King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 124

October 1991

Interpolation Mixed with I_s - Approximation

M.A. Bokhari

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Interpolation Mixed with l_2-Approximation

M.A. Bokhari

ABSTRACT

We consider sqn roots of unity and define a class $\mathcal{R}^{q}_{N^*}(U_s, f)$ of rational functions which interpolate a given analytic function f on U_s, a large subset of the roots of unity satisfying a congruence relation. f is then approximated over $\mathcal{R}^{q}_{N^*}(f, U_s)$ with respect to l_2-norm on the complement of U_s. We also discuss Walsh type equiconvrgence.

Key Words: Analytic functions, Lagrange interpolant, l_2-minimization, Equiconvergence.

AMS Classification: $30E10$, $41A20$

1 Introduction

Let π_s denote the family of all polynomials of degree $\leq s$, and let $L_{n-1}(z, f) \in \pi_{n-1}$ denote the Lagrange interpolant to a function f analytic in the region $|z| < \eta, \eta > 1$, at the n roots of $z^n = 1$. It is well-known [7] that the best l_2- approximant to f from π_{n-1} over the set of the n zeros of $z^n - 1$ is $L_{n-1}(z, f)$. During the last decade several papers have appeared on discrete least squares minimization problems considered over a large set of the primitive roots of unity. In [3] Rivlin noted that the $(n-1)th$ degree polynomial which solves the problem

$$\min_{p \in \pi_{n-1}} \sum_{k=0}^{qn-1} |f(\omega^k) - p(\omega^k)|^2, \; \omega^{qn} = 1, q \geq 1 \tag{1.1}$$

is essentially $S_{n-1}[z, L_{qn-1}(z, f)]$, the $(n-1)th$ degree Taylor section of the polynomial $L_{qn-1}(z, f)$. In a different direction Sharma and Ziegler considered the following question [4]:

If $\mathcal{L}(f, U_s)$ denote the class of all polynomials of degree $\leq nq(s-1) + n - 1$ interpolating f at the set

$$U_s = \{\omega^\nu : \nu = 1, 2, \ldots, sqn; \nu \not\equiv 0 (\text{mod } s); \omega^{sqn} = 1\}, \tag{1.2}$$

find the solution to the problem

$$\min_{Q \in \mathcal{L}(f, U_s)} \sum_{\nu=0}^{qn-1} |f(\lambda^\nu) - Q(\lambda^\nu)|^2 \; , \; \lambda^{sqn} = 1, \lambda \notin U_s. \tag{1.3}$$

They discovered that the solution $Q_n^*(z,f) \in \mathcal{L}(f,U_s)$ to (1.3) is given by

$$Q_n^*(z,f) = L^*(z,f) + W_s(z)S_{n-1}[z, L_{qn-1}(z,g)] \tag{1.4}$$

where $L^*(z,f)$ is the Lagrange interpolant of degree $nq(s-1)-1$ to f on U_s, $g(z) := s^{-1}[f(z) - L^*(z,f)]$ and

$$W_s(z) = (z^{sqn} - 1)/(z^{qn} - 1). \tag{1.5}$$

The aim of the present note is two fold. First, we develop a variant of minimization problem (1.3) replacing $\mathcal{L}(f,U_s)$ by a class of certain interpolatory rational functions. The second problem to be discussed here is related to Walsh-type equiconvergence. This topic has attracted many mathematicians in the last decade. For the background we refer the reader to [2] - [6].

2 Preliminaries and Statement of Problems

We denote by $A_\rho, 1 < \rho < \infty$, the class of functions analytic in $|z| < \rho$ with at least one singularity on $|z| = \rho$, and set

$$N = qn(s-1) \quad \text{and} \quad N^* = N + n + m \tag{2.1}$$

where $s \geq 1$, $q \geq 1$ and $m \geq -1$ are fixed integers. For a given $\sigma > 1$, let $\mathcal{R}_{\nu,n}^\sigma$ denote the class of the rational functions $r(z)$ of the form

$$r(z) = p(z)/(z^n - \sigma^n), \qquad p \in \pi_\nu. \tag{2.2}$$

With the set U_s defined in (1.2) and an $f \in A_\rho$, let $\mathcal{R}^\sigma_{\nu,n}(f, U_s)$ denote the subclass of the rational functions $r \in \mathcal{R}^\sigma_{\nu,n}$ which interpolate f on U_s.

