King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 129

October 1992

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FEEDBACK STABILIZATION OF NEUTRAL SYSTEMS VIA THE TRANSFORMATION TECHNIQUE

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Abstract

Within the class of delay systems, a neutral system may be distinguished by the possible presence of a neutral root chain - an infinite chain of eigenvalues in a vertical strip of the complex plane. Assuming that such a neutral root chain, if it exists, is stable, this work shows how our method of transformation can be extended to obtain feedback controllers for this class of neutral systems.

1 Introduction

Let $\mathcal{C}([-r,0];\mathbf{R}^n)$ denote the $\mathbf{R}^n$ valued continuous functions on $[-r,0]$, $\mathbf{BV}([-r,0];\mathbf{R}^{n\times n})$ denote the $n\times n$ matrix valued functions of bounded variation on $[-r,0]$ where $0<r<\infty$ and $\mathcal{C}^1((0,\infty);\mathbf{R}^n)$ denote the $\mathbf{R}^n$ valued differentiable functions on $(0,\infty)$. For a fixed $t\in(0,\infty)$, let the segment function, $x_t\in\mathcal{C}([-r,0];\mathbf{R}^n)$, be defined by $x_t(\theta)=x(t+\theta)$ where $\theta\in[-r,0]$. Following Hale(1977), define $\mathcal{D}:\mathcal{C}([-r,0];\mathbf{R}^n)\to\mathbf{R}^n$ by $\mathcal{D}x_t=$

$x(t)-\int_{-r}^{0}d\mu(\theta)x(t+\theta)$ where $\mu \in \mathbf{BV}([-r,0];\mathbf{R}^{n\times n})$ and $\lim_{s\to 0}\mathbf{Var}_{[-s,0]}\mu =$ 0. The last condition implies that μ is continuous at $\theta = 0$. Assuming that $\mathcal{D}x_t \in \mathcal{C}^1((0,\infty);\mathbf{R}^n)$, we are interested in the neutral functional differential equation(nfde) given by

$$\mathcal{S}_n : \qquad \frac{d}{dt}\mathcal{D}x_t = \int_{-r}^{0} d\alpha(\theta)x(t+\theta) + \int_{-h}^{0} d\beta(\theta)u(t+\theta) \qquad (1.1)$$

with initial function

$$\phi \in \mathcal{C}([-r,0];\mathbf{R}^n)$$

where $u \in \mathbf{L}_1^{loc}((0,\infty);\mathbf{R}^m)$ is a given control function, $x(t) \in \mathbf{R}^n$ and $\alpha,\beta \in \mathbf{BV}([-r,0];\mathbf{R}^{n\times n})$. By a solution of (1.1) is meant a function $x(\cdot) \in \mathcal{C}((0,\infty);\mathbf{R}^n)$ satisfying (1.1) such that $x_{t=0}(\theta) = \phi(\theta)$, $\theta \in [-r,0]$.

Let ν_0 denote a desired margin of stability and $\mathbf{C}^+_{-\nu_0} = \{s \in \mathbf{C} : \mathrm{Re}s \geq -\nu_0\}$ represent the right half of the complex plane and $\mathbf{C}^-_{-\nu_0} = \{s \in \mathbf{C} : \mathrm{Re}s < -\nu_0\}$ the left half. The spectrum of (1.1) is given by $\sigma(\mathcal{S}_n) = \{s \in \mathbf{C} : \det\Delta(s) = 0\}$ where

$$\Delta(s) = \left[s\left(I - \int_{-r}^{0} e^{s\theta}d\mu(\theta)\right) - \int_{-r}^{0} e^{s\theta}d\alpha(\theta)\right]$$

denotes the system characteristic quasi-polynomial matrix. In particular, the system unstable spectrum becomes $\sigma_u(\mathcal{S}_n) = \left\{s \in \mathbf{C}^+_{-\nu_0} : \det\Delta(s) = 0\right\}$. It is known (see, for example, Henry(1974), Hale(1977)) that the homogeneous part of $\mathcal{S}_n$ is exponentially stable if $\sigma(\mathcal{S}_n) \subset \mathbf{C}^-_{-\nu_0}$ provided that $\delta \leq \nu_0 < \infty$ where $\delta > 0$ is fixed. (Fixing $\delta > 0$ removes the pathological situation of an unstable neutral system whose eigenvalues are all in the left-half plane; see El'sgol'ts and Norkin(1973), Datko(1983) for illustrative examples). We shall exploit this result by determining a feedback controller, $u(t) = f(x_t, u_t)$, so that under the feedback closure $u : \mathcal{S}_n \to \overline{\mathcal{S}}_n$, we have $\sigma(\overline{\mathcal{S}}_n) \subset \mathbf{C}^-_{-\nu_0}$.

