Comments on a Paper of El-Qallali

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Let $S$ be a semigroup and let $L$ and $R$ be Green’s Relations on $S$. Define $(a,b) \in L^*$ if $(a,b) \in S \times S$ and $(a,b) \in L$ in some oversemigroup of $S$. $R^*$ is defined dually. El-Qallali and Fountain [3] term a semigroup $S$ quasi-adequate if each $L^*$-class of $S$ and each $R^*$-class of $S$ contains an idempotent and $E(S)$, the set of idempotents of $S$, is a subsemigroup. If, in addition, each $L^*$-class of $S$ contains precisely one idempotent, El-Qallali [4] terms $S$ an $L^*$-unipotent semigroup. The purpose of this note is to extend El-Qallali’s main result (Theorem 4.6) of [4] from $L^*$-unipotent semigroups to quasi-adequate semigroups (Theorem 4). More precisely, El-Qallali’s theorem states: Let $E$ be a band and $E = \cup(E_\alpha : \alpha \in Y)$ be its maximal semilattice decomposition. Suppose that for each $\alpha \in Y$, $E_\alpha$ is a right zero semigroup and to each $\alpha \in Y$ assign a cancellative monoid $M_\alpha$ such that $M_\alpha \cap M_\beta = \emptyset$ if $\alpha \neq \beta$. Further, suppose that for $\alpha > \beta$ there exists a homomorphism

$$\pi_{\alpha,\beta} : M_\alpha \to M_\beta$$

such that if $\alpha > \beta > \nu$, then $\pi_{\alpha,\beta} \pi_{\beta,\nu} = \pi_{\alpha,\nu}$. Set $\pi_{\alpha,\alpha}$ equal to the identity automorphism on $M_\alpha$. Let $S = \cup(E_\alpha \times M_\alpha : \alpha \in Y)$ and define a multiplication on $S$ by $(e,x)(f,y) = (ef, x\pi_{\alpha,\beta} y \pi_{\beta,\alpha})$ for any $(e,x) \in E_\alpha \times M_\alpha$ and $(f,y) \in E_\beta \times M_\beta$. Then, $S$ is an $L^*$-unipotent semigroup which is a band of cancellative monoids. Conversely, any $L^*$-unipotent semigroup which is a band of cancellative monoids can be constructed in this manner.
In Theorem 4, we show that we may omit "suppose that for each $\alpha \in Y$, $E_{\alpha}$ is a right zero semigroup" and that "$L^*$-unipotent semigroup" may be replaced by "quasi- adequate semigroup" in El-Qallali's theorem.

A semigroup $S$ is a band of cancellative monoids if $S = \cup(S_\alpha : \alpha \in B)$ where $S_\alpha$ is a cancellative monoid, $S_\alpha \cap S_\beta = \Box$ if $\alpha \neq \beta$, $B$ is a band (idempotent semigroup), and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. For definitions not given here see [1] or [6].

In the case of regular semigroups, Theorem 4 reduces to Yamada's characterization of orthodox semigroups which are bands of groups [9].

We will use the following fact without explicit mention: $(a, b) \in L^*$ if and only if for all $x, y \in S'$ (S with an appended identity) $ax = ay$ if and only if $bx = by$. The dual result is valid for $R^*$ (see [7], for example). Let $H^* = L^* \cap R^*$.

We will term a semigroup $S$ a quasi-orthogroup if $E(S)$ is a subsemigroup and each $H^*$-class of $S$ contains an idempotent. To prove Theorem 4, we will first need the "gross" structure of quasi-orthogroups (Lemma 1). The following terminology will be used in the proof of Lemma 1. Let $S$ be a semigroup and let $I$ and $J$ be sets and let $P : J \times I \to S$ with $(j, i)P = p_{ji}$. Let $M(S, I, J, P)$ denote $S \times I \times J$ under the multiplication $(a, i, j)(b, r, s) = (ap_{ji}, b, i, s)$. We term $M(S, I, J, P)$ a Rees matrix semigroup over $S$ with entries in $P$. We also need the following notation to state Lemma 1. Let $S$ be a semigroup. For $a \in S$, $L^*_a(S)$ will denote the $L^*$-class of $S$ containing $a$.