We shall consider the following problems:

(P1) *For a given $f \in A_\rho, \rho > 1$, find the rational function $R^*_{N^*,n}(z, f) \in \mathcal{R}^\sigma_{N^*,n}(f, U_s)$ which solves the problem*

$$R \in \mathcal{R}^{\sigma \, \min}_{N^*,n}(f, U_s) \sum_{\nu=0}^{q_{n-1}} |f(\lambda^\nu) - R(\lambda^\nu|^2, \quad \lambda^{sqn} = 1, \quad \lambda \notin U_s \tag{2.3}$$

(P2) *If $r_{N^*,n}(z, f) \in \mathcal{R}^\sigma_{N^*,n}$ minimizes $f \in A_\rho$ on $|z| = 1$ in the L_2- sense over the class $\mathcal{R}^\sigma_{N^*,n}$ ([5], (1.4)) and if $R^*_{N^*,n}(z, f)$ is the solution of (P_1), what is the region of convergence of the difference*

$$R^*_{N^*,n}(z, f) - r_{N^*,n}(z, f) \tag{2.4}$$

to zero as $n \to \infty$?

Remark 2.1. *When $s = 1$, the solutions to the problems (P_1) and (P_2) are provided in* [1]. *For the justification, it is enough to note that the set U_s which consists of the zeros of $W_s(z)$ is empty for $s = 1$.*

3 Solution of (P_1)

In order to solve the problem (P_1), we need an expression for the Lagrange interpolant of the function

$$h(z) := s^{-1}[f(z) - R^*_{N-1,n}(z,f,)] \tag{3.1}$$

where $R^*_{N-1,n}(z,f)$ is the rational function in $\mathcal{R}^{\sigma}_{N-1,n}$ which interpolates f on U_s. More precisely, we need the following Lemma:

Lemma 3.1 *If $s \geq 2$ and $q \geq 1$ are fixed integers and if*

$$L_{nq-1}(z,h) = \sum_{j=0}^{q-1} \sum_{\nu=0}^{n-1} c_{\nu+jn} z^{\nu+jn} \tag{3.2}$$

is the Lagrange polynomial of degree $qn-1$ which interpolates the function $h(z)$ at the roots of $z^{qn} = 1$, then

$$c_{\nu+jn} = \frac{1}{2\pi i} \int_\Gamma \left[\frac{\lambda_0(q) t^{n-\nu-1}}{\sigma^{(j+1)n}} + \frac{t^{(q-j)n-\nu-1)}}{t^{qn}-1} \right] \frac{f(t)}{W_s(t)} dt, \tag{3.3}$$

where Γ is a circle $|t| = \eta$, $1 < \eta < \rho$ and

$$\lambda_0(q) := \sigma^{qn}/(1-\sigma^{qn}). \tag{3.4}$$

Proof. In order to establish (3.4), we first find $L_{nq-1}(z, R^*_{N-1,n}(z,f))$. Since

$$R^*_{N-1,n}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n)}{W_s(t)} \cdot \frac{G(t,z)}{z^n - \sigma^n} f(t) dt. \tag{3.5}$$

where

$$G(t,z) := \frac{W_s(t) - W_s(z)}{t-z},$$

it is enough to evaluate $L_{qn-1}(z, \frac{G(t,z)}{z^n-\sigma^n})$. It is easy to see that

