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RINGS CHARACTERIZED BY THEIR FUZZY SUBMODULES

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Abstract The notions of pure ideals and pure submodules play an important role in the study of rings and their module categories. In this paper, we introduce pure fuzzy ideals and pure fuzzy submodules of an R-module. We establish their basic properties and use them to obtain fuzzy theoretic characterizations of right weakly regular rings. Among other results, we prove that a ring R is right weakly regular if and only if each fuzzy submodule of a cyclic R-module is fuzzy pure.

1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh in his classic paper [12] of 1965 provides a natural framework for generalizing some of the basic notions of algebra. In [11], Rosenfeld formulated the elements of a theory of fuzzy groups. N.Kuroki laid the foundations of a theory of fuzzy semigroups in his papers([5] and [6]). Fuzzy subgroups and fuzzy ideals of a ring have been investigated in [7] and [8], among others. Fu-Zheng Pan [9] initiated the study of fuzzy finitely generated modules. Recently, Golan [4] has considered the problem of assigning to each R-module the structure of a fuzzy submodule such that all R-homomorphisms become fuzzy R-homomorphisms in the sense of Fu-Zheng Pan [9].

In this paper, we introduce t-pure fuzzy ideals of a ring by extending the notion of (ordinary) pure ideals of a ring (cf. [1]). Moreover, as a generalization of t-pure fuzzy ideals, we define pure fuzzy submodules of an R-module and fuzzy normal modules. In section 2, we give preliminary definitions and state our notational conventions. In this section, we also establish basic properties of t-pure fuzzy ideals and discuss other related notions. In section 3, we prove two characterization theorems for weakly regular rings. Among other results, it is proved that a ring R is weakly regular if and only if each fuzzy ideal of R is t-pure (if and only if each cyclic R-module is fuzzy normal).

2. Preliminaries

Throughout this paper, R will denote a ring with an identity 1, and R-modules are right unitary. L will denote a complete lattice, that is, a set L with a partial order on it such that for any subset A of L, infimum and supremum of A, denoted by $\wedge_{a\in A} a$ and $\vee_{a\in A} a$, respectively, always exist. We also assume that L is distributive with a least element 0 and a greatest element 1, in which the infinite meet distributive law:

$$x\wedge(\vee_{a\in A} a) = \vee_{a\in A}(x\wedge a)$$

holds for any $A \subseteq L$ and $x\in L$. All fuzzy subsets of the set S of discourse are L-fuzzy subsets of S in the sense of Goguen [3], that is, a function from S into L. However, for convenience, we shall write fuzzy subset instead of L-fuzzy subset. If L is the unit interval [0,1] of real numbers, L-fuzzy subsets are fuzzy subsets in the usual sense (cf. [12]). A fuzzy subset $\lambda: R \to L$ is nonempty if it is not the constant map which assumes the value 0 of L. For any fuzzy subsets λ and μ of R, $\lambda \le \mu$ means that, for all x in R, $\lambda(x) \le \mu(x)$, in the ordering of L. The symbols $\lambda\wedge\mu$, $\lambda\vee\mu$, $\lambda o \mu$ will mean the following fuzzy subsets of R:

$$(\lambda\wedge\mu)(x) = \lambda(x)\wedge\mu(x)$$

$$(\lambda\vee\mu)(x) = \lambda(x)\vee\mu(x)$$

$$(\lambda o\mu)(x) = \bigvee_{\sum y_i z_i = x} \{\lambda(y_i)\wedge\mu(z_i)\}$$

A nonempty fuzzy subset A of R is called a fuzzy right (left) ideal of R if for any $x,y\in R$,

$$\lambda(x-y) \ge \lambda(x)\wedge\lambda(y) \text{ and } \lambda(xy)\ge\lambda(x) \ (\lambda(xy)\ge\lambda(y)).$$

A fuzzy subset of R which is both a fuzzy right ideal and a fuzzy left ideal of R is called a fuzzy ideal of R. It is easily seen that λ is a fuzzy ideal of R if and only if

$$\lambda(xy)\ge\lambda(x)\vee\lambda(y) \text{ for any } x,y\in R.$$

Analogous to the definition of the product IJ, for any ideals I and J of R, we define the product $\lambda\mu$, for any fuzzy ideals λ and μ of R, as the fuzzy subset defined by

$$(\lambda\mu)(x) = \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left(\bigwedge_{1\le i\le p} (\lambda(y_i)\wedge\mu(z_i))\right) \quad (p \text{ is an integer} \ge 1)$$

for any $x \in R$. It is easily seen that $\lambda\mu$ is a fuzzy ideal of R, whenever $\lambda\mu$ is nonempty.

Definition 2.1 [4] Let M_R be a right R-module. A function $\lambda: M \longrightarrow L$ is called a *fuzzy submodule* of M_R if the following conditions hold:

(1) $\lambda(0_M) = 1$;

(2) $\lambda(m+m') \geq \inf(\lambda(m), \lambda(m'))$ for all $m, m' \in M$.