For systems with discrete commensurate delays only, the feedback problem for neutral systems has been considered by O'Connor and Tarn(1983), Byrnes, Spong and Tarn(1984) and by Lu, Lee and Zak(1986). Employing the Hale decomposition, Pandolfi(1976) gave necessary and sufficient conditions for feedback stabilizability of neutral systems with distributed delay in the state. By frequency domain methods, Logeman(1986) has also given necessary and sufficient conditions for the existence of finite dimensional compensators for exponentially stabilizing neutral systems with distributed delay in both the state and control variables. Salamon(1984) focusses on a theoretical framework and gives feedback structures for very general neutral systems.

In all the above works, the principal difficulty stems from the fact that unlike a retarded functional differential equations(rfde) where $d\mu(\theta) = 0$, an nfde may possess in a vertical strip of the complex plane, an infinite number of eigenvalues called a *neutral root chain.* For, the possible presence of an insufficiently stable neutral root chain implies the necessity to reassign an infinite number of unstable eigenvalues. To illustrate, consider the simple neutral system

$$\frac{d}{dt}[x(t) - Hx(t-d)] = A_0x(t) + A_1x(t-r) + B_0u(t) \qquad (1.2)$$

where $H, A_0, A_1 \in \mathbf{R}^{n\times n}$, $B_0 \in \mathbf{R}^{n\times m}$ $0 < r, d < \infty$. It can be shown that the neutral root chain for (1.2) is given by

$$s_{\nu,k} = \frac{1}{d}\{\ln|\sigma_\nu(H)| + j[2k\pi + arg\sigma_\nu(H)]\} \tag{1.3}$$

$1 \le \nu \le n$, $k = 1, 2, \ldots$ where $\sigma_\nu(H) \neq 0$ and $\sigma_\nu(H)$ denotes the νth non-zero eigenvalue of H. See also O'Connor and Tarn(1983). It is thus evident from the real part of $s_{\nu,k}$ that unless $\sigma_\nu(H)$ belongs to the open unit disc, (1.3) will constitute an unstable neutral root chain. To eliminate such a neutral root chain, O'Connor and Tarn(1983), Byrnes, Spong and Tarn(1984) as well as Salamon(1984) have employed a preliminary feedback of the form

$$u(t) = v(t) - K\frac{d}{dt}x(t-d) \tag{1.4}$$

which transforms (1.2) to

$$\frac{d}{dt}[x(t) - (H - B_0K)x(t-d)] = A_0x(t) + A_1x(t-r) + B_0v(t). \tag{1.5}$$

Assuming that (H, B_0) is a completely controllable pair, one can then select $K \in \mathbf{R}^{m\times n}$ so that $\sigma(H - B_0K) \subset \left\{s \in \mathbf{C} : 0 < |s| < e^{-\nu_0 d}\right\}$. This gives $\text{Re}s_{\nu,k} < -\nu_0$ or $\mathbf{C}^+_{-\nu_0}$ is free of a neutral root chain. The remaining finite number of unstable eigenvalues may then be stabilized by a secondary feedback. A serious practical difficulty with (1.4), vis-a-vis implementation, is the presence of the derivative of the past state history.

To avoid this differentiation, it is assumed in this work that there is no neutral root chain in $\mathbf{C}^+_{-\nu_0}$ so that $\sharp\sigma_u(\mathcal{S}_n) = N$ where $0 \le N < \infty$ and $\sharp$ denotes cardinality. A criterion to check the validity of this assumption is given in Section 2. The finiteness of the unstable spectrum then facilitates the extension of the transformation method in Fiagbedzi and Pearson(1986,1987,1990) to stabilize the class of neutral systems described by (1.1). To this end, the notion of the generalized left characteristic matrix equation (glcme) is extended to neutral systems in Section 3. Section 4 introduces the transformation by which (1.1) is reduced to an ordinary differential equation. The stabilizing feedback controller which constitutes the principal result is then given. Concluding remarks follow in Section 5.