$$\begin{aligned}
&L_{qn-1}(z, G(t,z)) \\
&= \sum_{\nu=0}^{s-1} L_{qn-1}(z, \frac{t^{\nu qn} - z^{\nu qn}}{t-z}) = \frac{t^{qn} - z^{qn}}{t-z} \sum_{\nu=0}^{s-1} \frac{t^{\nu qn} - 1}{t^{qn} - 1} \\
&= \frac{t^{qn} - z^{qn}}{t-z} \left[\frac{t^{sqn} - 1}{(t^{qn} - 1)^2} - \frac{s}{t^{qn} - 1} \right] \\
&= \frac{t^{qn} - z^{qn}}{t-z} \left[\frac{W_s(t)}{t^{qn} - 1} - \frac{s}{t^{qn} - 1} \right].
\end{aligned}$$

Therefore,

$$L_{qn-1}(z, \frac{G(t,z)}{z^n - \sigma^n}) = S_1(t,z) - S_2(t,z)$$

where we have set

$$\left.\begin{aligned}
S_1(t,z) &= \tfrac{W_s(t)}{t^{qn}-1}\, L_{qn-1}(z, \tfrac{t^{qn}-z^{qn}}{t-z} \cdot \tfrac{1}{z^n-\sigma^n}) \\
S_2(t,z) &= \tfrac{s}{t^{qn}-1} L_{qn-1}(z, \tfrac{t^{qn}-z^{qn}}{t-z} \cdot \tfrac{1}{z^n-\sigma^n}).
\end{aligned}\right\} \tag{3.6}$$

It follows from (3.5) that

$$L_{qn-1}(z, R^*_{N-1,n}(z,f)) = I_1(z) - I_2(z) \tag{3.7}$$

where

$$I_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n) f(t)}{W_s(t)}\, S_1(t,z)\, dt,$$

and

$$I_2(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n) f(t)}{W_s(t)}\, S_2(t,z)\, dt.$$

Since

$$\frac{(t^{qn} - z^{qn})(t^n - \sigma^n)}{(t-z)(z^n - \sigma^n)} = \frac{t^{qn} - z^{qn}}{t-z} + \frac{(t^n - z^n)(t^{qn} - z^{qn})}{(t-z)(z^n - \sigma^n)},$$

we have

$$I_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(f(t)(t^{qn}-z^{qn})}{(t-z)(t^{qn}-1)}dt + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{qn}-1}\, L_{qn-1}\left(z, \frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)}\right)\, dt. \tag{3.8}$$

The second integral in (3.8) vanishes because

$$\frac{f(t)}{t^{qn}-1}\, L_{qn-1}\,(z, \frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)}) = f(t)L_{qn-1}\,(z, \frac{t^n-z^n}{(t-z)(z^n-\sigma^n)}).$$

Therefore,

$$I_1(z) = L_{qn-1}(z, f). \tag{3.9}$$

Similarly,

$$I_2(z) = I_3(z) + I_4(z) \tag{3.10}$$

where

$$\begin{aligned} I_3(z) &= \frac{s}{2\pi i}\int_\Gamma \frac{f(t)}{t^{sqn}-1} L_{qn-1}(z, \frac{t^{qn}-z^{qn}}{t-z})dt \\ &= \frac{s}{2\pi i}\sum_{\nu=0}^{qn-1} z^\nu \int_\Gamma \frac{f(t)}{t^{\nu+1}}\cdot\frac{t^{qn}}{t^{sqn}-1}dt, \end{aligned} \tag{3.11}$$

and

$$I_4(z) = \frac{s}{2\pi i}\int_\Gamma \frac{f(t)}{t^{sqn}-1}\, L_{qn-1}\,(z, \frac{(t^{qn}-z^{qn})(t^n-z^n)}{(t-z)(z^n-\sigma^n)})\, dt$$