(3) $\lambda(ma) \geq \lambda(m)$ for all $m \in M$ and all $a \in R$.

From the above definition it follows that, if λ is a fuzzy submodule of R, then $\lambda(0) = 1$. In the sequel, fuzzy submodules of R_R are called fuzzy right ideals of R. Fuzzy left ideals of R are defined analogously. By a fuzzy ideal of R we shall mean a fuzzy ideal which is both a fuzzy right and a fuzzy left ideal of R.

Definition 2.2 Let λ be a fuzzy submodule of M and μ a fuzzy ideal of R. Then we define the fuzzy subset $\lambda\mu$ of M by

$$(\lambda\mu)(x) = \bigvee_{x=\sum_{i=1}^{p} y_i z_i \ (p \in \mathbb{N})} \left[\bigwedge_{1 \leq i \leq p} \{\lambda(y_i) \wedge \mu(z_i)\} \right]$$

where $y_i \in M$ and $z_i \in R$ for each $x \in M$.

Proposition 2.1 $\lambda\mu$ is a fuzzy submodule of M.

Proof We have

$$(\lambda\mu)(x) = \bigvee_{x=\sum_{i=1}^{p} y_i z_i \ (p \text{ is an integer} \geq 1)} \left(\bigwedge_{1 \leq i \leq p} (\lambda(y_i) \wedge \mu(z_i)) \right) \text{ for any } x \in M.$$

where $y_i \in M$ and $z_i \in R$, for any $x \in M$.

Thus $(\lambda\mu)(0_M) = \bigvee_{0=\sum_{i=1}^{p} y_i z_i} \left(\bigwedge_{1 \leq i \leq p} (\lambda(y_i) \wedge \mu(z_i)) \right)$

$$\geq \wedge(\lambda(0_M), \mu(0)) = \wedge(1,1) = 1$$

Thus $(\lambda\mu)(0_M) = 1$.

Again, $(\lambda\mu)(m) = \bigvee_{m=\sum_{j=1}^{q} y'_j z'_j} \left(\bigwedge_{1\le j\le q} (\lambda(y'_j)\wedge\mu(z'_j))\right)$

and $(\lambda\mu)(m') = \bigvee_{m'=\sum_{k=1}^{r} y''_k z''_k} \left(\bigwedge_{1\le k\le r} (\lambda(y''_k)\wedge\mu(z''_k))\right)$

where $m, m' \in M$.

Thus $[(\lambda\mu)(m)\wedge(\lambda\mu)(m')]$

$$=\inf \left[\bigvee_{m=\sum_{j=1}^{q} y'_j z'_j} \left(\bigwedge_{1\le j\le q} (\lambda(y'_j)\wedge\mu(z'_j))\right), \bigvee_{m'=\sum_{k=1}^{r} y''_k z''_k} \left(\bigwedge_{1\le k\le r} (\lambda(y''_k)\wedge\mu(z''_k))\right)\right]$$

$$\le \bigvee_{m+m'=\sum_{l=1}^{s} y'''_l z'''_l} \left[\bigwedge_{1\le l\le s} (\lambda(y'''_l)\wedge\mu(z'''_l))\right] = (\lambda\mu)(m+m').$$

Thus $(\lambda\mu)(m+m') \ge [(\lambda\mu)(m)\wedge(\lambda\mu)(m')]$ for $m, m' \in M$.

Also, $(\lambda\mu)(m) = \bigvee_{m=\sum_{j=1}^{q} y'_j z'_j} \left(\bigwedge_{1\le j\le q} (\lambda(y'_j)\wedge\mu(z'_j))\right)$

$$\le \bigvee_{m=\sum_{j=1}^{q} y'_j z'_j} \left(\bigwedge_{1\le j\le q}(\lambda(y'_j)\wedge\mu(z'_j a))\right)$$

$$\le \bigvee_{ma=\sum_{n=1}^{t} y''_n z''_n} \left(\bigwedge_{1\le n\le t} (\lambda(y''_n)\wedge\mu(z''_n))\right)$$

$$= (\lambda\mu)(ma).$$

Thus $(\lambda\mu)(ma) \ge (\lambda\mu)(m)$ for $m\in M$ and $a\in R$.

Hence $\lambda\mu$ is a fuzzy submodule of M.

We now recall some definitions and results from ring theory.

Definition 2.3 ([1]) A two-sided ideal I of R is called *right t-pure* if, for each $x \in I$, there exists an element $y \in I$ such that $x=xy$ (t stands for the two sidedness of I).

Proposition 2.2 ([1]) A two sided ideal I of R is right t-pure if and only if $J \cap I = JI$ for any right ideal J of R.

Extending the above notion to arbitary modules we obtain the following definition.

