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A NOTE ON THE INTEGRAL REPRESENTATIONS OF $J_\nu(z)K_\mu(z)$

By

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Abstract

In this paper, integral representations of the product $J_\nu(z)K_\mu(z)$ when $\mu = \nu$ have been obtained. The problem is still open for $\mu \neq \nu$.

INTRODUCTION

The integral representation of product of Bessel functions is useful in the applied problems of diffraction of electric waves by perfectly conducting right circular cones and wedges [6, 7]. The integral representations of the products $J_\nu(z)J_\mu(z)$, $I_\nu(z)K_\mu(z)$ and $K_\nu(z)K_\mu(z)$ of the Bessel functions are known [1, 3, 4, 5, 6]. However, the integral representation of the product $J_\nu(z)K_\mu(z)$ does not seem to be known in the literature. In this paper we have found integral representations of $J_\nu(z)K_\mu(z)$ when $\mu = \nu$. The problem is still open for $\mu \neq \nu$.

THEOREM 1. For $\mathrm{Re}(z) > 0$

$$\begin{aligned} J_\nu(\sqrt{2}z)K_\nu(\sqrt{2}z) &= \int_0^{\pi/2} J_\nu[z/\sqrt{\sec\theta + \tan\theta}]K_\nu[z\sqrt{\sec\theta + \tan\theta}]d\{J_0(z\sqrt{2\tan\theta})\} \\ &= \int_1^\infty J_\nu[z/\sqrt{t}]K_\nu[z\sqrt{t}]d\{J_0(z\sqrt{t - 1/t})\} \\ &= \int_0^\infty J_\nu[z\sqrt{\cosh\tau}]K_\nu[z\sqrt{\cosh\tau}]d\{J_0(z\sqrt{\tanh\tau \sec h\tau})\} \end{aligned}$$

PROOF. Consider a function,

$$f(x) = (1/2)\exp(-\alpha^2/2x)J_\nu(x), \quad 0 < x < \infty, \quad Re\ \alpha > 0.$$

Then, according to [4, p 719]

$$L\{x^{-1}f(x);p\} = J_\nu\{\alpha[\sqrt{p^2+1} - p]^{1/2}\}K_\nu\{\alpha[\sqrt{p^2+1} + p]^{1/2}\}, \tag{1}$$

where L is the Laplace transform operator.

Using the properties [2, p 129] and [2, p 132] of the Laplace transformation we get

$$(2)\int_0^\infty J_\nu\{\alpha[\sqrt{x^2+1}-x]^{1/2}K_\nu\{\alpha[\sqrt{x^2+1}+x]^{1/2}\}J_{-1}(2\sqrt{px})\frac{dx}{\sqrt{x}}=\frac{1}{2\sqrt{p}}L\{x^{-1}J_\nu(1/x);\ p+\alpha^2/2\}$$

Using the result [2, p 191] we get

$$\int_0^\infty J_\nu\{\alpha[\sqrt{x^2+1}-x]^{1/2}K_\nu\{[\alpha[\sqrt{x^2+1}+x]^{1/2}\}J_1(2\sqrt{px})\frac{dx}{\sqrt{x}}$$

$$(3)\qquad =\frac{1}{\sqrt{p}}J_\nu(\sqrt{2p+\alpha^2})K_\nu(\sqrt{2p+\alpha^2})\quad (\text{ Re } \ p>0)$$

However, [4, p 968]

$$(4)\qquad \frac{d}{dx}J_0[2\sqrt{px}]=-J_1(2\sqrt{px})\frac{\sqrt{p}}{\sqrt{x}}=J_{-1}(2\sqrt{px})\frac{\sqrt{p}}{\sqrt{x}}$$

Substituting $2p=\beta^2$ in (3) and using (4) we get

$$(5)\int_0^\infty J_\nu\{\alpha[\sqrt{x^2+1}-x]^{1/2}\}K_\nu\{\alpha[\sqrt{x^2+1}+x]^{1/2}\}d\{J_0(\beta\sqrt{2x})\}=J_\nu(\sqrt{\alpha^2+\beta^2})K_\nu(\sqrt{\alpha^2+\beta^2})$$

The substitution $x=\tan\theta$ in (5) yields

$$(6)\qquad J_\nu(\sqrt{\alpha^2+\beta^2})K_\nu(\sqrt{\alpha^2+\beta^2})=\int_0^{\pi/2}J_\nu\{\alpha/\sqrt{\sec\theta+\tan\theta}\}$$

$$K_\nu\{\alpha\sqrt{\sec\theta+\tan\theta}\}d\{J_0(\beta\sqrt{2\tan\theta})\}.$$

The substitution $\sec\theta+\tan\theta=t$ in (6) yields

$$(7)\qquad J_\nu(\sqrt{\alpha^2+\beta^2})K_\nu(\sqrt{\alpha^2+\beta^2})=\int_1^\infty J_\nu(\alpha/\sqrt{t})K_\nu(\alpha\sqrt{t})\,d\{J_0(\beta\sqrt{t-1/t})\}.$$

Substituting $t=\cosh\tau$ in (7) we obtain

$$(8)\ \ J_\nu(\sqrt{\alpha^2+\beta^2})K_\nu(\sqrt{\alpha^2+\beta^2})=\int_0^\infty J_\nu(\alpha/\sqrt{\cosh\tau})K_\nu(\alpha\sqrt{\cosh\tau})d\{J_0(\beta\sqrt{\tanh\tau\sec h\tau})\}$$

Substituting $\alpha=\beta=z$ in (6), (7) and (8) we get the proof of the theorem.

COR. For $a>0$ and $0<x<\pi/2$

$$\begin{aligned}J_\nu(a)K_\nu(a) &= \int_0^{\pi/2}J_\nu(a\cos x/\sqrt{\sec\theta+\tan\theta})K_\nu(a\cos x\sqrt{\sec\theta+\tan\theta})d\{J_0(a\sin x\sqrt{2\tan\theta})\}\\ &= \int_1^\infty J_\nu(a\cos x/\sqrt{t})K_\nu(a\cos x\sqrt{t})d\{J_0(a\sin x\sqrt{t-1/t})\}\\ &= \int_0^\infty J_\nu(a\cos x/\sqrt{\cosh\tau})K_\nu(a\cos x\sqrt{\cosh\tau})d\{J_0(a\sin x\sqrt{\tanh\tau\text{sech}\,\tau})\}\end{aligned}$$

PROOF. This follows from (6), (7) and (8) when we substitute $\alpha = a\cos x$ and $\beta = a\sin x$, $a > 0$, $0 < x < \pi/2$.

REMARK. The positions of the sine and cosine functions in the above corollary can be interchanged without changing the result.

THEOREM 2. Let

$$a_{n,\nu}(x) = \int_1^\infty [(-t)^{n-1} + t^{-n-1}] J_\nu(x/\sqrt{t}) K_\nu(x\sqrt{t}) dt, \ x > 0 \quad (n = 1, 2, 3, \ldots).$$

Then,

$$J_\nu(\sqrt{2}x) K_\nu(\sqrt{2}x) + \sum_{n=1}^\infty n^{-1} a_{n,\nu}(x) J_n(x) I_n(x) = 0.$$

PROOF. This follows from (7) and [3, p 100].

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