Since $L_{qn-1}(z, \frac{1}{z^n-\sigma^n}) = \frac{\sigma^{qn-n}}{1-\sigma^{qn}} \sum_{j=0}^{q-1} \frac{z^{jn}}{\sigma^{jn}}$, it is easy to see that

$$I_4(z) = \frac{s}{2\pi i}\lambda_0(q)\int_\Gamma \frac{f(t)}{W_s(t)}\sum_{j=0}^{q-1}\sum_{\nu=0}^{n-1}\frac{t^{n-\nu-1}z^{jn+\nu}}{\sigma^{nj+n}}dt \tag{3.12}$$

where $\lambda_0(q)$ is defined in (3.4). Combining (3.7) with (3.9) – (3.10) we obtain

$$L_{qn-1}(z, R^*_{N-1,n}(z,f)) = L_{qn-1}(z,f) - I_3(z) - I_4(z).$$

Using (3.1), we observe that

$$\begin{aligned} L_{qn-1}(z,h) &= s^{-1}[L_{qn-1}(z,f) - L_{qn-1}(z, R^*_{N-1,n}(z,f))] \\ &= s^{-1}[I_3(z) + I_4(z)]. \end{aligned}$$

Finally, the substitution of the values of $I_3(z)$ and $I_4(z)$ from (3.11) - (3.12) in the above relation establishes the equation (3.2) for which the coefficients are given in (3.3). □

Now we proceed to determine solution of the problem (P_1). First we note that any rational function $R(z) \in \mathcal{R}^\sigma_{N^*,n}(f, U_s)$ can be expressed as

$$R(z) = R^*_{N-1,n}(z,f) + W_s(z)B(z)$$

for some $B(z) \in \mathcal{R}^\sigma_{n+m,n}$. Here $R^*_{N-1,n}(z,f)$ is the rational function used in (3.1). Since $W_s(\lambda^\nu) = s$ for any solution λ of $z^{sqn} = 1$, it follows that

$$\begin{aligned} & |f(\lambda^\nu) - R(\lambda^\nu)|^2 \\ = \; & |f(\lambda^\nu) - R^*_{N-1,n}(\lambda^\nu, f) - sB(\lambda^\nu)|^2 \\ = \; & |sh(\lambda^\nu) - sB(\lambda^\nu)|^2. \end{aligned}$$

Thus the problem (P_1) is equivalent to minimizing

$$\sum_{\nu=0}^{qn-1} |h(\omega^\nu) - B(\omega^\nu)|^2 \ , \ \omega^{qn} = 1 \ , \tag{3.13}$$

over all rational functions $B \in \mathcal{R}^{\sigma}_{n+m,n}$. This problem had been solved by the author in [1]. In fact, if $L_{qn-1}(z,h) := \sum_{\nu=0}^{qn-1} c_\nu z^\nu$ then ([1], Proposition 1)

$$B^*_{n+m,n}(z,h) := \sum_{\nu=0}^{n+m} \tau_\nu z^\nu/(z^n - \sigma^n) \tag{3.14}$$

will be the minimizer of (3.13) over $\mathcal{R}^{\sigma}_{n+m,n}$ where

$$\tau_\nu = \begin{cases} -c_\nu\sigma^n + \frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \sum_{j=1}^{q-1} \sigma^{-jn} c_{\nu+jn}, & 0 \le \nu \le m \\ \frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}} \sum_{j=0}^{q-1} \sigma^{-jn} c_{\nu+jn}, & m+1 \le \nu \le n-1 \\ c_{\nu-n} - \frac{\sigma^{2(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \sum_{j=1}^{q-1} \sigma^{-jn} c_{\nu+(j-1)n}, & n \le \nu \le n+m \end{cases} \tag{3.15}$$

Now the description of c_j's given in Lemma 3.1 can be applied to τ_ν's for the explicit representation of $B^*_{n+m,n}(z,h)$. Thus

$$R^*_{N^*,n}(z,f) := R^*_{N-1,n}(z,f) + W_s(z) B^*_{n+m,n}(z,h) \tag{3.16}$$

is the rational function in $\mathcal{R}^{\sigma}_{N^*,n}(f, U_s)$ which provides the desired solution of (P_1).

