lemma 9, the comments following Lemma 9, Lemma 10, and Lemma 11.

Lemma 1.13 *(X,V,ϕ) is a semigroup obeying the term condition.*

Proof: By (1) and (2), (X,V,ϕ) is a semigroup. It is easily seen that G,I, and J obey the term condition. Thus, $GXIXJ$ (algebraic direct product) obeys the term condition. Hence, V obeys the term condition. Suppose $q((x_1,a_1),(x_2,a_2),\ldots,(x_n,a_n),(x,u)) = q((y_1,c_1),(y_2,c_2),\ldots,(y_n,c_n),(x,u))$. Thus, $\phi((a_1,a_2,\ldots,a_n,u),y_u) = \phi((c_1,c_2,\ldots,c_n,u),y_u)$ and $p(a_1,a_2,\ldots,a_n,u) = p(c_1,c_2,\ldots,c_n,u)$ (note the proof of Lemma 10). Hence, using (3) and the fact that V obeys the term condition, $q((x,a_1),(x_2,a_2),\ldots,(x_n,a_n),(y,w)) = q((y_1,c_1),(y_2,c_n),\ldots,(y_n,c_n),(y,w))$ for all $(y,w) \in (X,V,\phi)$. So, (X,V,ϕ) obeys the term condition.

Theorem 1.14 *A semigroup R obeys the term condition if and only if $R \cong (X,V,\phi)$ for some X,V, and ϕ.*

Proof. Combine Lemma 1.13 and Lemma 1.14.

2 Periodic Semigroups Obeying the Term Condition

In this section, we determine the structure of periodic semigroups obeying the term condition (Theorem 2.11).

We need the following definitions. Let V be a semigroup. An element $a \in V$ is said to be of finite order if there exists a positive integer $n(a)$ such that $a^{n(a)}$ is an idempotent. V is termed periodic if every element of V is of finite order. An element $a \in V$ is termed a regular element if there exists $x \in V$ such that $axa = a$. Let V_{Reg} denote the set of regular elements of V and let $E(V)$ denote the set of idempotents of V. V is termed regular if $V_{\text{Reg}} = V$ and V is called a band if $E(V) = V$. A rectangular band is a band obeying the additional law $xyz = xz$ for all $x, y, z \in V$. V is termed a right zero (left zero) semigroup if $xy = y (xy = x)$ for all $x, y \in V$.

In Lemmas 2.1 – 2.5, 2.7, 2.8, and 2.10 and in Remark 2.9, S will denote a semigroup obeying the term condition.

Lemma 2.1 *If $x, y \in S$ and $e \in E(S)$, then $xey = xy$.*

Proof. Choose the term $t(u, y) = uy$. Then, $t(e, ey) = e(ey) = ey = t(e, y)$. So, $t(x, ey) = t(x, y)$ or $xey = xy$.

Lemma 2.2 *If $E(S) \neq \Box$, $E(S)$ is a rectangular band and $E(S) \cong E(S)eXeE(S)$ for any fixed $e \in E(S)$. $E(S)e$ is a left zero semigroup and $eE(S)$ is a right zero semigroup.*

Proof. Let $e, f \in E(S)$. Using Lemma 2.1, $efe = e$. Hence, $efef = ef$. Using Lemma 2.1, $E(S)$ is a rectangular band. Fix an element $e \in E(S)$.

Then, it is easily checked that $E(S)e(eE(S))$ is a left (right) zero semigroup and $x\phi = (xe, ex)$ defines an isomorphism of $E(S)$ onto $E(S)eXeE(S)$. (see also, [6, page 115]).

Lemma 2.3 *Every regular element of S is contained in a subgroup of S.*

Proof. Let $a \in S_{\text{Reg}}$. Thus, $axa = a$ for some $x \in S$. Hence, $ax, xa \in E(S)$. Let $g = ax^2a = axxa \in E(S)$. Note, $axg = axax^2a = g$, $gax = axxaax = ax$, $xag = xaaxxa = xa$, and $gxa = ax^2axa = axxaxa = g$. Thus, $ga = (gax)a = axa = a$, $ag = a(xag) = axa = a$, $a(gxg) = axg = g$, and $(gxg)a = gxa = g$. Thus, a is contained in the group of units of gSg (i.e., the maximal subgroup of S with identity g).

