King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 122

September 1991

Comments on a Paper of El-Qallali

Warne, R.J.

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Comments on a Paper of El-Qallali

R.J. Warne

Let S be a semigroup and let L and R be Green's Relations on S. Define $(a,b) \in L^*$ if $(a,b) \in S \times S$ and $(a,b) \in L$ in some oversemigroup of S. R^* is defined dually. El-Qallali and Fountain [3] term a semigroup S quasi-adequate if each L^*-class of S and each R^*-class of S contains an idempotent and $E(S)$, the set of idempotents of S, is a subsemigroup. If, in addition, each L^*-class of S contains precisely one idempotent, El-Qallali [4] terms S an L^*-unipotent semigroup. The purpose of this note is to extend El- Qallali's main result (Theorem 4.6) of [4] from L^*-unipotent semigroups to quasi-adequate semigroups (Theorem 4). More precisely, El-Qallali's theorem states: Let E be a band and $E = \cup(E_\alpha : \alpha \in Y)$ be its maximal semilattice decomposition. Suppose that for each $\alpha \in Y$, E_α is a right zero semigroup and to each $\alpha \in Y$ assign a cancellative monoid M_α such that $M_\alpha \cap M_\beta = \Box$ if $\alpha \neq \beta$. Further, suppose that for $\alpha > \beta$ there exists a homomorphism

$$\pi_{\alpha,\beta} : M_\alpha \to M_\beta$$

such that if $\alpha > \beta > \nu$, then $\pi_{\alpha,\beta}\,\pi_{\beta,\nu} = \pi_{\alpha,\nu}$. Set $\pi_{\alpha,\alpha}$ equal to the identity automorphism on M_α. Let $S = \cup(E_\alpha \times M_\alpha : \alpha \in Y)$ and define a multiplication on S by $(e,x)(f,y) = (ef,\, x\pi_{\alpha,\alpha\beta}\, y\pi_{\beta,\alpha\beta})$ for any $(e,x) \in E_\alpha \times M_\alpha$ and $(f,y) \in E_\beta \times M_\beta$. Then, S is an L^*-unipotent semigroup which is a band of cancellative monoids. Conversely, any L^*-unipotent semigroup which is a band of cancellative monoids can be constructed in this manner.

In Theorem 4, we show that we may omit "suppose that for each $\alpha \in Y$, E_α is a right zero semigroup" and that "L^*-unipotent semigroup" may be replaced by "quasi- adequate semigroup" in El-Qallali's theorem.

A semigroup S is a band of cancellative monoids if $S = \cup(S_\alpha : \alpha \in B)$ where S_α is a cancellative monoid, $S_\alpha \cap S_\beta = \Box$ if $\alpha \neq \beta$, B is a band (idempotent semigroup), and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. For definitions not given here see [1] or [6].

In the case of regular semigroups, Theorem 4 reduces to Yamada's characterization of orthodox semigroups which are bands of groups [9].

We will use the following fact without explicit mention: $(a, b) \in L^*$ if and only if for all $x, y \in S^1$ (S with an appended identity) $ax = ay$ if and only if $bx = by$. The dual result is valid for R^* (see [7], for example). Let $H^* = L^* \cap R^*$.

We will term a semigroup S a quasi-orthogroup if $E(S)$ is a subsemigroup and each H^*-class of S contains an idempotent. To prove Theorem 4, we will first need the "gross" structure of quasi-orthogroups (Lemma 1). The following terminology will be used in the proof of Lemma 1. Let S be a semigroup and let I and J be sets and let $P\colon J \times I \to S$ with $(j, i)P = p_{ji}$. Let $M(S, I, J, P)$ denote $S \times I \times J$ under the multiplication $(a, i, j)(b, r, s) = (ap_{jr}b, i, s)$. We term $M(S, I, J, P)$ a Rees matrix semigroup over S with entries in P. We also need the following notation to state Lemma 1. Let S be a semigroup. For $a \in S$, $L_a^*(S)$ will denote the L^*-class of S containing a.