Remark 3.2. *The relation* (3.15) *reduces to* $\tau_\nu = (1-\sigma^n)c_\nu,\ 0 \le \nu \le n-1$ *when* $q = 1$ *and* $m = -1$. *Thus, the rational function* $B^*_{n-1,n}(z,h)$ *turns out to be* $[(1-\sigma^n)/(z^n - \sigma^n)]L_{n-1}(z,h)$. *Consequently, the solution* (3.16) *to* (P_1) *, in this case, bases entirely on the interpolatory character of the rational functions* $R^*_{N-1,n}(z,f)$ *and* $B^*_{n-1,n}(z,h)$.

4 Solution of (P_2)

The problem (P_2) deals with Walsh-type equiconvergence. Here we shall provide its solution and note that it extends an earlier result due to Sharma and Ziegler ([4], $Theorem 1$). In order to avoid lengthy expressions in the calculations, we shall discuss the problem (P_2) for $m = -1$. However, the solution stands valid for any integer $m > -1$. More precisely, we prove

Theorem 4.1. *Let $s \geq 2$ and $q \geq 2$ be fixed integers, and let $N := (s-1)qn$. If $f \in A_\rho, 1 < \rho < \infty$, and $\sigma > 1$ then (cf.(2.4))*

$$\lim_{n\to\infty} \left\{ R^*_{N+n-1,n}(z,f) - r_{N+n-1,n}(z,f) \right\} = 0, \quad \forall\, z \in D_\sigma \tag{4.1}$$

where D_σ for $q \geq 3$ is given by

(i) $\left\{ z \in C : |z| < \rho^{1+\frac{1}{s-1}} \right\}$ *if* $\sigma \geq \rho^{q-1}$

(ii) $\left\{ z \in C : |z| < \rho^{1+\frac{1}{(s-1)q}} \cdot \sigma^{\frac{1}{(s-1)q}}, \quad |z| \neq \sigma \right\}$ *if* $\rho \leq \sigma < \rho^{q-1}$

(iii) $\left\{ z \in C : |z| < \rho \cdot \sigma^{\frac{2}{(s-1)q}}, \quad |z| \neq \sigma \right\}$ *if* $\sigma < \rho$,

and for $q = 2$, D_σ is given by

(i) $\left\{ z \in C : |z| < \rho^{1+\frac{1}{s-1}} \right\}$ *if* $\sigma \geq \rho^s$

(ii) $\left\{ z \in C : |z| < \rho^{1+\frac{3}{2s-3}} \cdot \sigma^{\frac{1}{3-2s}}, |z| \neq \sigma \right\}$ *if* $\rho \leq \sigma < \rho^s$

(iii) $\left\{ z \in C : |z| < \rho \cdot \sigma^{\frac{1}{s-1}}, |z| \neq \sigma \right\}$ *if* $\sigma < \rho$.

Moreover, the convergence is uniform and geometric in any compact subset of the region D_σ.

The proof of the above theorem requires integral representations of the rational functions $R^*_{N+n-1,n}(z,f)$ and $r_{N+n-1,n}(z,f)$ together with some estimates which we describe below as remarks.

Remark 4.1. *It is known that* ([5] ,(1.4))

$$r_{N+n-1,n}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{t^n-\sigma^n}{z^n-\sigma^n}\cdot\frac{f(t)}{t-z}\cdot\frac{k(t)-k(z)}{k(t)}\,dt \qquad (4.2)$$

where Γ *is a circle* $|t| = \rho',\ 1 < \rho' < \rho,$ *and*

$$k(t) = t^N(t^n - \sigma^{-n}). \qquad (4.3)$$

Remark 4.2. *If we set*

$$\delta_1 = \frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}}\ ,\ \delta_2 = \frac{1-(\sigma t)^{-qn}}{1-(\sigma t)^{-n}} \qquad (4.4)$$

