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Abstract

We consider the problem of heat flow across a semi-infinite plane contact in a two-layered plate in which the contact between the layers takes place in one part of the interface while the other part is perfectly insulated. The Wiener-Hopf technique has been employed to obtain the solution in a closed form.

Introduction.

The problem of heat flow across an interface between conducting bodies is of interest in many practical situations. One-dimensional problem concerning the contact of two half-spaces and some interesting relevant cases can be found in literature (see Carslaw and Jaeger [2]). A more general and interesting situation has been recently considered by Georgiadis, Barber and Ben Ammar [3] who study part-contact of two dissimilar half-spaces while part of the interface is perfectly insulated so that no heat exchange takes place. The method used is Jones' modification [4] of the so-called Wiener-Hopf technique which has been used in a variety of situations concerning electromagnetic waves [6], elastic waves [1], [9] and acoustic waves [6].

A mathematically similar problem that fits this model is that of diffusion of two fluids at a semi-infinite interface. The analysis carried out in [3] can easily be re-interpreted in this physical situation also.

In this paper we consider the problem considered in [3] with the difference that instead of interface of two half-spaces we consider two layers of different conductivities with the same uniform thickness H in contact at a semi-infinite interface. As mentioned above, this could also describe the model of two layers of different fluids in contact at a semi-infinite interface. We shall, however, describe it as a contact problem between two conducting layers – the mathematical problem in the two cases being the same. It is assumed that the two layers are perfectly insulated at part of the interface chosen to be $x < 0$, while at the remaining interface ($x > 0$) the two layers are in perfect contact. With y-axis chosen to be directed upwards, we assume the outer surfaces of the plate at $y = \pm H$ kept at zero temperature. The conductivity, temperature at time t and the heat flux for the upper layer is denoted, respectively by $k_1, T_1(x,y,t)$ and $q_{y_1}(x,y,t)$ while we use the subscript 2 for these quantities for the lower layer of the plate. We assume the initial temperature at $t = 0$ in the upper layer to be zero and in the lower layer to be T_0. The current temperature at time t in the lower layer is however re-scaled as $\tilde{T}_2(x,y,t) = T_2(x,y,t) - T_0$ so that $\tilde{T}_2(x,y,0) = 0$ is obtained at $t = 0$. In what follows we agree to suppress $\tilde{}$ and write $T_2(x,y,t)$ to denote $\tilde{T}_2(x,y,t)$. The problem can now be stated as

$$\nabla^2 T_j = \frac{1}{k_j}\frac{\partial T_j}{\partial t}, \quad j = 1,2, \tag{1}$$

where ∇^2 is the two-dimensional Laplace operator. The heat flux $q_{yj}(x,y,t)$ is given by

$$q_{yj}(x,y,t) = -k_j \frac{\partial T_j}{\partial y}. \tag{2}$$

The physical considerations induce the following boundary and initial conditions

(a) At $t = 0$,

$$T_1(x,y,0) = T_2(x,y,0) = 0. \tag{3a}$$

(b) At $y = 0, \quad -\infty < x < 0$,

$$q_{yj}(x,0,t) = 0, \tag{3b}$$

or equivalently, $\frac{\partial T_j}{\partial y}(x,0,t) = 0, \quad j = 1,2.$

(c) At $y = 0, \quad 0 < x < \infty$,

$$T_1(x,0,t) = T_2(x,0,t) + T_0,$$

and (3c)

$$q_{y_1}(x,0,t) = q_{y_2}(x,0,t).$$

which can be written as

$$k_1 \frac{\partial T_1(x,0,t)}{\partial y} = k_2 \frac{\partial T_2(x,0,t)}{\partial y}.$$

(d) At $y = \pm H, \quad -\infty < x < \infty$

$$T_j(x, \pm H, t) = 0, \quad j = 1, 2. \tag{3d}$$

The Wiener-Hopf Method.