Lemma 1 *A semigroup S is a quasi-orthogroup if and only if S is a semi-*

lattice $Y = S/J^*$ (see [5] for definition of J^*. If S is a regular semigroup, $J^* = J$) of semigroups $(S_y : y \in Y)$ where $S_y = T_y \times E(S_y)$ where T_y is a cancellative monoid and $E(S_y)$ is a rectangular band, $L^*_a(S) = L^*_a(S_y)$ and $R^*_a(S) = R^*_a(S_y)$ for $y \in Y$ and $a \in S_y$ and $E(S)$ is a semilattice Y of rectangular bands $(E(S_y) : y \in Y)$.

Proof. Utilizing [5, Theorem 6.8 and its proof and Corollary 5.2], we obtain the above theorem (except the statement about $E(S)$) with $S_y = M(T_y, I_y, J_y, P_y)$, a Rees matrix semigroup over a cancellative monoid T_y where the entries of P_y are units U of T_y. As is easily shown, [1, Lemma 3.6] is valid for the above matrix semigroups if we require the mappings to have range U. Using this lemma we may "normalize" P_y such that all the elements in a given row and a given column are the identity e of T_y. Then using the assumption that $E(S)$ is a subsemigroup, we may show $p_{ji} = e$, the identity of T_y, for all $j \in J_y$ and $i \in I_y$. Hence $M(T_y, I_y, J_y, P_y) = T_y \times E(S_y)$ where $E(S_y)$ is a rectangular band.

In the proof of Theorem 4, we will need a quasi-orthogroup analogue to the minimum inverse semigroup congruence of an orthogroup (an orthodox union of groups) (Proposition 3).

For (g,i,j), $(h,r,s) \in S$, a quasi-orthogroup define $(g,i,j)\delta(h,r,s)$ if (g,i,j), $(h,r,s) \in S_y$, say, and $g = h$.

We show (Proposition 3) that δ is the smallest good congruence on S (aL^*b implies $a\delta L^* b\delta$ and aR^*b implies $a\delta R^*b$)

such that $E(S/\delta)$ is a semilattice and, furthermore, that S/δ is a strong semilattice Y of the cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 1).

To show δ is a congruence relation, we will need the following lemma.

Lemma 2 *Let $S_y = T_y \times E_y$ and $S_z = T_z \times I_z \times J_z$ where T_y and T_z are cancellative monoids, E_y is a rectangular band, I_z is a left zero semigroup and J_z is a right zero semigroup. Assume there exists*

a) A left representation $A \to \lambda_A$ of S_y by transformations of I_z.

b) A right representation $A \to \rho_A$ of S_y by transformation of J_z.

c) A homomorphism ϕ of T_y into T_z.

Define a binary operation on $S_y \cup S_z$ extending the given ones on S_y and S_z by defining products of $A = (a, e) \in S_y$ and $(b, i, j) \in S_z$ as follows

$$(a, e)(b, i, j) = (a\phi b,\ \lambda_A i, j)$$

$$(b, i, j)(a, e) = (b(a\phi), i, j\rho_A).$$

Then $S_y \cup S_z$ becomes a semigroup with S_z as an ideal.

Conversely, every possible binary associative operation on $S_y \cup S_z$ extending the given ones on S_y and S_z, and such that S_z is an ideal, can be constructed in the above manner.

Proof. Lemma 2 has been established by Clifford [2, Lemma 2.5] in the case T_y and T_z are groups. Clifford's proof is easily seen to be valid when T_y and T_z are just cancellative monoids.