Definition 2.4 A submodule N of an R-module M is *pure* in M if and only if $N \cap MI = NI$ for each ideal I of R. M is called *normal* if each submodule of M is pure.

The following definitions extend the above notions to the case of fuzzy subsets.

Definition 2.5 A fuzzy ideal λ of R is called a *t-pure fuzzy ideal* of R if $\lambda \cap \mu = \lambda\mu$, for each fuzzy right ideal μ of R.

Definition 2.6 Let λ be a fuzzy submodule of an R-module M. Then λ will be called a *pure fuzzy submodule* of M if for each fuzzy ideal μ of R, $\lambda \cap (\mathcal{M} \circ \mu) = \lambda\mu$, where $\mathcal{M}$ denotes the fuzzy submodule of M_R defined by $\mathcal{M}(x) = 1$ for each $x \in M$. M_R will be called *fuzzy normal* if each fuzzy submodule of M_R is a pure fuzzy submodule of M. In particular, R will be called fuzzy normal if R_R is fuzzy normal.

The following proposition shows that the notion of a t-pure fuzzy ideal is an extension of an ordinary t-pure ideal of a ring, as defined in [1].

Proposition 2.3 Let A be an ideal of R. Then the following statements are equivalent:

(1) A is right t-pure in R;

(2) The characteristic function of A, denoted by δ_A, is a t-pure fuzzy ideal of R.

Proof Suppose A is (right) t-pure in R

Now, since A is a two sided ideal of R, δ_A is obviously a fuzzy ideal of R. To prove that δ_A is fuzzy t-pure we must show that, for any fuzzy right ideal ξ of R, $\xi \cap \delta_A = \xi \circ \delta_A$. Let $x \in A$. Now,

$$(\xi \circ \delta_A)(x) = \bigvee_{x=\sum y_i z_i} [\bigwedge_i (\xi(y_i) \wedge \delta_A(z_i))]$$

$$\leq \bigvee_{x=\sum y_i z_i} [\bigwedge_i (\xi(y_i z_i) \wedge \delta_A(y_i z_i))]$$

$$= \bigvee_{x=\sum y_i z_i} [(\bigwedge_i \xi(y_i z_i)) \wedge (\bigwedge_i \delta_A(y_i z_i))]$$

$$\leq \bigvee_{x=\sum y_i z_i} [\xi(x) \wedge \delta_A(x)]$$

$$= \xi(x) \wedge \delta_A(x) = (\xi \cap \delta_A)(x)$$

Now consider the case $x \notin A$. We have

$$(\xi \cap \delta_A)(x) = \xi(x) \wedge \delta_A(x) = 0$$

$$\leq (\xi \circ \delta_A)(x).$$

For the case $x \in A$ we have $(\xi \cap \delta_A)(x) = \xi(x) \wedge \delta_A(x) = \xi(x) \wedge \delta_A(t)$. Here $t \in A$ such that $x = xt$.

(This follows, since A is (right) t-pure, sc for any $x \in A$ there exists $t \in A$ such that $x = xt$, and since $t \in A$, so $\delta_A(x) = 1 = \delta_A(t)$).

Thus $(\xi\cap\delta_A)(x)=\xi(x)\wedge\delta_A(t)$

$$\leq \bigvee_{x=\sum y_i z_i}[\bigwedge_i(\xi(y_i)\wedge\delta_A(z_i))] = (\xi o\delta_A)(x).$$

Thus, for any fuzzy right ideal ξ of R, $\xi\cap\delta_A=\xi o\delta_A$. Thus, δ_A is a t-pure fuzzy ideal of R.

Now, suppose that δ_A is a t-pure fuzzy ideal of R. We show that A is (right) t-pure in R. We show that for each right ideal B of R, $A\cap B = BA$. Since B is a right ideal of R, the characteristic function, δ_B, of B is a fuzzy right ideal of R. Since δ_A is (right) t-pure fuzzy, we have $\delta_A\cap\delta_B=\delta_B o\delta_A$. This implies that $\delta_{A\cap B}=\delta_{BA}$. From here it follows that $A\cap B = BA$. Hence A is a (right) t-pure ideal of R.

Proposition 2.4 The following assertions are true:

(1) If λ_1 and λ_2 are t-pure fuzzy ideals of R then so is $\lambda_1\wedge\lambda_2$;

(2) If $\{\lambda_i : i\in I\}$ is a family of t-pure fuzzy ideals of R, then so is $\bigvee_{i\in I}\lambda_i$.

Proof (1) λ_1 and λ_2 are (right) t- pure fuzzy ideals of R. We have to show that $\lambda_1\wedge\lambda_2$ is a (right) t-pure fuzzy ideal of R. That is, for each fuzzy right ideal μ of R $\mu o(\lambda_1\wedge\lambda_2)=\mu\wedge(\lambda_1\wedge\lambda_2)$.