Lemma 2.4 *S_{Reg} is a regular subsemigroup of S or $S_{Reg} = \square$.*

Proof. Let $a, b \in S_{\text{Reg}}$. Let $a^{-1}(b^{-1})$ be the group inverse of $a(b)$ in the maximal subgroup of S containing $a(b)$. (Lemma 2.3). Thus, using Lemma 2.1 and Lemma 2.2, $abb^{-1}a^{-1}ab = a(bb^{-1}a^{-1}a)b = ab$. Hence, $ab \in S_{\text{Reg}}$. Since $a \in S_{\text{Reg}}$ implies $a^{-1} \in S_{\text{Reg}}$, S_{Reg} is a regular subsemigroup of S.

Lemma 2.5 *$S_{Reg} \cong GXIXJ$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup or $S_{Reg} = \square$.*

Proof. For $g \in E(S)$, let H_g denote the maximal subgroup of S with identity g. Thus, using Lemma 2.3, $S_{\text{Reg}} = U(H_g : g \in E(S))$. Using Lemma 1.3, each H_g is an abelian group. Select and fix $e \in E(S)$. Let $I = E(S)e$, $J = eE(S)$, and $G = H_e$. Using Lemma 2.2, I is a left zero semigroup and J is a right zero semigroup. Let $a \in S_{\text{Reg}}$. Then, $a \in H_f$, say. Hence, using

Lemma 2.1, $a = faf = feaf = feaef = fe(eae)ef$. Let a^{-1} denote the group inverse of a in H_f. Thus, using Lemma 2.1, $(eae)(ea^{-1}e) = eaa^{-1}e = efe = e$ and $(ea^{-1}e)(eae) = ea^{-1}ae = efe = e$. Hence, $eae \in G$. So, $a = igj$ where $i \in I$ and $j \in J$. Suppose $a = i'g'j'$ where $i' \in I$, $g' \in G$, and $j' \in J$. Thus, $eigje = ei'g'j'e$. So, $g = g'$. Since $ia = i'a$, $if = i'f$. Thus, $ifi = i'fi$. Hence $i = i'i = i'$. Similarly, $j = j'$. Thus every element of S_{Reg} may be uniquely expressed in the form igj where $i \in I$, $g \in G$, and $j \in J$. Let $b \in S_{\text{Reg}}$. Then, $b = phq$ where $p \in I$, $g \in G$ and $q \in J$. Thus, using lemma 2.1 and Lemma 2.4, $ab = igjphq = ighq = ipghjq \in S_{\text{Reg}}$. Hence, $(igj)\phi = (i, g, j)$ defines an isomorphism of S_{Reg} onto $GXIXJ$.

Corollary 2.6 *A regular semigroup S obeys the term condition if and only if $S \cong GXIXJ$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup.*

Proof. Utilize Lemma 2.5, the proof of Lemma 1.13, and a routine calculation to establish regularity of GXIXJ.

Let F denote the set of elements of S of finite order.

Lemma 2.7 *F is a subsemigroup of S or $F = \Box$.*

Proof. Let $a, b \in F$. Thus, $a^r \in E(S)$ and $b^s \in E(S)$ for some positive integers r and s. Hence, using Lemma 1.3, $(ab)^{rs} = a^{rs}b^{rs} \in E(S)$. So, $ab \in F$.

Lemma 2.8 *Let $e \in E(S)$ and $a \in S$. If $ae \in F$, then $ae \in S_{Reg}$. If $ea \in F$, then $ea \in S_{Reg}$. In particular, if $a \in F$, ae and $ea \in S_{Reg}$.*

Proof. Let $b = ea$. So, $eb = e(ea) = ea = b$. Furthermore, $b^r \in E(S)$ for some positive integer r. Let $b^r = g$. Thus, $eg = g$. Hence, $ge = ege = e$. Thus, $geb = eb$. So, using Lemma 2.1, $gb = b$. Hence, $b^{r+1} = b$. Thus, using the proof of Lemma 2.5, $b \in H_{b^r} \subseteq S_{\text{Reg}}$. Dually, $ae \in S_{\text{Reg}}$. The last sentence of the lemma is a consequence of the first sentence and Lemma 2.7.