Proposition 3 *Let S be a quasi-orthogroup. Then, δ is the smallest good congruence on S such that $E(S/\delta)$ is a semilattice. Furthermore, S/δ is isomorphic to a semigroup $T = \bigcup(T_y : y \in Y)$ which is a strong semilattice Y of the cancellative monoids $(T_y : y \in Y)$ under the isomorphism $(g,i,j)\delta\tau = g$.*

Proof. We first show that δ is a congruence relation on S. Let $\bar{\delta}$ denote the smallest congruence relation on S containing δ. Suppose $a\bar{\delta}b$. Then, there exists $a = a_1, a_2, \ldots, a_n = b \in S$ such that $a_i = x_i u_i y_i$, $a_{i+1} = x_i v_i y_i$ where $x_i, y_i \in S^1$ and $(u_i, v_i) \in \delta$ for $1 \leq i \leq n-1$. Suppose $x_i = (w,p,q)_y \in S_y, y_i = (h,c,d)_z \in S_z$, $u_i = (g,m,n)_t \in S_t$ and $v_i = (g,r,s)_t \in S_t$. Let $\theta = yzt$. Hence, $a_i = (A,a,b)_\theta \in S_\theta$ and $a_{i+1} = (B,e,f)_\theta \in S_\theta$, say. So,

$$(A,a,b)_\theta = (w,p,q)_y(g,m,n)_t(h,c,d)_z$$

$$(B,e,f)_\theta = (w,p,q)_y(g,r,s)_t(h,c,d)_z.$$

If we multiply both of the above equations on the left and the right by $(e,a,b)_\theta$ where e is the identity of T_θ, we obtain

$$\left.\begin{aligned}(A,a,b)_\theta &= (\bar{w},\bar{p},\bar{q})_\theta(g,m,n)_t(\bar{h},\bar{c},\bar{d})_\theta \\ (B,a,b)_\theta &= (\bar{w},\bar{p},\bar{q})_\theta(g,r,s)_t(\bar{h},\bar{c},\bar{d})_\theta.\end{aligned}\right\}. \tag{1}$$

Using Lemma 2, we obtain

$$\left.\begin{aligned}(\bar{w},\bar{p},\bar{q})_\theta(g,m,n)_t &= (\bar{w}(g\phi_{t,\theta}),\ \bar{p}, \bar{q}\rho(g,m,n)_t)_\theta \\ (\bar{w},\bar{p},\bar{q})_\theta(g,r,s)_t &= (\bar{w}(g\phi_{t,\theta}),\ \bar{p}, \bar{q}\rho(g,r,s)_t)_\theta\end{aligned}\right\} \tag{2}$$

where $\phi_{t,\theta}$ is a homomorphism of T_t into T_θ and $A \to \rho_A$ is a right representation of S_t by transformations of J_θ. Combining (1) and (2), we obtain $A = B$. Thus, $a_i\delta a_{i+1}$ for $1 \leq i \leq n-1$. Thus, $a\delta b$. Hence, $\delta = \bar{\delta}$, and thus

δ is a congruence relation on S. Since $(g,i,j)\delta\tau = g$ defines a one-to-one mapping of S/δ onto $T = \cup(T_y : y \in Y)$, it is easily checked that T becomes a semilattice Y of cancellative monoids $(T_y : y \in Y)$ under the multiplication $ab = (a\tau^{-1}b\tau^{-1})\tau$ and τ defines an isomorphism of S/δ onto T. For $a \in T_z$ and $z \geq y$ define $aC_{z,y} = ae_y$ where e_y is the identity of T_y. It is routine to verify that $C_{z,y}$ is a homomorphism of T_z into T_y; $C_{y,y}$ is the identity automorphism of T_y; and for $a \in T_q$, $b \in T_r$, $ab = aC_{q,qr}bC_{r,qr}$. Since $e_ye_z = e_{yz}$ for all $y, z \in Y$, $C_{y,z}C_{z,w} = C_{y,w}$ for $y \geq z \geq w$. So, T is a strong semilattice Y of the T_y. Suppose aR^*b. Then, $a, b \in S_y$, say. Hence, $a\delta$, $b\delta \in T_y$. Thus, using the fact T is a strong semilattice Y of the T_y, $a\delta R^* b\delta$. Similarly, aL^*b implies $a\delta L^* b\delta$. So, δ is a good congruence. (So, each H^*-class of S/δ contains an idempotent). Clearly, $E(S/\delta)$ is a semilattice. If λ is another such congruence, use the fact $E(S_y)\lambda$ is a singleton to show $\delta \leq \lambda$.