then using (3.3) *we can write*

$$\sum_{j=0}^{q-1} \sigma^{-jn}\ c_{\nu+jn} = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{\nu+1}W_s(t)}\{\frac{t^n}{\sigma^n}\lambda_0(q) + \frac{t^{qn}}{t^{qn}-1}\delta_2\}\ dt$$

where $\lambda_0(q)$ *is given by* (3.4). *Thus the numerator of the rational function* $B^*_{n-1,n}(z,h)$ (*cf.*(3.14)) *can be expressed as*

$$\sum_{\nu=0}^{n-1}\tau_\nu z^\nu = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{W_s(t)}\{\frac{t^n}{\sigma^n}\delta_1\lambda_0(q) + \frac{t^{qn}}{t^{qn}-1}\delta_1\delta_2\}\frac{t^n-z^n}{t^n(t-z)}dt. \qquad (4.5)$$

Remark 4.3. *Due to the interpolatory proprties of the rational function* $R^*_{N-1,n}(z,f)$ (*cf.*(3.1)) *we have the following integral representation:*

$$R^*_{N-1,n}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{t^n - z^n}{z^n - \sigma^n}\cdot\frac{W_s(t) - W_s(z)}{W_s(t)}\cdot\frac{f(z)}{t-z}dt. \tag{4.6}$$

Remark 4.4. *Since* $\sigma > 1$, *we have the following estimates* (*cf.*(4.4), (3.4))*:*

$$\left.\begin{array}{l}\delta_1\lambda_0(q) = \sigma^n(1-\sigma^{-2n}) + O(\sigma^{-2qn+n}) \\ \delta_2\lambda_0(q) = \frac{t^n\sigma^n(\sigma^{-2n}-1)}{t^n-\sigma^{-n}}) + O(\sigma^{-qn+n}).\end{array}\right\} \tag{4.7}$$

The above remarks bring us to the

Proof of Theorem 4.1. From (4.2), (4.5) and (4.6) we can write

$$\begin{aligned} &R^*_{N+n-1}(z,f) - r_{N+n-1}(z,f) \\ &\quad = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\{K_1(t,z,\sigma) + K_2(t,z,\sigma) - K_3(t,z,\sigma)\}dt \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} K_1(t,z,\sigma) &= \frac{k(z)}{k(t)}\cdot\frac{t^n-\sigma^n}{z^n-\sigma^n}, \\ K_2(t,z,\sigma) &= \frac{W_s(z)}{W_s(t)}\cdot\frac{1}{z^n-\sigma^n}\left\{\frac{t^n}{\sigma^n}\delta_1\lambda_0(q) + \frac{t^{qn}}{t^{qn}-1}\delta_1\delta_2 - (t^n-\sigma^n)\right\}, \\ K_3(t,z,\sigma) &= \frac{W_s(z)}{W_s(t)}\cdot\frac{1}{z^n-\sigma^n}\left\{\frac{t^n}{\sigma^n}\delta_1\lambda_0(q) + \frac{t^{qn}}{t^{qn}-1}\delta_1\delta_2\right\}\frac{z^n}{t^n}. \end{aligned}$$

After the cancellation of suitable terms and using (4.7) together with the estimate $W_s(z)/W_s(t) = O(z^N/t^N)$ we notice that

$$K_2(t,z,\sigma) = \frac{1}{z^n-\sigma^n}O\left(\frac{z^N}{t^N}\max\left\{\frac{|t|^n}{\sigma^{2n}}, \frac{\sigma^n}{|t|^{qn}}, \frac{1}{\sigma^n}\right\}\right). \tag{4.9}$$