Now, since λ_2 is (right) t-pure fuzzy in R therefore $\lambda_1 o\lambda_2=\lambda_1\wedge\lambda_2$. Hence

$$\mu o(\lambda_1\wedge\lambda_2)=\mu o(\lambda_1 o\lambda_2) \qquad (1)$$

Also,

$$\mu\wedge(\lambda_1\wedge\lambda_2)=(\mu\wedge\lambda_1)\wedge\lambda_2=(\mu o\lambda_1)\wedge\lambda_2 \qquad (2)$$

Now we show that $\mu o\lambda_1$ is a fuzzy right ideal of R,

i.e. $(\mu o\lambda_1)(xx') \geq (\mu o\lambda_1)(x)$

and $(\mu o\lambda_1)(x-x') \geq [(\mu o\lambda_1)(x)\wedge(\mu o\lambda_1)(x')]$ for any $x,x'\in R$.

Now $(\mu o\lambda_1)(x) = \bigvee_{x=\sum y_i z_i} [\bigwedge_i (\mu(y_i)\wedge\lambda_1(z_i))]$

$$\leq \bigvee_{x=\sum y_i z_i} [\bigwedge_i (\mu(y_i)\wedge\lambda_1(z_i x'))]$$

$$\leq \bigvee_{xx'=\sum y'_i z'_i} [\bigwedge_i (\mu(y'_i)\wedge\lambda_1(z'_i))] = (\mu o\lambda_1)(xx')$$

Also, $[(\mu o\lambda_1)(x)\wedge(\mu o\lambda_1)(x')]$

$$=\wedge[\bigvee_{x=\sum y_i z_i} [\bigwedge_{1\leq i\leq p} (\mu(y_i)\wedge\lambda_1(z_i))], \bigvee_{x'=\sum y'_j z'_j} [\bigwedge_{1\leq j\leq q} (\mu(y'_j)\wedge\lambda_1(z'_j))]]$$

$$\leq \bigvee_{x-x'=\sum y''_k z''_k} [\bigwedge_{1\leq k\leq r} (\mu(y''_k),\lambda_1(z''_k))] = (\mu o\lambda_1)(x-x')$$

Thus (2) gives us

$$\mu\wedge(\lambda_1\wedge\lambda_2)=(\mu o\lambda_1)o\lambda_2=\mu o(\lambda_1 o\lambda_2)$$

by the associativity of the operation `o´.

Hence (1) and (2) give $\mu o(\lambda_1\wedge\lambda_2)=\mu\wedge(\lambda_1\wedge\lambda_2)$.

(2) Now suppose $\{\lambda_i : i\in I\}$ is a family of (right) t-pure fuzzy ideals of R. Let μ be any fuzzy right ideal of R. We need to show that $\mu o(\bigvee_{i\in I}\lambda_i)=\mu\wedge(\bigvee_{i\in I}\lambda_i)$.

Thus, for each $x\in R$ we have

$$[\mu o(\bigvee_{i\in I}\lambda_i)](x) = \bigvee_{x=\sum y_j z_j}[\bigwedge_j (\mu(y_j)\wedge(\bigvee_{i\in I}\lambda_i)(z_j))]$$

$$\leq \bigvee_{x=\sum y_j z_j}[\bigwedge_j (\mu(y_j z_j)\wedge(\bigvee_{i\in I} \lambda_i)(y_j z_j))]$$

$$= \bigvee_{x=\sum y_j z_j}[\wedge [\bigwedge_j \mu(y_j z_j), \bigwedge_j (\bigvee_{i\in I} \lambda_i)(y_j z_j)]]$$

$$\leq \bigvee_{x=\sum y_j z_j}[\wedge [\mu(x), (\bigvee_{i\in I} \lambda_i)(x)]]$$

$$= \wedge[\mu(x), (\bigvee_{i\in I} \lambda_i)(x)]$$

$$= (\mu\wedge(\bigvee_{i\in I} \lambda_i))(x)$$

Therefore $(\mu o \bigvee_{i\in I} \lambda_i) \leq \mu\wedge(\bigvee_{i\in I} \lambda_i)$

Again, $[\mu\wedge(\bigvee_{i\in I} \lambda_i)](x) = [\mu(x)\wedge(\bigvee_{i\in I}\lambda_i)(x)]$

$$= [\mu(x)\wedge \bigvee_{i\in I} (\lambda_i(x))]$$

$$= \bigvee_{i\in I}[\mu(x)\wedge\lambda_i(x)]$$ (by the infinite meet distributive law.)

$$= \bigvee_{i\in I}[(\mu \wedge \lambda_i)(x)] = \bigvee_{i\in I}[(\mu o\lambda_i)(x)]$$

because λ_i are (right) pure fuzzy ideals of R.