Remark 2.9 *Let $e \in E(S)$. Then, $ae \in F$ if and only if $a \in F$. Suppose $ae \in F$. Thus, $(ae)^{2k} = (ae)^k$ for some positive integer k. Thus, using Lemma 1.3, $a^{2k}e = e^k e$. Hence, $a^{2k}ea = a^k ea$. Thus, using Lemma 2.1, $a^{2k+1} = a^{k+1}$. Hence, $a \in F$.*

Lemma 2.10 *If $a, b \in F$, then $ab \in F \cap S_{Reg}$.*

Proof. Using Lemma 2.7, $ab \in F$. Let $e \in E(S)$. Then, using Lemma 2.1, Lemma 2.7, Lemma 2.8, and Lemma 2.4, $ab = aeb = (ae)(eb) \in S_{\text{Reg}}$. So, $ab \in F \cap S_{\text{Reg}}$.

Let us next state the main theorem (Theoerem 2.11) of this section which we now are in a position to prove.

Let G be an abelian group, let I, J, and X be sets, let ϕ be a function of X into G, let $A \to c_A$ be a function of X into I, and let $A \to d_A$ be a function of X into J. We denote

$$XU(GXIXJ))$$

under the multiplication

$$AoB = (A\phi B\phi,\ c_A,\ d_B) \quad \text{where } A, B \in X$$

$$A_0(g,i,j) = (A\phi g, c_A, j) \quad \text{where } A \in X,\ g \in G,\ i \in I,\ j \in J$$

$$(g,i,j)oA = (g \cdot A\phi,\ i, d_A) \quad \text{where } A \in X,\ g \in G,\ i \in I,\ j \in J$$

$$(g,i,j)o(h,k,s) = (gh,i,s) \quad \text{where } g,h \in G,\ i,k \in I,\ j,s \in J \text{ by } (X,G,I,J,\phi,c,d).$$

Theorem 2.11 *S is a periodic semigroup satisfying the term condition if and only if $S = (X,G,I,J,\phi,c,d)$ for some (X,G,I,J,c,d) with G periodic.*

Proof. Let S be a periodic semigroup obeying the term condition. Let $X = S - S_{\text{Reg}}$ and denote the elements of X by capital roman letters. Using Lemma 2.5, $S_{\text{Reg}} = GXIXJ$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup. Since $g \to (g, i_0, j_0)\,(i_0 \in I,\ j_0 \in J$, fixed) defines an isomorphism of G onto $GX\{i_0\}X\{j_0\}$, G is a periodic group. Let $(e,i,j) \in E(S_{\text{Reg}})$. Using Lemma 2.8, $A(e,i,j)$, $(e,i,j)\,A \in S_{\text{Reg}}$. Thus,

$$A(e,i,j) = (a',i',j')\,(a' \in G,\ i' \in I,\ j' \in J).$$

Hence,

$$A(e,i,j) = (A(e,i,j))\,(e,i,j) = (a',i',j')\,(e,i,j) = (a',i',j).$$

Suppose

$$A(e,i,s) = (a'',i'',s).$$

Thus,

$$A(e,i,j) = (A(e,i,s))\,(e,i,j) = (a'',i'',s)\,(e,i,j) = (a'',i'',j).$$

Hence

$$a' = a'' \text{ and } i' = i''.$$

Thus, we may write

$$A(e, i, j) = (A\phi_i,\ s\rho_A,\ j)$$

where $i \to \phi_i$ is a mapping of I into $F(X, G)$, the set of mappings of X into G and $A \to \rho_A$ is a mapping of X into $F(I, I)$. Similarly,

$$(e, i, j)A = (A\psi_j,\ i, j\lambda_A)$$

where $j \to \psi_j$ is a mapping of X into $F(X, G)$ and $A \to \lambda_A$ is a mapping of X into $F(J, J)$. Hence

$$\begin{aligned} A(g, i, j) &= (A(e, i, j))\,(g, i, j) \\ &= (A\phi_i, i\rho_A, j)(g, i, j) \\ &= (A\phi_i g, i\rho_A, j). \end{aligned}$$