Lemma 1 A semigroup $S$ is a quasi-orthogroup if and only if $S$ is a semi-
lattice $Y = S/J^*$ (see [5] for definition of $J^*$). If $S$ is a regular semigroup, $J^* = J$) of semigroups $(S_y : y \in Y)$ where $S_y = T_y \times E(S_y)$ where $T_y$ is a cancellative monoid and $E(S_y)$ is a rectangular band, $L_y^*(S) = L_y^*(S_y)$ and $R_y^*(S) = R_y^*(S_y)$ for $y \in Y$ and $a \in S_y$ and $E(S)$ is a semilattice $Y$ of rectangular bands $(E(S_y) : y \in Y)$.

Proof. Utilizing [5, Theorem 6.8 and its proof and Corollary 5.2], we obtain the above theorem (except the statement about $E(S)$) with $S_y = M(T_y, I_y, J_y, P_y)$, a Rees matrix semigroup over a cancellative monoid $T_y$ where the entries of $P_y$ are units $U$ of $T_y$. As is easily shown, [1, Lemma 3.6] is valid for the above matrix semigroups if we require the mappings to have range $U$. Using this lemma we may "normalize" $P_y$ such that all the elements in a given row and a given column are the identity $e$ of $T_y$. Then using the assumption that $E(S)$ is a subsemigroup, we may show $p_{ji} = e$, the identity of $T_y$, for all $j \in J_y$ and $i \in I_y$. Hence $M(T_y, I_y, J_y, P_y) = T_y \times E(S_y)$ where $E(S_y)$ is a rectangular band.

In the proof of Theorem 4, we will need a quasi-orthogroup analogue to the minimum inverse semigroup congruence of an orthogroup (an orthodox union of groups) (Proposition 3).

For $(g, i, j), (h, r, s) \in S$, a quasi-orthogroup define $(g, i, j) \delta (h, r, s)$ if $(g, i, j), (h, r, s) \in S_y$, say, and $g = h$.

We show (Proposition 3) that $\delta$ is the smallest good congruence on $S$ $(aL^*b$ implies $a\delta L^*b\delta$ and $aR^*b$ implies $a\delta R^*b$)

such that $E(S/\delta)$ is a semilattice and, furthermore, that $S/\delta$ is a strong
semilattice $Y$ of the cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 1).

To show $\delta$ is a congruence relation, we will need the following lemma.

Lemma 2 Let $S_y = T_y \times E_y$ and $S_z = T_z \times I_z \times J_z$ where $T_y$ and $T_z$ are cancellative monoids, $E_y$ is a rectangular band, $I_z$ is a left zero semigroup and $J_z$ is a right zero semigroup. Assume there exists

a) A left representation $A \rightarrow \lambda_A$ of $S_y$ by transformations of $I_z$.

b) A right representation $A \rightarrow \rho_A$ of $S_y$ by transformation of $J_z$.

C) A homomorphism $\phi$ of $T_y$ into $T_z$.

Define a binary operation on $S_y \cup S_z$ extending the given ones on $S_y$ and $S_z$ by defining products of $A = (a, e) \in S_y$ and $(b, i, j) \in S_z$ as follows

$$(a, e)(b, i, j) = (a \phi b, \lambda_A i, j)$$

$$(b, i, j)(a, e) = (b(a \phi), i, j \rho_A).$$

Then $S_y \cup S_z$ becomes a semigroup with $S_z$ as an ideal.

Conversely, every possible binary associative operation on $S_y \cup S_z$ extending the given ones on $S_y$ and $S_z$, and such that $S_z$ is an ideal, can be constructed in the above manner.