Now, $(\mu o\lambda_i)(x) = \bigvee_{x=\sum y_j z_j}[\bigwedge_j [\mu(y_j), \lambda_i(z_j)]]$.

We have, $\lambda_i(z_j) \leq (\vee\lambda_i)(z_j)$

therefore, $(\mu o\lambda_i)(x) \leq \bigvee_{x=\sum y_j z_j}[\wedge [\mu(y_j), (\vee\lambda_i)(z_j)]]$

$$=[\mu o(\vee\lambda_i)](x)$$

therefore, $\bigvee_{i\in I}[(\mu o\lambda_i)(x)] \leq [\mu o(\vee\lambda_i)](x)$

Thus, $[\mu\wedge(\bigvee_{i\in I} \lambda_i)](x) \leq [\mu o(\bigvee_{i\in I}\lambda_i)](x)$

Thus $\mu\wedge(\bigvee_{i\in I}\lambda_i) \leq \mu o(\bigvee_{i\in I}\lambda_i)$.

Hence $\bigvee_{i\in I}\lambda_i$ is also a t-pure fuzzy ideal of R.

3. Characterizations of rings by the properties of their fuzzy submodules

First we recall a well known definition from ring theory. A ring R is called regular (in the sense of Von Neumann) if for each $x\in R$, there exists an element $y\in R$ such that $x=xyx$. In [2], Brown and McCoy considered the related notion of weakly regular rings. These rings were later studied by Ramamurthy [10] and others. R is called right weakly regular if, for each $x\in R$, $x\in(xR)^2$. Thus, if R is commutative and weakly regular, then R is regular (in the sense of Von Neumann).

We now state a characterization theorem for right weakly regular rings, part of whose proof is due to Ramamurthy [12]. We include a complete proof for the sake of convenience.

Proposition 3.1 For a ring R the following assertions are equivalent:

(1) R is right weakly regular;

(2) $B^2=B$ for any right ideal B of R;

(3) Each (two sided) ideal of R is t-pure.

Proof (1) $\Rightarrow$ (2) Let B be a right ideal of R. Clearly, $B^2\subseteq B$. Let $x\in B$. Then $x\in(xR)(xR)\subseteq BB=B^2$. Hence $B=B^2$.

(2) $\Rightarrow$ (3) Let A be a two sided ideal of R. In order to prove that A is right t-pure, we must show (using Proposition 2.2) that $B\cap A=BA$ for each right ideal B of R. Clearly, $BA\subseteq B\cap A$. To prove the reverse inclusion, let

$x\in B\cap A$. Since $x\in xR=(xR)(xR)=x(RxR)\subseteq xA\subseteq BA$, we have $B\cap A\subseteq BA$ and so $B\cap A=BA$.

(3) $\Rightarrow$ (1) Let $x\in R$ and $A=RxR$ be the two-sided ideal generated by x. Let B be the right ideal xR generated by x. Then $x\in B\cap A=BA$, since A is right t-pure. But $BA=(xR)(RxR)\subseteq xR^2xR\subseteq(xR)(xR)$. Hence $x\in(xR)(xR)$, showing that R is right weakly regular.

Next we prove a Lemma.

Lemma 3.1 Let R be a right weakly regular ring and I an ideal of R. Then for any finite number of elements $a_1,a_2,\ldots,a_n\in I$, there exists an element $b\in I$ such that $a_k=a_kb$ for $k=1,\ldots,n$.

Proof [1] We prove the Lemma by induction on n. The result is valid for n=1, since then I is right pure by Proposition 3.1 (3). Now suppose that the result is valid for n and let $a_1,\ldots,a_{n+1}$ be n+1 elements in I. Since I is right pure, we can choose an element $t\in I$ such that $a_{n+1}=a_{n+1}t$. By induction hypothesis, there exists $t'\in I$ such that for $k=1,\ldots,n$,

$$(a_k-a_kt)t' = (a_k-a_kt)$$

Hence $$a_{n+1}(t+t'-tt') = a_{n+1}t + a_{n+1}t' - a_{n+1}tt'$$

$$= a_{n+1} + a_{n+1}t' - a_{n+1}t' = a_{n+1}$$

and for $k=1,\ldots,n$; we have

$$a_k(t+t'-tt') = a_kt+a_kt'-a_ktt' = a_kt+(a_k-a_kt)t'$$

$$= a_kt+a_k-a_kt=a_k.$$

This completes the inductive argument.

We now establish a module theoretic characterization of weakly regular rings.

Proposition 3.2 The following assertions for R are equivalent:

(1) R is right weakly regular;

(2) R_R is normal;

(3) Each R-module is normal.

Proof (1) ⇒ (2) In order to prove that R_R is normal, we show, by Definition 2.4, that each right ideal J of R is pure, that is, $J \cap RI = JI$ for each two sided ideal I of R. But $J \cap RI = J \cap I = JI$ by condition (3) of Proposition 3.1.