2 The Neutral Root Chain

Definition 2.1 *Corresponding to an eigenvalue $s_k \in \sigma(\mathcal{S}_n)$, a non-zero row vector $q_k \in \mathbf{C}_n$ is a left eigenvector of $\mathcal{S}_n$ if $q_k\Delta(s_k) = 0$.*

Definition 2.2 *A neutral root chain of $\mathcal{S}_n$ is an infinite sequence, $\{s_k\}$, of eigenvalues of $\mathcal{S}_n$ with $|\mathrm{Re}\, s_k| < \infty$ and $|\mathrm{Im}\, s_k| \to \infty$ as $k \to \infty$.*

It turns out that the existence of a neutral root chain is governed by μ only as will soon be evident. Following Kolmogorov and Fomin(1970, pg.341), a function of bounded variation can be represented as the sum of a jump function, an absolutely continuous function and a singular function. Thus,

$$\mu(\theta) = \sum_{\theta_j<\theta} H_j + \int^{\theta} N(\theta) + S(\theta) \tag{2.1}$$

where $H_j \in \mathbf{R}^{n\times n}$ denotes the jump at θ_j, $N \in \mathbf{L}_1([-r,0];\mathbf{R}^{n\times n})$ and $S(\cdot)$ denotes the singular function. By the last reference, if $S(\cdot)$ is absent in (2.1), then one can write

$$\int_{-r}^{0} e^{s\theta}d\mu(\theta) = \sum_{j=1}^{\infty} e^{-d_j s}H_j + \int_{-r}^{0} e^{s\theta}N(\theta)d\theta$$

where $\theta_j = -d_j$ and $0 < d_1 < d_2 < \ldots.. \leq r$ represent the jumps in μ at d_j . It is assumed that $\sum_{j=1}^{\infty} \|H_j\| < \infty$. In order to obtain an easily applied criterion for practical models exhibiting point delays, it is assumed that the Lebesgue decomposition of μ does not contain the singular part, $S(\cdot)$. In that case, (1.1) becomes

$$\mathcal{S}_n : \quad \frac{d}{dt}[x(t) - \sum_{j=1}^{\infty} H_j x(t-d_j) - \int_{-r}^{0} N(\theta)x(t+\theta)d\theta] = \int_{-r}^{0} d\alpha(\theta)x(t+\theta) + \int_{-h}^{0} d\beta(\theta)u(t+\theta) \tag{2.2}$$

with characteristic matrix

$$\Delta(s) = s\left(I - \sum_{j=1}^{\infty} e^{-d_j s} H_j - \int_{-r}^{0} e^{s\theta} N(\theta)d\theta\right) - \int_{-r}^{0} e^{s\theta} d\alpha(\theta). \tag{2.3}$$

The following theorem can now be proved.

Theorem 2.1 *If*

$$0 < \nu_0 < \frac{-1}{d_{\max}} \ln\left(\sum_{j=1}^{\infty} \|H_j\|\right) < \infty, \tag{2.4}$$

then (2.2) has no neutral root chain in the right half plane, $\mathbf{C}^{+}_{-\nu_0}$. *By* $\|H_j\|$ *is meant the Euclidean norm of* H_j *and* $d_{\max} = \max d_j$.

Proof. Let $\{s_k\}$ be a neutral root chain of eigenvalues of $\mathcal{S}_n$ in $\mathbf{C}^{+}_{-\nu_0}$ where ν_0 satisfies (2.4) . Let $q_k \in \mathbf{C}_n$ be the corresponding sequence of eigenvectors of $\mathcal{S}_n$. Then by definition, $0 = q_k\Delta(s_k)$ where $\Delta(s)$ is given by (2.3) since ν is assumed not to contain a singular part. For $s_k \neq 0$, this yields

$$0 = q_k\left(I - \sum_{j=1}^{\infty} e^{-d_j s_k} H_j - \int_{-r}^{0} e^{s_k\theta} N(\theta)d\theta\right) - \frac{1}{s_k}\int_{-r}^{0} e^{s_k\theta} q_k d\alpha(\theta).$$