Similarly, a detailed analysis with appropriate cancellation of certain terms of higher order leads us to

$$\begin{aligned} K_1(t,z,\sigma) &- K_3(t,z,\sigma) \\ &= \frac{1}{z^n-\sigma^n}\left\{O\left(\frac{z^N}{t^N}\max\{\frac{1}{\sigma^n},\frac{1}{|t|^n}\}\right)\right. \\ &+ O\left(\frac{z^{N+n}}{t^N}\max\left\{\frac{1}{|t|^{sqn}},\frac{1}{|t|^n\sigma^n},\frac{\sigma^n}{|t|^{sqn+n}},\frac{1}{\sigma^{2n}}\right\}\right) \\ &+ \left. O\left(\frac{z^{N-qn+n}}{t^N}\max\left\{1,\frac{\sigma^n}{|t|^n}\right\}\right)\right\}. \end{aligned} \tag{4.10}$$

Thus, from (4.9) and (4.10) we can write

$$\begin{aligned} K_1(t,z,\sigma) &+ K_2(t,z,\sigma)-K_3(t,z,\sigma) \\ &= \frac{1}{z^n-\sigma^n}\{O(T_1)+O(T_2)+O(T_3)\} \end{aligned}$$

where

$$\begin{aligned} T_1 &= \frac{z^{(s-2)qn+n}}{t^{(s-1)1n}}\max\left\{1,\frac{\sigma^n}{|t|^n}\right\}, \\ T_2 &= \frac{z^{(s-1)qn}}{t^{sqn}}\max\left\{\frac{|t|^{(q+1)n}}{\sigma^{2n}},\frac{|t|^{qn}}{\sigma^n},|t|^{(q-1)n},\sigma^n\right\}, \\ T_3 &= \frac{z^{(s-1)qn+n}}{t^{(s-1)qn}}\max\left\{\frac{1}{|t|^{sqn}},\frac{1}{|t|^n\sigma^n},\frac{\sigma^n}{|t|^{sqn+n}},\frac{1}{\sigma^{2n}}\right\}. \end{aligned}$$

After considering various cases for σ separately for the integers $q \geq 3$ and $q = 2$, we analyze the order of the terms T_1, T_2, T_3. This leads us to the determination of different regions of convergence for (4.8) as desired in the theorem. □.

Remark 4.5. *Theorem 4.1 does not consider the case when* $s = 1$ *or* $q = 1$. *These cases are already settled in [1] and [5] respectively. It is worth mentioning that the problem* (P_1) *entirely deals with* l_2*-minimization if* $s = 1$, *whereas it reduces to interpolation problems when* $q = 1$.

Remark 4.6. *If we let* $\sigma \to \infty$ *in Theorem 4.1, we retrieve a result of Sharma-Ziegler ([4], Theorem 1).*

Acknowledgement. The proof of Lemma (3.1) in the present form is due to Professor Ambikeshwar Sharma. The author is grateful for his suggestions.

REFERENCES

1. M.A. Bokhari, *On certain sequences of least squares approximants*, Bull. Austral. Math. Soc. 38(1988), 415-442.

2. A.S. Cavaretta Jr., A. Sharma and R.S. Varga, *Interpolation in the roots of unity: an extension of a Theorem of J.L. Walsh*, Resultate der Mathematik 3(1981), 155-191.

3. T.J. Rivlin, *On Walsh equiconvergence*, J. Approx. Theory 36(1982), 334-345.

4. A. Sharma and Z. Ziegler, *Walsh equiconvergence for best l_2-Approximates*, Studia Math., T. LXXVII (1984), 523-528.

5. E.B. Saff and A. Sharma, "*On equiconvergence of certain sequences of rational interpolants*", Proc. Rational Approximations and Interpolation (Lecture Notes in Math 1105. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984), 256-271.

6. R.S. Varga, *Topics in polynomial and rational approximation*. Séminare, de Math. Supérieures Les Presses de L'Univ. de Montreál, 1982, 69-93.

7. J.L. Walsh, "*Interpolation and approximation by rational functions in the complex domain*", Amer. Math. Soc. Colloq. Publ., Vol. XX, 5th ed. 1969.

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arbaia