(2) ⇒ (1) Let $x \in R$. If $B = xR$ is the right ideal of R generated by x and $A = RxR$ the two sided ideal of R generated by x, then, by the hypothesis, $B \cap RA = B \cap A = BA$. Since $x \in B \cap A$, it follows that $x \in BA = (xR)(RxR) = xR^2xR \subseteq (xR)(xR)$. Thus R is right weakly regular.

(1) ⇒ (3) Let M be a right R-module. To show that M is normal we must show that every submodule N of M is pure in the sense of Definition 2.4, that is, $N \cap MI = NI$ for each two side ideal I of R. Clearly, $NI \subseteq N \cap MI$. So, we prove that $N \cap MI \subseteq NI$. Let $x \in N \cap MI$. Since $x \in MI$, we can write $x = \sum_{i=1}^{n} m_i a_i$, where $m_i \in M$ and $a_i \in I$. By Lemma 3.1, there exists $b \in I$ such that $a_i = a_i b$ $(i=1,\dots,n)$.

Hence $x = \sum_{i=1}^{n} m_i a_i b = (\sum_{i=1}^{n} m_i a_i)b = xb \in NI$.

(3) ⇒ (1) Since each right R-module is normal, so in

particular, R_R is normal. Hence, by (2)⇒(1) proved above, it follows that R is weakly regular.

We now prove the following theorem.

Theorem 3.1 The following assertions for R are equivalent:

(1) R is right weakly regular;

(2) $B^2=B$ for each right ideal B of R;

(3) Each (two sided) ideal of R is right t-pure;

(4) R_R is normal;

(5) Each right R-module is normal;

(6) Each fuzzy right ideal of R is idempotent (A fuzzy right ideal λ of R is called idempotent if $\lambda.\lambda=\lambda^2=\lambda$);

(7) Each fuzzy ideal of R is a t-pure fuzzy ideal;

If R is commutative, then the above assertions are equivalent to:

(8) R is (Von Neumann) regular.

Proof (1) ⟺ (2) ⟺ (3) ⟺ (4): This follows from Propositions 3.1 and 3.2. Also, (1) ⟺ (8) is clear.

(1) ⇒ (6): Let δ be a fuzzy right ideal of R. Then, for any $x \in R$,

$$(\delta o \delta)(x) = \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\wedge_i(\delta(y_i)\wedge\delta(z_i)].$$

$$\leq \bigvee_{x=\sum y_i z_i} [\wedge_i(\delta(y_i z_i)\wedge\delta(z_i))]$$

$$= \bigvee_{x=\sum y_i z_i} [\wedge_i[(\wedge_i\delta(y_i z_i))\wedge\delta(z_i)]]$$

$$\leq \bigvee_{x=\sum y_i z_i} [\wedge_i [\delta(x) \wedge \delta(z_i)]]$$

$$\leq \bigvee_{x=\sum y_i z_i} [\delta(x)] = \delta(x)$$

Thus $(\delta o \delta)(x) \leq \delta(x)$. Hence $\delta o \delta \subseteq \delta$.

Again, $\delta(x) = \delta(x) \wedge \delta(x) \leq \delta(xa) \wedge \delta(xb)$,

where $a, b \in R$ are such that $x = xaxb$

$$\leq \bigvee_{x=\sum y_i z_i} [\wedge_i (\delta(y_i) \wedge \delta(z_i))] = (\delta o \delta)(x).$$

Hence $\delta(x) \leq (\delta o \delta)(x)$. Therefore $\delta \subseteq \delta o \delta$. It follows that $\delta o \delta = \delta$.

(6) → (1): Let $x \in R$. We show that $x \in (xR)^2$. Let $A = xR$ be the right ideal generated by x. Let δ_A be the characteristic function of A. δ_A is a fuzzy right ideal of R. Hence $\delta_A = \delta_A o \delta_A = \delta_{A^2}$. This implies that $A = A^2$. Since $x \in A$ it follows that $x \in A^2 = (xR)^2$. Hence R is (right) weakly regular.

(1) → (7). Let δ be a fuzzy ideal of R and μ a fuzzy right ideal of R. We show that $\mu o \delta = \mu \cap \delta$. Now, for any $x \in R$

$$(\mu o \delta)(x) = \bigvee_{x=\sum y_i z_i} [\wedge_i (\mu(y_i) \wedge \delta(z_i))] \leq \bigvee_{x=\sum y_i z_i} [\wedge_i (\mu(y_i z_i) \wedge \delta(y_i z_i))]$$

$$= \bigvee_{x=\sum y_i z_i} [[(\wedge_i \mu(y_i z_i)) \wedge (\wedge_i \delta(y_i z_i))]]$$

$$\leq \bigvee_{x=\sum y_i z_i} [[\mu(x) \wedge \delta(x)]] = \mu(x) \wedge \delta(x) = (\mu \cap \delta)(x).$$

Again, $(\mu \cap \delta)(x) = \mu(x) \wedge \delta(x) \leq \mu(xa) \wedge \delta(xb)$

where $a, b \in R$ s.t. $x = xaxb$.

$$\leq \bigvee_{x=\sum y_i z_i} [\wedge_i [\mu(y_i) \wedge \delta(z_i)]] = (\mu o \delta)(x).$$

Thus $\mu o \delta = \mu \cap \delta$. Hence δ is fuzzy pure.