Proof. Lemma 2 has been established by Clifford [2, Lemma 2.5] in the case $T_y$ and $T_z$ are groups. Clifford’s proof is easily seen to be valid when $T_y$ and $T_z$ are just cancellative monoids.
Proposition 3 Let $S$ be a quasi-orthogroup. Then, $\delta$ is the smallest good congruence on $S$ such that $E(S/\delta)$ is a semilattice. Furthermore, $S/\delta$ is isomorphic to a semigroup $T = \bigcup(T_y : y \in Y)$ which is a strong semilattice $Y$ of the cancellative monoids $(T_y : y \in Y)$ under the isomorphism $(g, i, j)\delta_T = g$.

Proof. We first show that $\delta$ is a congruence relation on $S$. Let $\bar{\delta}$ denote the smallest congruence relation on $S$ containing $\delta$. Suppose $a\bar{\delta}b$. Then, there exists $a = a_1, a_2, \ldots, a_n = b \in S$ such that $a_i = x_iu_iy_i$, $a_{i+1} = x_iv_iy_i$ where $x_i, y_i \in S^1$ and $u_i, v_i \in \delta$ for $1 \leq i \leq n - 1$. Suppose $x_i = (w, p, q)_y \in S_y, y_i = (h, c, d)_z \in S_z, u_i = (g, m, n)_i \in S_i$ and $v_i = (g, r, s)_i \in S_i$. Let $\theta = yzt$. Hence, $a_i = (A, a, b)_\theta \in S_\theta$ and $a_{i+1} = (B, e, f)_\theta \in S_\theta$, say. So,

$$(A, a, b)_\theta = (w, p, q)_y(g, m, n)_i(h, c, d)_z,$$

$$(B, e, f)_\theta = (w, p, q)_y(g, r, s)_i(h, c, d)_x.$$

If we multiply both of the above equations on the left and the right by $(e, a, b)_\theta$ where $e$ is the identity of $T_\theta$, we obtain

\begin{align*}
(A, a, b)_\theta &= (\bar{\omega}, \bar{p}, \bar{q})_\theta(g, m, n)_i(h, \bar{c}, \bar{d})_\theta \\
(B, a, b)_\theta &= (\bar{\omega}, \bar{p}, \bar{q})_\theta(g, r, s)_i(h, \bar{c}, \bar{d})_\theta.
\end{align*}

(1)

Using Lemma 2, we obtain

\begin{align*}
(\bar{\omega}, \bar{p}, \bar{q})_\theta(g, m, n)_i &= (\bar{\omega}(g, \phi_\theta), \bar{p}, \bar{q}, \rho(g, m, n)_i)_\theta \\
(\bar{\omega}, \bar{p}, \bar{q})_\theta(g, r, s)_i &= (\bar{\omega}(g, \phi_\theta), \bar{p}, \bar{q}, \rho(g, r, s)_i)_\theta
\end{align*}

(2)

where $\phi_\theta$ is a homomorphism of $T_i$ into $T_\theta$ and $A \rightarrow \rho_\theta$ is a right representation of $S_i$ by transformations of $J_\theta$. Combining (1) and (2), we obtain