Remark The statement and proof of Proposition 3 is contained in the proof of [8, Theorem 6].

We are now in a position to establish Theorem 4.

Theorem 4 *Let E be a band and $E = \cup(E_\alpha : \alpha \in Y)$ be its maximal semilattice decomposition. To each $\alpha \in Y$ assign a cancellative monoid M_α such that $M_\alpha \cap M_\beta = \square$ if $\alpha \neq \beta$. Furthermore, suppose that for $\alpha > \beta$ there exists a homomorphism*

$$\pi_{\alpha,\beta} : M_\alpha \to M_\beta$$

such that if $\alpha > \beta > \nu$ then $\pi_{\alpha,\nu} = \pi_{\alpha,\beta}\pi_{\beta,\nu}$. Set $\pi_{\alpha,\alpha}$ equal to the identity automorphism on M_α. Let $S = \cup((E_\alpha \times M_\alpha) : \alpha \in Y)$ and define a multiplication on S by $(e,x)(f,y) = (ef, x\pi_{\alpha,\alpha\beta}y\pi_{\beta,\alpha\beta})$ for any $(e,x) \in$

$E_\alpha \times M_\alpha$, $(f,y) \in E_\beta \times M_\beta$. Then, S is a quasi-adequate semigroup which is a band of cancellative monoids. Conversely, any quasi-adequate semigroup which is a band of cancellative monoids can be constructed in this manner.

Proof. Let S be a quasi-adequate semigroup which is a band of cancellative monoids. Then, using [4, Lemma 4.1], each H^*-class of S contains an idempotent and H^* is a congruence relation on S. Thus, S is a quasi-orthogroup on which H^* is a congruence relation. By Proposition 3, S/δ is isomorphic to $T = \cup(T_y : y \in Y)$, a strong semilattice Y of cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 1). Thus, $T_y \cap T_z = \square$ if $y \neq z$ and for $y > z$, there exists a homomorphism $\pi_{y,z} : T_y \to T_z$ such that for $y > z > w$, $\pi_{y,z}\pi_{z,w} = \pi_{y,w}$ and $\pi_{y,y}$ is the identity automorphism on T_y for $y \in Y$. Furthermore, $ab = a\pi_{y,yz}\, b\pi_{z,yz}$ for $a \in T_y$ and $b \in T_z$. Let $P = \cup(E(S_y) \times T_y) : y \in Y)$ under the multiplication $(e,x)(f,q) = (ef, xq)$ where $(e,x) \in E(S_y) \times T_y$ and $(f,q) \in E(S_z) \times T_z$. We will show that $(g,i,j)_y\lambda = ((e_y,i,j)_y, g)$ where $(g,i,j)_y \in S_y$ and e_y is the identity of T_y defines an isomorphism of S onto P. Clearly, λ defines a one-to- one mapping of S onto P. We will next show that λ defines a homomorphism of S onto P. Let $(g,i,j)_y \in S_y$ and $(h,r,s)_z \in S_z$. Thus, $(g,i,j)_y H^*(e_u,r,s)_u$, say. Hence, it is easily checked that $u = y$. Using the fact that H^* is a congruence relation, $(g,i,j)_y H^*(e_y,i,j)_y$. Similarly, $(h,r,s)_z H^*(e_z,r,s)_z$. Suppose $(g,i,j)_y(h,r,s)_z = (t,k,p)_{yz}$. Hence, $(t,k,p)_{yz} H^*(e_y,i,j)_y(e_z,r,s)_z$. Thus, $(e_y,i,j)_y(e_z,r,s)_z = (e_{yz},k,p)_{yz}$. Hence, $((g,i,j)_y(h,r,s)_z)\lambda = (t,k,p)_{yz}\lambda = ((e_{yz},k,p)_{yz}, t)$ while $(g,i,j)_y\lambda(h,r,s)_z\lambda = ((e_y,i,j)_y, g)((e_z,r,s)_z, h) = ((e_{yz},k,p)_{yz}, gh)$. Using Proposition 3 and its notation, $(g,i,j)_y\delta(h,r,s)_z\delta = (t,k,p)_{yz}\delta$. So, $(g,i,j)_y\delta\tau \cdot (h,r,s)_z\delta\tau = (t,k,p)_{yz}\delta\tau$. Thus, $gh = t$. Hence, λ