Let $<\cdot,\cdot>$ denote the standard inner product on $\mathbf{C}^n$. Then

$$< q_k, q_k > \;=\; \sum_{j=1}^{\infty} e^{-d_j s_k} < q_k H_j, q_k > + < q_k \int_{-r}^{0} e^{s_k\theta} N(\theta) d\theta, q_k > + \frac{1}{s_k} < q_k \int_{-r}^{0} e^{s_k\theta} d\alpha(\theta), q_k >$$

so that, on division by $< q_k, q_k >= \|q_k\|^2$, we obtain

$$1 \;=\; \sum_{j=1}^{\infty} e^{-d_j s_k} \frac{< q_k H_j, q_k >}{< q_k, q_k >} + \frac{< q_k \int_{-r}^{0} e^{s_k\theta} N(\theta) d\theta, q_k >}{< q_k, q_k >} + \frac{1}{s_k} \frac{< q_k \int_{-r}^{0} e^{s_k\theta} d\alpha(\theta), q_k >}{< q_k, q_k >} \tag{2.5}$$

By triangle and Cauchy-Schwarz inequalities,

$$|\sum_{j=1}^{\infty} e^{-d_j s_k} \frac{< q_k H_j, q_k >}{< q_k, q_k >}| \le \sum_{j=1}^{\infty} e^{-d_j \sigma_k} \frac{|< q_k H_j, q_k >|}{< q_k, q_k >} \le \sum_{j=1}^{\infty} e^{-d_j \sigma_k} \|H_j\|$$

where $\sigma_k = \mathrm{Re} s_k$. Similarly,

$$\frac{|< q_k \int_{-r}^{0} e^{s_k\theta} N(\theta) d\theta, q_k >|}{< q_k, q_k >} \le \| \int_{-r}^{0} e^{s_k\theta} N(\theta) d\theta \|$$

and

$$\frac{|< q_k \int_{-r}^{0} e^{s_k\theta} d\alpha(\theta), q_k >|}{< q_k, q_k >} \le \| \int_{-r}^{0} e^{s_k\theta} d\alpha(\theta) \|.$$

Using triangle inequality on (2.5) and making use of the above results yield

$$1 \le \sum_{j=1}^{\infty} e^{-d_j \sigma_k} \|H_j\| + \| \int_{-r}^{0} e^{s_k\theta} N(\theta) d\theta \| + \frac{1}{|s_k|} \| \int_{-r}^{0} e^{s_k\theta} d\alpha(\theta) \|. \tag{2.6}$$

It is known (Hale(1977)) that if $s_k \in \sigma(\mathcal{S}_n)$, then there exists a $\nu_{\max} < \infty$ such that $\sigma_k \le \nu_{\max}$. Therefore, if $s_k \in \mathbf{C}^{+}_{-\nu_0}$, then $\{\sigma_k\} \subset [-\nu_0, \nu_{\max}]$ and by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{\sigma_{k_l}\}$. Thus,

$$\lim_{l\to\infty} \sum_{j=1}^{\infty} e^{-d_j \sigma_{k_l}} \|H_j\| \le \sum_{j=1}^{\infty} e^{-d_j \lim_{l\to\infty} \sigma_{k_l}} \le \sum_{j=1}^{\infty} e^{\nu_0 d_{\max}} \|H_j\|.$$

Next, observe that $e^{\sigma_k \theta} N(\theta) \in \mathbf{L}_1([-r,0];\mathbf{R}^{n\times n})$ so that by the Riemann-Lebesgue lemma,

$$\lim_{k\to\infty} \int_{-r}^{0} e^{s_k \theta} N(\theta) d\theta = \lim_{k\to\infty} \int_{-r}^{0} e^{i\omega_k \theta} [e^{\sigma_k \theta} N(\theta)] d\theta = 0 \tag{2.7}$$

Therefore,

$$\begin{aligned} 0 \le \lim_{l\to\infty} \| \int_{-r}^{0} e^{s_{k_l} \theta} N(\theta) d\theta \|^2 &= \lim_{l\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} | \int_{-r}^{0} e^{i\omega_{k_l} \theta} [e^{\sigma_{k_l} \theta} N_{i,j}(\theta)] d\theta |^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \lim_{l\to\infty} | \int_{-r}^{0} e^{i\omega_{k_l} \theta} [e^{\sigma_{k_l} \theta} N_{i,j}(\theta)] d\theta |^2 \\ &= 0 \end{aligned}$$

by (2.7) ; then by the sandwich theorem,

$$\lim_{l\to\infty} \| \int_{-r}^{0} e^{s_{k_l} \theta} N(\theta) d\theta \| = 0.$$