$A = B$. Thus, $a_i\delta a_{i+1}$ for $1 \leq i \leq n - 1$. Thus, $a\delta b$. Hence, $\delta = \bar{\delta}$, and thus
\( \delta \) is a congruence relation on \( S \). Since \((g,i,j)\delta r = g\) defines a one-to-one mapping of \( S/\delta \) onto \( T = \cup(T_y : y \in Y) \), it is easily checked that \( T \) becomes a semilattice \( Y \) of cancellative monoids \( (T_y : y \in Y) \) under the multiplication \( ab = (ar^{-1}br^{-1})r \) and \( r \) defines an isomorphism of \( S/\delta \) onto \( T \). For \( a \in T_x \) and \( z \geq y \) define \( aC_{z,y} = ae_y \) where \( e_y \) is the identity of \( T_y \). It is routine to verify that \( C_{z,y} \) is a homomorphism of \( T_z \) into \( T_y \); \( C_{y,y} \) is the identity automorphism of \( T_y \); and for \( a \in T_y, b \in T_r \), \( ab = ac_{y,w}bC_{z,w} \). Since \( e_y e_z = e_y \) for all \( y, z \in Y \), \( C_{y,y}C_{z,w} = C_{y,w} \) for \( y \geq z \geq w \). So, \( T \) is a strong semilattice \( Y \) of the \( T_y \). Suppose \( aR^*b \). Then, \( a, b \in S_y \), say. Hence, \( a\delta, b\delta \in T_y \). Thus, using the fact \( T \) is a strong semilattice \( Y \) of the \( T_y \), \( a\delta R^*b\delta \). Similarly, \( aL^*b \) implies \( a\delta L^*b\delta \). So, \( \delta \) is a good congruence. (So, each \( H^* \)-class of \( S/\delta \) contains an idempotent). Clearly, \( E(S/\delta) \) is a semilattice. If \( \lambda \) is another such congruence, use the fact \( E(S_y)\lambda \) is a singleton to show \( \delta \leq \lambda \).

Remark The statement and proof of Proposition 3 is contained in the proof of [8, Theorem 6].

We are now in a position to establish Theorem 4.

**Theorem 4** Let \( E \) be a band and \( E = \cup(E_\alpha : \alpha \in Y) \) be its maximal semilattice decomposition. To each \( \alpha \in Y \) assign a cancellative monoid \( M_\alpha \) such that \( M_\alpha \cap M_\beta = \emptyset \) if \( \alpha \neq \beta \). Furthermore, suppose that for \( \alpha > \beta \) there exists a homomorphism

\[ \pi_{\alpha,\beta} : M_\alpha \rightarrow M_\beta \]

such that if \( \alpha > \beta > \nu \) then \( \pi_{\alpha,\nu} = \pi_{\alpha,\beta} \pi_{\beta,\nu} \). Set \( \pi_{\alpha,\alpha} \) equal to the identity automorphism on \( M_\alpha \). Let \( S = \cup((E_\alpha \times M_\alpha) : \alpha \in Y) \) and define a multiplication on \( S \) by \((e,x)(f,y) = (ef,x\pi_{\alpha,\alpha}y\pi_{\beta,\alpha})\) for any \((e,x) \in E_\alpha \times M_\alpha \) and \((f,y) \in E_\beta \times M_\beta \).
$E_{\alpha} \times M_{\alpha}, (f, y) \in E_{\beta} \times M_{\beta}$. Then, $S$ is a quasi-adequate semigroup which is a band of cancellative monoids. Conversely, any quasi-adequate semigroup which is a band of cancellative monoids can be constructed in this manner.