defines an isomorphism of S onto P. Thus, we have established the converse of Theorem 4.

Next, we establish the direct part of Theorem 4. It is easily checked that S is a semigroup. Let $e \in E_\alpha$. Then, $\{e\} \times M_\alpha$ is a cancellative monoid with identity (e, e_α) where e_α is the identity of M_α. Let $M_e = \{e\} \times M_\alpha$. Thus, $S = \cup(M_e : e \in E)$ and S is the band E of cancellative monoids $(M_e : e \in E)$. Clearly, $E(S) = \{(e, e_\alpha) : e \in E_\alpha, \alpha \in Y\}$. Since $e_\alpha e_\beta = e_\alpha \pi_{\alpha,\alpha\beta} e_\beta \pi_{\beta,\alpha\beta} = e_{\alpha\beta} e_{\alpha\beta} = e_{\alpha\beta}$, it follows that $E(S)$ is a subsemigroup of S. To complete the proof, we will show that $(e, g)H^*(e, e_y)$ where $e \in E_y$ and $g \in M_y$. We will show that $(e, g)L^*(e, e_y)$. Dually, $(e, g)R^*(e, e_y)$. Suppose $(e, g)(a, b) = (e, g)(c, d)$ where $a \in E_z$, $b \in M_z$, $c \in E_t$ and $d \in M_t$. Thus, $ea = ec$ and $gb = gd$. So, $g\pi_{y,yz} b\pi_{z,yz} = g\pi_{y,yt} d\pi_{t,yt}$ with $yz = yt$. Hence, $b\pi_{y,yz} = d\pi_{t,yt}$. Thus, $e_y \pi_{y,yz} b\pi_{y,yz} = e_y \pi_{y,yt} d\pi_{t,yt}$. Hence, $e_y b = e_y d$. So, $(e, e_y)(a, b) = (e, e_y)(c, d)$.

Acknowledgement: The author gratefully acknowledges the support of King Fahd University of Petroleum and Minerals.

Bibliography

1. A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys Amer. Math. Soc. 7, Vol I (AMS, Providence, R.I., 1961).

2. A.H. Clifford, *A Structure Theorem For Orthogroups*, J. Pure and Appl. Algebra 8 (1976), 23-50.

3. A. El-Qallali and J.B. Fountain, *Quasi-Adequate Semigroups*, Proc. Roy. Soc. Edinburgh, Series A, 91(1981), 91-99.

4. Abdulsalam El- Qallali, *L^*-unipotent semigroups*, J. Pure Appl. Algebra 62(1989), 9-33.

5. J.B. Fountain, *Abundant Semigroups*, Proc. London Math. Soc. (3), 44 (1982), 103-129.

6. J.M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).

7. D.B. McAlister, *One-to-one Partial Right Translations of a Right Cancellative Semigroup*, J. Algebra 43(1976), 231-251.

8. R.J. Warne, *Orthogroups and Generalizations*, Proc. of the International Symposium on The Semigroup Theory and its Related Fields, Kyoto, Japan, 1990, 233-241.

9. M. Yamada, *Strictly Inverse Semigroups*, Bull. Shimane Univ. 13(1964), 128-138.

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia