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A NOTE ON THE SERIES PRESENTATION OF SPHERICAL FUNCTIONS OF THE SECOND KIND

By

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ABSTRACT.

In this paper we have found a new series representation of the spherical functions of the second kind.

INTRODUCTION.

Some series representations of the spherical functions $Q_\nu^\mu(z)$ are known when $|Re(z)| > 1$. See [1], [2, pp 131-139] and [4, pp 999-1012]. In this paper we have found a new representation of the function $Q_\nu^\mu(z)$ for $\frac{3}{2\sqrt{2}} < Re\, z < \infty$. The new representation is useful in describing the asymptotic behaviour of the function for large $|\nu|$.

THEOREM.

Let ν and $\nu+\mu$ be non-integers and $\nu+1 > |\mu|$. Then, for $0 < x < \frac{1}{\sqrt{2}}$,

$$Q_\nu^\mu[\frac{1}{2}(x+x^{-1})] = \sqrt{\pi}\frac{\sin[(\nu+\mu)\pi]}{\sin(\nu\pi)}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}F(\nu+\mu+1,\nu-\mu+1;\nu+3/2;\, -(\frac{x^2}{1-x^2})(\frac{x}{1-x^2})^{\nu+1}$$

Proof: Let us define the integral function $I_\mu(\nu,b;p)$ by

$$I_\mu(\nu,b;p) = L\{x^\nu K_\mu(bx);p\} \tag{1}$$

where L is the Laplace transform operator [3, p 129]. Then, according to [3, p 198]

$$I_\mu(\nu,b;p) = \frac{\sin(\nu\pi)\Gamma(\nu-\mu+1)}{\sin[(\mu+\nu)\pi](p^2-b^2)^{(\nu+1)/2}}Q_\nu^\mu(\frac{p}{\sqrt{p^2-b^2}}),\ p > -b. \tag{2}$$

We assume that ν and $\mu+\nu$ in (2) are non-integers. The substitution $\beta = 2\sqrt{2b}$ and $\alpha = a$ in [4, p 725] leads to

$$\int_0^\infty x^{-1/2}\exp(-b/x-ax)K_\alpha(b/x)dx = \sqrt{4\pi}a^{-1/2}K_{2\alpha}(2\sqrt{2ab}),\ a > 0,\, b > 0. \tag{3}$$

We can write (3) in the operational form

$$(4) \qquad L\{x^{-1/2}\exp(-b/x)K_\alpha(b/x);a\} = \sqrt{4\pi}a^{-1/2}K_{2\alpha}(2\sqrt{2}ab)$$

By using the property [3, p 132]

$$L\{x^{\nu-1}f(1/x);p\} = p^{-\nu/2}\int_0^\infty x^{\nu/2}J_\nu(2\sqrt{px})F(x)dx$$

of the Laplace transformation, we get from (4) that

$$(5) \qquad \int_0^\infty t^\nu J_\nu(2\sqrt{pt})K_\alpha(2\sqrt{b}t)dt = \frac{p^{\nu/2}}{4\sqrt{\pi}}I_{\alpha/2}(\frac{2\nu-1}{2},\frac{b}{2};p+b/2)$$

But we also have [4, p 693]

$$(6) \quad \int_0^\infty t^\nu J_\nu(2\sqrt{p}\,t)K_\alpha(2\sqrt{b}\,t)dt = \frac{p^{\nu/2}\Gamma(\frac{2\nu+\alpha+1}{2})\Gamma(\frac{2\nu-\alpha+1}{2})}{4b^{(2\nu+1)/2}\Gamma(\nu+1)}F(\frac{2\nu+\alpha+1}{2},\frac{2\nu-\alpha+1}{2};\nu+1;-p/b),$$

$$2\nu+1>|\alpha|.$$

Therefore, from (5) and (6) we get

$$(7) \quad I_{\alpha/2}(\frac{2\nu-1}{2},\,b/2;\,p+b/2) = \sqrt{\pi}\frac{\Gamma(\frac{2\nu+\alpha+1}{2})\Gamma(\frac{2\nu-\alpha+1}{2})}{b^{(2\nu+1)/2}\Gamma(\nu+1)}F(\frac{2\nu+\alpha+1}{2},\frac{2\nu-\alpha+1}{2};\nu+1;-p/b),$$