Proof. Let $S$ be a quasi-adequate semigroup which is a band of cancellative monoids. Then, using [4, Lemma 4.1], each $H^*$-class of $S$ contains an idempotent and $H^*$ is a congruence relation on $S$. Thus, $S$ is a quasi-orthogroup on which $H^*$ is a congruence relation. By Proposition 3, $S/\delta$ is isomorphic to $T = \cup(T_y : y \in Y)$, a strong semilattice $Y$ of cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 1). Thus, $T_y \cap T_z = \emptyset$ if $y \neq z$ and for $y > z$, there exists a homomorphism $\pi_{y, z} : T_y \rightarrow T_z$ such that for $y > z > w$, $\pi_{y, z} \pi_{z, w} = \pi_{y, w}$ and $\pi_{y, y}$ is the identity automorphism on $T_y$ for $y \in Y$. Furthermore, $ab = a\pi_{y, z} b\pi_{z, y}$ for $a \in T_y$ and $b \in T_z$. Let $P = \cup(E(S_y) \times T_y) : y \in Y$) under the multiplication $(e, x)(f, q) = (ef, xq)$ where $(e, x) \in E(S_y) \times T_y$ and $(f, q) \in E(S_z) \times T_z$. We will show that $(g, i, j)_y \lambda = ((e_y, i, j)_y, g)$ where $(g, i, j)_y \in S_y$ and $e_y$ is the identity of $T_y$ defines an isomorphism of $S$ onto $P$. Clearly, $\lambda$ defines a one-to-one mapping of $S$ onto $P$. We will next show that $\lambda$ defines a homomorphism of $S$ onto $P$. Let $(g, i, j)_y \in S_y$ and $(h, r, s)_z \in S_z$. Thus, $(g, i, j)_y H^*(e_u, r, s)_u$, say. Hence, it is easily checked that $u = y$. Using the fact that $H^*$ is a congruence relation, $(g, i, j)_y H^*(e_u, i, j)_y$. Similarly, $(h, r, s)_z H^*(e_z, r, s)_z$. Suppose $(g, i, j)_y (h, r, s)_z = (t, k, p)_z$. Hence, $(t, k, p)_z H^*(e_y, i, j)_y (e_z, r, s)_z$. Thus, $(e_y, i, j)_y (e_z, r, s)_z = (e_y, k, p)_z$. Hence, $((g, i, j)_y (h, r, s)_z) \lambda = (t, k, p)_z \lambda = ((e_y, k, p)_z, t)$ while $(g, i, j)_y (h, r, s)_z \lambda = ((e_y, i, j)_y, g)((e_z, r, s)_z, h) = ((e_z, k, p)_z, gh)$. Using Proposition 3 and its notation, $(g, i, j)_y \delta(h, r, s)_z \delta = (t, k, p)_z \delta$. So, $(g, i, j)_y \delta \tau \cdot (h, r, s)_z \delta \tau = (t, k, p)_z \delta \tau$. Thus, $gh = t$. Hence, $\lambda$
defines an isomorphism of $S$ onto $P$. Thus, we have established the converse of Theorem 4.

Next, we establish the direct part of Theorem 4. It is easily checked that $S$ is a semigroup. Let $e \in E_\alpha$. Then, $\{e\} \times M_\alpha$ is a cancellative monoid with identity $(e, e_\alpha)$ where $e_\alpha$ is the identity of $M_\alpha$. Let $M_e = \{e\} \times M_\alpha$. Thus, $S = \bigcup(M_e : e \in E)$ and $S$ is the band $E$ of cancellative monoids $(M_e : e \in E)$. Clearly, $E(S) = \{(e, e_\alpha) : e \in E_\alpha, \alpha \in Y\}$. Since $e_\alpha e_\beta = e_\alpha \pi_{\alpha, \alpha \beta} e_\beta \pi_{\beta, \alpha \beta} = e_{\alpha \beta} e_{\alpha \beta} = e_{\alpha \beta}$, it follows that $E(S)$ is a subsemigroup of $S$. To complete the proof, we will show that $(e, g) H^*(e, e_y)$ where $e \in E_y$ and $g \in M_y$. We will show that $(e, g) L^*(e, e_y)$. Dually, $(e, g) R^*(e, e_y)$. Suppose $(e, g)(a, b) = (e, g)(c, d)$ where $a \in E_z$, $b \in M_z$, $c \in E_t$ and $d \in M_t$. Thus, $e_a = e_c$ and $g_b = g_d$. So, $g \pi_{y,t} b \pi_{z,t} = g \pi_{y,t} d \pi_{t,y}$ with $yz = yt$. Hence, $b \pi_{y,z} = d \pi_{t,y}$. Thus, $e_y \pi_{y,z} b \pi_{z,y} = e_y \pi_{y,t} d \pi_{t,y}$. Hence, $e_y b = e_y d$. So, $(e, e_y)(a, b) = (e, e_y)(c, d)$.

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Bibliography


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