Let us define the Laplace transform pair in time as

$$L\{f(t)\} = \overline{f}(p) = \int_0^\infty f(t)e^{-pt}\,dt, \tag{4a}$$

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \overline{f}(p)\,e^{pt}\,dp, \tag{4b}$$

where α is chosen in such a way that the integration path lies within the domain of convergence (see Noble [8] for details).

We denote the two-sided Laplace transform of the function $f(x, y, t)$ with respect to x as $f^*(\xi, y, t)$ and define it as

$$f^*(\xi, y, t) = \int_{-\infty}^{\infty} f(x, y, t)e^{-\xi x}\,dx, \tag{5a}$$

while the inverse transform is defined as

$$f(x, y, t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} f^*(\xi, y, t)e^{\xi x}d\xi, \tag{5b}$$

where β is chosen in such a way that the integration path lies in the domain of convergence. We split the range of integration in (5a) to define the transforms $f_+^*(\xi, y, t)$ and $f_-^*(\xi, y, t)$ of $f(x, y, t)$ as

$$f_+^*(\xi, y, t) = \int_0^\infty f(x, y, t)e^{-\xi x}dx, \tag{6a}$$

$$f_-^*(\xi, y, t) = \int_{-\infty}^0 f(x, y, t)e^{-\xi x}dx, \tag{6b}$$

so that

$$f^*(\xi, y, t) = f^*_+(\xi, y, t) + f^*_-(\xi, y, t). \tag{7}$$

If $\xi = \sigma + i\tau$ and $|f| \leq A\ \exp(\sigma_- x)$ as $x \to \infty$ and $|f| \leq B \exp(\sigma_+ x)$ as $x \to -\infty$ A, B constants, then $f^*_+(\xi, y, t)$ is analytic in the right half-plane $\sigma > \sigma_-$ and $f^*_-(\xi, y, t)$ is analytic for $\sigma < \sigma_+$. Thus, $f^*(\xi, y, t)$ is regular in the region defined by $-\eta < \sigma < \eta$ where $\sigma_+ > \sigma_-$. It seems reasonable to have $\sigma_+ = \eta$ and $\sigma_- = -\eta$ so that $f^*(\xi, y, t)$ is regular in $\sigma_- < \sigma < \sigma_+$ (Noble [8]).

Applying the Laplace transform in time and the two-sided Laplace transform in x to the partial differential equation (1), we obtain

$$\frac{\partial^2 \overline{T}^*_j(\xi, y, t)}{\partial y^2} - \left(\frac{p}{k_j} - \xi^2\right) \overline{T}^*_j = 0, \quad j = 1, 2. \tag{8}$$

The boundary conditions (3a - 3d) transform into the following using (5a) and (6a)

$$\overline{T}^*_j(\xi, \pm H, p) = 0, \quad j = 1, 2, \tag{9a}$$

$$\overline{T}'^*_-(\xi, 0, p) = 0, \quad j = 1, 2. \tag{9b}$$

$$k_1 \overline{T}'^*_{1+}(\xi, 0, p) = k_2\, \overline{T}'^*_{2+}(\xi, 0, p), \tag{9c}$$

and

$$\overline{T}^*_1(\xi, 0, p) = \overline{T}^*_2(\xi, 0, p) + \overline{T}^*_0(\xi, 0, p). \tag{9d}$$

Here, $'$ denotes differentiation with regard to y.