(7) → (1): We show that R is (right) weakly regular. Let $x \in R$ and let $A = RxR$ be the two sided ideal generated by x. Let δ_A be the characteristic function of A. Then δ_A is a fuzzy ideal of R. Hence, by the hypothesis, δ_A is fuzzy pure. Hence, A is pure in R, by Proposition 2.3. Since $x \in A$ and A is pure in R, therefore, there exists $y \in A$ such that $x = xy$. This means that $x \in xA = x(RxR) = (xR)^2$. Hence, R is (right) weakly regular. This completes the proof of the theorem.

Next we prove the following Lemmas.

Lemma 3.2 If M is a module over a weakly regular ring R then for any fuzzy submodule λ of M and any fuzzy ideal μ of R,

$$(\lambda o \mu)(x) = \bigvee_{\substack{x=\sum y_i z_i \\ y_i \in M \\ z_i \in R}} (\wedge_i [\lambda(y_i z_i) \wedge \mu(z_i)]) \quad \text{for all } x \in M.$$

Proof Note that for any $x \in M$,

$$(\lambda o \mu)(x) \overset{\text{defn.}}{=} \bigvee_{x=\sum y_i z_i} (\wedge_i [\lambda(y_i) \wedge \mu(z_i)]).$$

Since R is weakly regular therefore, for each z_i, there exists a_i and $b_i \in R$ such that $z_i = z_i a_i z_i b_i$. Note that

$$\lambda(y_i z_i) \leq \lambda(y_i z_i a_i) \leq \lambda(y_i z_i a_i z_i b_i) = \lambda(y_i z_i).$$

Thus $\lambda(y_i z_i) = \lambda(y_i z_i a_i)$.

Also, $\mu(z_i)\le\mu(z_ib_i)\le\mu(z_ia_iz_ib_i)=\mu(z_i)$.

Thus $\mu(z_i)=\mu(z_ib_i)$.

Hence, we have,

$$(\lambda o\mu)(x)\le \bigvee_{x=\sum y_iz_i} (\wedge_i[\lambda(y_iz_i)\wedge\mu(z_i)])= \bigvee_{x=\sum y_iz_i} (\wedge_i[\lambda(y_iz_ia_i)\wedge\mu(z_ib_i)])$$

$$\le \bigvee_{\substack{x=\sum y'_iz'_i\\ y'_i\in M\\ z'_i\in R}} (\wedge_i[\lambda(y'_i)\wedge\mu(z'_i)]) = (\lambda o\mu)(x).$$

It follows that $(\lambda o\mu)(x)= \bigvee_{\substack{x=\sum y_iz_i\\ y_i\in M\\ z_i\in R}} (\wedge_i[\lambda(y_iz_i)\wedge\mu(z_i)])$

for all $x\in M$.

Lemma 3.3 Let M be a cyclic module over a weakly regular ring R. Then any element $\sum_i y_iz_i$, where $y_i\in M$ and $z_i\in R$, of M can be written as yz where y is the generator of M and z is an element of R which satisfies the inequality

$$\mu(z) \ge \wedge_i \mu(z_i)$$

for all fuzzy ideals μ of R.

Proof We have $y_i=yz'_i$, where y is the generator of M and $z'_i\in R$. Thus, $\sum_i y_iz_i=\sum_i (yz'_i)z_i=\sum_i y(z'_iz_i)=y\sum_i z'_iz_i$.

$\sum_i z'_iz_i\in R$, call it z.

Thus $\sum_i y_iz_i$ can be written as yz.

Again, $\mu(z)=\mu(\sum_i z'_iz_i)\ge \wedge_i \mu(z'_iz_i)\ge \wedge_i \mu(z_i)$.

Thus $\mu(z)\ge \wedge_i\mu(z_i)$ for all fuzzy ideals μ of R.

Proposition 3.3 If M is a cyclic module over a weakly regular ring R then M is fuzzy normal over R.