Similarly,

$$(g, i, j)A = (g \cdot A\psi_j, i, j\lambda_A).$$

However,

$$((e, i, j)A)(e, k, s) = (A\psi_j, i, j\lambda_A)(e, k, s) = (A\psi_j, i, s)$$

and

$$(e, i, j)(A(e, k, s)) = (e, i, j)(A\phi_k, k\rho_A, s) = (A\phi_k, i, s).$$

So, $\psi_j = \phi_k$ for every $j \in J$ and $k \in I$. Let $j_0 \in J$ and $k_0 \in I$. Thus $\phi_k = \psi_{j_0}$ for all $k \in I$ and $\psi_j = \phi_{k_0}$ for all $j \in J$. Let $\psi_{j_0} = \phi_{k_0} = \phi$. Thus,

(1) $A(g, i, j) = (A\phi g, i\rho_A, j)$

(2) $(g, i, j)A = (gA\phi, i, j\lambda_A)$ where $\phi \in F(X, G)$.

Using Lemma 2.1,

$$A(e,i,j)B = AB.$$

Thus, using (1) and (2),

$$\begin{aligned}(A(e,i,j))A &= (A\phi, i\rho_A, j)A = (A\phi \cdot A\phi, i\rho_A, j\lambda_A)\\ &= AA.\end{aligned}$$

So, fix $i_0 \in I$ and $j_0 \in J$. Thus,

$$A(e,i,j)A = A(e,i_0,j_0)A.$$

Hence, $i\rho_A = i_0\rho_A$ for all $i \in I$ and $j\lambda_A = j_0\lambda_A$ for all $j \in J$. Let $i_0\rho_A = c_A \in I$ and $j_0\lambda_A = d_A \in J$. Thus, $(A \to c_A) \in F(X,I)$ and $(A \to d_A) \in F(X,J)$ and $i\rho_A = c_A$ for all $i \in I$ and $j\lambda_A = d_A$ for all $j \in J$.

Thus, using (1) and (2),

(3) $A(g,i,j) = (A\phi g, c_A, j)$

(4) $(g,i,j)A = (g \cdot A\phi, i, d_A)$

Furthermore, using (3) and (4)

(5) $AB = (A(e,i,j))B = (A\phi, c_A, j)B = (A\phi B\phi, c_A, d_A)$.

Conversely, let us consider $S = (G, I, J, \phi, c, d)$ where G is a periodic group. By a routine calculation, S is a semigroup. We next establish the term condition. Note, $AoB = (A\phi B\phi, c_A, d_A) = (A\phi, c_A, d_A)(B\phi, c_B, d_B)$; $A_0(g,i,j) = (A\phi g, c_A, j) = (A\phi, c_A, d_A)(g,i,j)$; and $(g,i,j)oA = (g,i,j)(A\phi, c_A, d_A)$. If $A \in X$, let $\bar{A} = (A\phi, c_A, d_A)$. If $(g,i,j) \in GXIXJ$, let $\overline{(g,i,j)} = (g,i,j)$. So, if $p(u, x_1, x_2, \ldots, x_n)$ is a term in S, then $p(u, x_1, x_2, \ldots, x_n) = \bar{p}(\bar{u}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ where $\bar{p}(\bar{u}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is corresponding term in

$GXIXJ$. As in the proof of Lemma 1.13, $GXIXJ$ obeys the term condition. Hence, as is easily seen, S obeys the term condition. Since $A^k = ((A\phi)^k, c_A, d_A)$, $(g, i, j)^k = (g^k, i, j)$ for any positive integer $k \geq 2$ and G is a periodic group, it easily follows that S is a periodic semigroup.

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