The solution of (8) that satisfies (9a) is

$$\begin{aligned} \overline{T}^*_1 &= A(\xi, p) \sinh \gamma_1 (y - H), \quad y > 0, \\ \overline{T}^*_2 &= C(\xi, p) \sinh \gamma_2 (y + H), \quad y < 0, \end{aligned} \tag{10}$$

where

$$\gamma_j^2 = \left(\frac{p}{k_j} - \xi^2\right), \quad j = 1, 2. \tag{11}$$

Differentiating (1) and (2), putting $y = 0$ and eliminating the unknowns $A(\xi, p)$ and $C(\xi, p)$, we get

$$\begin{aligned} \overline{T}_1^*(\xi, 0, p) &= \tfrac{-\tanh \gamma_1 H}{\gamma_1} \overline{T}_1^{*'}(\xi, 0, p), \\ \overline{T}_2^*(\xi, 0, p) &= \tfrac{\tanh \gamma_2 H}{\gamma_1} \overline{T}_2^{*'}(\xi, 0, p), \end{aligned} \tag{12}$$

Using the decomposition (7) and keeping in mind $\overline{T}_{j-}^*(\xi, 0, p) = 0, \quad j = 1, 2$, we obtain

$$\overline{T}_{1+}^*(\xi, 0, p) + \overline{T}_{1-}^*(\xi, 0, p) = \frac{-\tanh \gamma_1 H}{\gamma_1} \overline{T}_{1+}^{*'}(\xi, 0, p) \tag{13a}$$

$$\overline{T}_{2+}^*(\xi, 0, p) + \overline{T}_{2-}^*(\xi, 0, p) = \frac{k_1}{\gamma_2 k_2} \tanh \gamma_2 H \, \overline{T}_{1+}^{*'} \tag{13b}$$

where in (13) use of (9b) and (9c) has been made.

Now subtracting (13a) and (13b) and using $\overline{T}_{1+}^* = \overline{T}_{2+}^* + \frac{T_0}{\xi p}$, we arrive at

$$\overline{T}_{2-}^*(\xi, 0, p) - \overline{T}_{1-}^*(\xi, 0, p) + \frac{T_0}{\xi p} = \overline{T}_{1+}^{*'}(\xi, 0, p) R(\xi, P). \tag{14}$$

Equation (14) is the Wiener-Hopf equation. We can obtain another equation of the above type by a similar manipulation but we need only one of these to find the required unknowns. In (14), $R(\xi, p)$ is given by

$$R(\xi, p) = \frac{k_1}{\gamma_2 k_2} \tanh \gamma_2 H + \frac{1}{\gamma_1} \tanh \gamma_1 H. \tag{15}$$

We now need to factorize the function $R(\xi, p)$ into functions that are analytic in the left (right) half-planes only. Following the usual factorization methods we have demonstrated in the appendix that

$$R(\xi, p) = \frac{L_+(\xi, p)}{L_-(\xi, p)}, \tag{16}$$

where $L_\pm(\xi, p)$ are given by (A9) and (A10).

Equation (14) now becomes

$$\left\{\overline{T}^*_{2-}(\xi,0,p) - \overline{T}^*_{1-}(\xi,0,p)\right\} L_-(\xi,p) + \frac{T_0}{\xi p} L_-(\xi,p) = \overline{T}^{*'}_{1+}(\xi,0,p) L_+(\xi,p). \qquad (17)$$

The mixed term on the left hand side can be decomposed using the decomposition theorem given by Noble [8] page 15-16. We therefore write

$$\frac{T_0}{\xi p} L_-(\xi,p) = \overline{M}^*_+(\xi,p) + \overline{M}^*_-(\xi,p). \qquad (18)$$

Explicit expressions for $\overline{M}^*_\pm(\xi,p)$ can be found using the known $T_0(x,y,t)$. The special case of constant T_0 shall be taken up in the end. We now have

$$\left\{\overline{T}^*_{2-}(\xi,0,p) - \overline{T}^*_{1-}(\xi,0,p)\right\} L_-(\xi,p) + \overline{M}^*_-(\xi,p) = \overline{T}^{*'}_{1+}(\xi,0,p) L+(\xi,p) - \overline{M}^*_+(\xi,p). \quad (19)$$

The left hand side of the equation (19) is analytic in the left half plane $\sigma < \sigma_+$ while the right hand side is analytic in the right half plane $\sigma > \sigma_-$. Due to the common strip of analyticity $\sigma_- < \sigma < \sigma_+$, both sides define an entire function of ξ. Due to the asymptotic behaviour

$$\overline{T}^{*'}_j(\xi,0,p) \to \xi^{-1/2} \text{ as } |\xi| \to \infty$$

(Georgiadis, Barber and Ben Ammar [3]).