Proof We have to prove that for all $x \in M$

$(\lambda o \mu)(x) = [\lambda \wedge (\mathcal{M} o \mu)](x)$ where λ is any fuzzy submodule of M and μ is any ideal of R. Note that $\mathcal{M}$ denotes the fuzzy function

$$\mathcal{M}(x)=1 \text{ for all } x \in M.$$

From Lemma 3.2 we have

$$(\lambda o \mu)(x) = \bigvee_{x=\sum y_i z_i} (\wedge_i [\lambda(y_i z_i) \wedge \mu(z_i)]) \tag{3}$$

$$= \bigvee_{x=\sum y_i z_i} ((\wedge_i [\lambda(y_i z_i)]) \wedge (\wedge_i [\mu(z_i)]))$$

$$\leq \bigvee_{x=\sum y_i z_i} (\lambda(x) \wedge (\wedge_i \mu(z_i))).$$

Now, using Lemma 3.3, we have

$$(\lambda o \mu)(x) \leq \bigvee_{x=yz} (\lambda(yz) \wedge \mu(z)) \tag{4}$$

where $x=yz$ and $\mu(z) \geq \wedge_i \mu(z_i)$.

Finally, from (4), $(\lambda o \mu)(x) \leq \bigvee_{x=yz} (\lambda(yz) \wedge \mu(z))$

$$\leq \bigvee_{x=\sum y_i z_i} (\wedge_i [\lambda(y_i z_i) \wedge \mu(z_i)]) = (\lambda o \mu)(x).$$

Therefor, $(\lambda o \mu)(x) = \bigvee_{x=yz} (\lambda(yz) \wedge \mu(z))$

$$= \bigvee_{x=yz} (\lambda(x) \wedge \mu(z)) = \lambda(x) \wedge [\bigvee_{x=yz} \mu(z)] \tag{5}$$

by using the infinite meet distributive law.

Now, $\bigvee_{x=\sum y_i z_i} (\wedge_i \mu(z_i)) \leq \bigvee_{x=yz} (\mu(z))$

and $A = \{\wedge_i \mu(z_i) : x = \sum y_i z_i\}$, $A' = \{\mu(z) : x = yz\}$.

Since A′ is contained in A, therefore,

$$\bigvee_{x=\sum y_i z_i}(\wedge_i \mu(z_i))=\bigvee_{x=yz}\mu(z))$$

Thus, (5) becomes $(\lambda o \mu)(x)=\lambda(x)\wedge[\bigvee_{x=\sum y_i z_i}(\wedge_i \mu(z_i))]$ (6)

We have,

$$(\mathcal{M} o \mu)(x)=\bigvee_{x=\sum y_i z_i}(\wedge_i[\mathcal{M}(y_i)\wedge\mu(z_i)])$$

$$=\bigvee_{x=\sum y_i z_i}(\wedge_i[1\wedge\mu(z_i)])=\bigvee_{x=\sum y_i z_i}(\wedge_i \mu(z_i)) \quad (7)$$

Substituting (7) in (6), we have,

$$(\lambda o \mu)(x)=\lambda(x)\wedge(\mathcal{M} o \mu)(x)=(\lambda\wedge\mathcal{M} o \mu)(x).$$

Thus, M is fuzzy normal over R.

We now add a remark.

Remark. If $\mathcal{R}$ denotes the fuzzy function defined by $\mathcal{R}(x)=1$ for all $x\in R$, then, for all fuzzy ideals μ of a ring R, $\mathcal{R} o \mu=\mu$. This holds, since, for $x\in R$, we have

$$(\mathcal{R} o \mu)(x)=\bigvee_{x=\sum_{i=1}^{n} y_i z_i}[\wedge_i(\mathcal{R}(y_i)\wedge\mu(z_i))]$$

$$=\bigvee_{x=\sum_{i=1}^{n} y_i z_i}[\wedge_i(1\wedge\mu(z_i))]=\bigvee_{x=\sum_{i=1}^{n} y_i z_i}[\wedge_i \mu(z_i)]$$

Since $x=1x$ is one of the factorizations of x and also, since $\mu(x)=\mu(\sum_i y_i z_i)\geq \wedge_i \mu(y_i z_i)\geq \wedge_i \mu(z_i)$, it follows that $(\mathcal{R} o \mu)(x)=\mu(x)$.

Finally, we prove the following characterization theorem for weakly regular rings.

Theorem 3.2 The following assertions for R are equivalent:

(1) R is (right) weakly regular;

(2) Each cyclic right R-module is fuzzy normal;

(3) R_R is fuzzy normal.

Proof (1) ⇒ (2). This follows from Proposition 3.3

(2) ⇒ (3). This is immediate.

(3) ⇒ (1) Let μ be a fuzzy ideal of R. We prove that μ is t-pure, that is, for any fuzzy right ideal λ of R, $\lambda\cap\mu=\lambda\mu$, by Definition 2.5. Since R_R is fuzzy normal, λ is pure in the sense of Definition 2.6. Hence $\lambda\cap(R\circ\mu)=\lambda\circ\mu$. Since $R\circ\mu=\mu$, it follows that $\lambda\circ\mu=\lambda\cap(R\circ\mu)=\lambda\cap\mu$. Hence μ is t-pure and so, by Theorem 3.1, R is weakly regular.

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