$$2\nu+1>|\alpha|.$$

Replacing $\alpha/2$ by μ and $(2\nu-1)/2$ by ν in (7) we get

$$(8) \quad I_\mu(\nu,b/2;p+b/2) = \sqrt{\pi}\frac{\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}{b^{\nu+1}\Gamma(\nu+3/2)}F(\nu+\mu+1,\nu-\mu+1;\nu+3/2;-p/b),$$

$$\nu+1>|\mu|.$$

From (2) and (8) we get

$$(9) \quad Q_\nu^\mu(\frac{p+b/2}{\sqrt{p(p+b)}}) = \sqrt{\pi}\frac{\sin[(\nu+\mu)\pi]}{\sin(\nu\pi)}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}\left\{\frac{p(p+b)}{b^2}\right\}^{(\nu+1)/2}F(\nu+\mu+1;\nu-\mu+1;p+3/2;-p/b),$$

$$\nu+1>|\mu|.$$

Let us take

$$(10) \qquad p = b\tan^2\phi.$$

Since $p>-b$, it follows that $0<\phi<\pi/4$. From (9) and (10) we get

$$(11) \qquad Q_\nu^\mu[\frac{1}{2}(\sin\phi+\csc(\phi)] = \sqrt{\pi}\frac{\sin[(\nu+\mu)\pi]}{\sin(\nu\pi)}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}(\tan\phi\sec\phi)^{\nu+1}\times$$

$$F(\nu+\mu+1, \nu-\mu+1; \nu+3/2; -\tan^2\phi).$$

Substituting $x = \sin\phi$ in (11) we get the proof of the theorem.

COROLLARY 1. $Q_\nu^\mu(-z) = -e^{i\pi\nu}\, Q_\nu^\mu(z)$ for $\frac{3}{2\sqrt{2}} < Re\ z < \infty$.

Proof: This follows from (1) when we take $z = \frac{1}{2}(x + x^{-1})$.

COROLLARY 2. For $0 < x < 1/\sqrt{2}$

$$Q_\nu[(\frac{x+x^{-1}}{2})] = \sqrt{\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}F(\nu+1, \nu+1; \nu+3/2; -\frac{x^2}{1-x^2})(\frac{x}{1-x^2})^{\nu+1}.$$

Proof: This follows from (1) when we take $\mu = 0$ and use the fact that $Q_\nu^0(z) = Q_\nu(z)$.

COROLLARY 3.

(12) $$Q_\nu^\mu(\cosh\alpha) = -e^{i\pi\nu}\frac{\sqrt{\pi}}{2^{\nu+1}}\frac{\sin[(\nu+\mu)\pi]}{\sin(\nu\pi)}\frac{\Gamma(\mu+\mu+1)}{\Gamma(\nu+3/2)}\times$$

$$F(\nu+\mu+1; \nu-\mu+1; \nu+3/2; \tfrac{1}{1-e^{2\alpha}})(\operatorname{csc} h\alpha)^{\nu+1}$$

where ν and $\nu+\mu$ are not integers and $\alpha > \frac{\ln 2}{2}$.

(P1004)

Proof: Put $x = e^{-\alpha}$ in (1). We get the proof for $\alpha > \frac{\ln 2}{2}$.

COROLLARY 4. For $\nu > -1$ and $\alpha > \frac{\ln 2}{2}$

(13) $$Q_\nu[\cosh\alpha] = -e^{i\pi\nu}\frac{\sqrt{\pi}}{\sqrt{2\sinh\alpha}}e^{-(\nu+1/2)\alpha}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}F(1/2, 1/2; \nu+3/2; \frac{1}{1-e^{2\alpha}})$$

PROOF: This follows from (12) when we take $\mu = 0$ and use the transformation formula [4, p 1043]

$$F(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta; \gamma; z).$$

COROLLARY 5.

$$|Q_\nu(\cosh\alpha)| = \frac{\sqrt{\pi}}{\sqrt{2\nu\sinh\alpha}}e^{-(\nu+1/2)\alpha}[1 + 0(|\nu|^{-1})],\quad |\nu| \to \infty,\ \frac{\ln 2}{2} < \alpha < \infty.$$

PROOF. This follows from (13) when we use the fact that $\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)} \to \frac{1}{\sqrt{\nu}}$ as $\nu \to \infty$.

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