We can employ the Liouville theorem to conclude that this entire function is zero. Hence.

$$\overline{T}^{*'}_+(\xi,0,p) = \frac{\overline{M}^*_+(\xi,p)}{L_+(\xi,p)}. \qquad (20)$$

Using (9c), we can write

$$\overline{T}^{*'}_{2+}(\xi,0,p) = \frac{k_1}{k_2} \frac{\overline{M}^*_+(\xi,p)}{L_+(\xi,p)}. \qquad (21)$$

Hence we find

$$\overline{T}^*_1(\xi,y,p) = \frac{-\tanh\gamma_1 H}{\gamma_1} \frac{\overline{M}^*_+(\xi,p)}{L_+(\xi,p)}, \qquad (22)$$

$$\overline{T}^*_2(\xi,y,p) = k_1 \frac{\tanh\gamma_2 H}{k_2\gamma_2} \frac{\overline{M}^*_+(\xi,p)}{L_+(\xi,p)}, \qquad (23)$$

Heat Flux Across the Contact.

In practical problem, we are interested in heat flux at the contact rather than the temperature distribution in the body. Recalling our definition of the heat flux $q_{yj}(x, y, t)$, we can use (2) and (20) to write

$$\overline{q}^*_{yj}(\xi, 0, p) = -k_j \frac{\overline{M^*_+}(\xi, p)}{L_+(\xi, p)}. \tag{24}$$

We can substitute for $\overline{M^*_+}$ and $L_+(\xi, p)$ through (18) and (A9) and perform inversion of the integral transforms to obtain the heat flux.

To simplify the calculations, we assume that the initial temperature $T_0 =$ constant. A quick calculation then shows that

$$\overline{M^*_+} = \frac{T_0 L_-(0, p)}{\xi P L_+(\xi, p)}. \tag{25}$$

We therefore get

$$\overline{q}_{yj}(x, 0, p) = -\frac{T_0}{2\pi i p} \int_{c-i\infty}^{c+i\infty} \frac{L_-(0, p) e^{\xi x}}{\xi L_+(\xi, p)} d\xi, \tag{26}$$

where c is chosen such that $\sigma_- < c < \sigma_+$.

Noting that $L_+(\xi, p) = R(\xi, p)\, L_-(\xi, p)$, we notice that if we choose the contour in the left half plane, then the singularities of the integrand that lie in the appropriate half plane are $\xi = 0$, $\xi = -\alpha_m$, $m = 1, 2, \ldots$ (see A1) and the branch points at $\xi = -b_1$, $-b_2$ where $b_j^2 = \frac{p}{k_j}$, $j = 1, 2$. The contribution $\overline{q}^{(1)}_{yj}(x, 0, p)$ from poles to the integral in (26) is

$$\overline{q}^{(1)}_{yj}(x, 0, p) = -\frac{T_0 L_-(0, p)}{p} \left[\frac{1}{L_+(0, p)} + \sum_{m=1}^{\infty} \frac{e^{-\alpha_m x} / \alpha_m}{\frac{d}{d\xi} [R(\xi, p)]_{\xi = \alpha_m}} \right]. \tag{27}$$

The first term is independent of x and upon taking an inverse transform in p will give rise to a term indicating a one-dimensional heat flux across the interface.

The contribution due to the branch points at $\xi = -b_1$, $-b_2$, denoted by $\overline{q}^{(2)}_{yj}(x, 0, p)$ can be calculated by deforming the contour of integration so as to go around the branch cut $(-\infty, -b_1)$ where $b_1 < b_2$ is assumed. The integral along the branch cut line gives

$$\overline{q}_{yj}^{(2)}(x,0,p) = -\frac{T_0}{2\pi ip}\left[\int_{-\infty}^{-b_1} \frac{L_-(0,p)e^{\xi x}}{\xi L_+(\xi,p)}d\xi + \int_{-b_1}^{-\infty} \frac{L_-(0,p)e^{\xi x}}{\xi L_+(\xi,p)}d\xi\right]. \qquad (28)$$

Since $L(\xi,p)$ takes complex-conjugate values along $(-\infty,-b_1)$ and $(-b_1,-\infty)$, we can decompose the path into $(-\infty,-b_2)$ and $(-b_2,-b_1)$ and write the above integral

$$\overline{q}_{yj}^{(2)}(x,0,p) = -\frac{T_0 L_-(0,p)}{\pi ip}\left\{\int_{-\infty}^{-b_2} \frac{e^{\xi x}}{\xi L_+(\xi,p)}d\xi + \int_{-b_2}^{-b_1} \frac{e^{\xi x}}{\xi L_+(\xi,p)}d\xi\right\}. \qquad (29)$$

These integrals can be evaluated once $L_+(\xi,p)$, which has been found in the Appendix through (A9), is inserted.

We have found the solution in the p-plane and in order to determine it in the un-transformed variables, we can take Laplace inverse transform in p given by (4b). However, if we assume the heat conduction to be harmonic in time, we may use these as solutions in the un-transformed plane by simply inserting the usual exponential time-dependence factor frequently used in time-harmonic problems.

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Appendix

Let

$$R(\xi,P) = \frac{k_1\gamma_1 \tanh\gamma_2 H + \gamma_2 k_2 \tanh\gamma_1 H}{\gamma_1\gamma_2 k_2} = \frac{F(\xi,p)}{\gamma_1\gamma_2 k_2}. \quad \text{(A1)}$$

Let $\pm\alpha_m$ $m = 1, 2, \ldots$ be the zeros of $F(\xi,p)$, then

$$F(\xi,p) = \prod_{m=1}^{\infty}(\xi^2 - \alpha_m^2)G(\xi,p), \quad \text{(A2)}$$

where

$$G(\xi,p) = F(\xi,p)\Big/ \prod_{m=1}^{\infty}\left(\xi^2 - \alpha_m^2\right) \quad \text{(A3)}$$

is a function free of zeros.

We can now follow Jones [6] p 573, and write

$$G(\xi,p) = \frac{G_+(\xi,p)}{G_-(\xi,p)}, \quad \text{(A4)}$$

where

$$\ln(G_+(\xi,p) = -\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} \frac{\ln G(\xi,p)}{\lambda - \xi} d\lambda. \quad \text{(A5)}$$

The contour of integration is as described in [6].

$$G_+(\xi,p)G_-(\xi,p) = 1. \quad \text{(A6)}$$

It may be noted that $G_\pm(\xi,p)$ have no zeros in their respective domain of regularity.

Now

$$\gamma_j^2 = (b_j^2 - \xi^2), \quad = (b_j + \xi)(b_j - \xi),\ b_j^2 = \frac{p}{k_j}.$$

We can, therefore, write

$$R(\xi,p) = \prod_{m=1}^{\infty} \frac{(\xi+\alpha_m)G_+(\xi,p)(\xi-\alpha_m)}{(b_j+\xi)(b_j-\xi)G_-(\xi,p)} \tag{A7}$$

$$= \frac{L_+(\xi,p)}{L_-(\xi,p)}, \tag{A8}$$

where

$$L_+(\xi,p) = \frac{(\xi+\alpha_m)G_+(\xi,p)}{(\xi+b_j)}, \tag{A9}$$

$$L_-(\xi,p) = -\frac{(\xi-b_j)}{(\xi-\alpha_m)G_-(\xi,p)}. \tag{A10}$$

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