The fact that $\alpha(\cdot)$ is of bounded variation readily yields

$$\lim_{l\to\infty} \frac{1}{|s_{k_l}|} \| \int_{-r}^{0} e^{s_{k_l} \theta} d\alpha(\theta) \| = 0.$$

Therefore, taking subsequential limits in (2.6) as $l \to \infty$, and applying the above results, we obtain $1 \le \sum_{j=1}^{\infty} e^{\nu_0 d_{max}} \|H_j\|$ which, on taking logarithms, yields $\nu_0 \ge \frac{-1}{d_{max}} \ln \left(\sum_{j=1}^{\infty} \|H_j\| \right)$ in contradiction of (2.4). This proves the result.

3 Extension of the glcme

The notion of the glcme is documented in Fiagbedzi and Pearson(1990) with background material in Fiagbedzi and Pearson(1986,1987). We now extend it to cover neutral systems. It is assumed throughout the rest of this work that all eigenvalues in $\mathbf{C}^{+}_{-\nu_0}$ are simple.

Given $\mathcal{S}_n$, suppose that there is no neutral root chain in $\mathbf{C}^{+}_{-\nu_0}$ so that $\sharp\sigma_u(\mathcal{S}_n) = N < \infty$. Put $\sigma_u(\mathcal{S}_n) = \Lambda_u^+ \cup \Lambda_u^- \cup \Lambda_u^r$ where Λ_u^+ denotes a set of cardinality n_c consisting of the complex eigenvalues with positive imaginary part, Λ_u^- denotes the complex conjugate of Λ_u^+ and Λ_u^r denotes a set of cardinality n_r consisting of the real eigenvalues in $\mathbf{C}^{+}_{-\nu_0}$. Thus, $N = 2n_c + n_r$. Let $D_k^c = \begin{pmatrix} \sigma_k & -\omega_k \\ \omega_k & \sigma_k \end{pmatrix}$ denote the Jordan sub-block corresponding to $s_k = \sigma_k + i\omega_k \in \Lambda_u^+$, $i = \sqrt{-1}$, $k = 1, 2, .., n_c$ and $D_l^r = s_l$, $l = 1, 2, ..., n_r$, correspond to the entries of Λ_u^r. Then $J \in \mathbf{R}^{N\times N}$ constructed as

$$J = (\oplus_{l=1}^{n_r} D_l^r) \oplus (\oplus_{k=1}^{n_c} D_k^c) \tag{3.1}$$

is a Jordan matrix whose spectrum coincides with that of $\sigma_u(\mathcal{S}_n)$, i.e., $\sigma(J) = \sigma_u(\mathcal{S}_n)$. The glcme of $\mathcal{S}_n$ can now be introduced as

$$JQ = \int_{-r}^{0} e^{J\theta} \left[JQd\mu(\theta) + Qd\alpha(\theta)\right] \tag{3.2}$$

where $Q \in \mathbf{R}^{N\times n}$ constitutes the unknown. The solution of this equation is given by the following theorem.

Theorem 3.1 *$Q \in \mathbf{R}^{N\times n}$ is a solution of (3.2) if and only if*

$$Q = row\left[q_1^r, q_2^r, ..., q_{n_r}^r, \mathrm{Re}q_1^c, \mathrm{Im}q_1^c, \mathrm{Re}q_2^c, \mathrm{Im}q_2^c, ..., \mathrm{Re}q_{n_c}^c, \mathrm{Im}q_{n_c}^c\right] \tag{3.3}$$

where q_l^r , $l = 1, 2, ..., n_r$ are the left eigenvectors of $\mathcal{S}_n$ corresponding to the real eigenvalues (assumed simple) in $\mathbf{C}^{+}_{-\nu_0}$ and q_k^c, $k = 1, 2, ..., n_c$ are the left eigenvectors of S_n corresponding to the complex eigenvalues (assumed simple) in the upper half of $\mathbf{C}^+_{-\nu_0}$.

Proof. Necessity. Suppose that $Q \in \mathbf{R}^{N\times n}$ is a solution of (3.2). Since $N = n_r + 2n_c$, let $Q = row\left[q_1^r, q_2^r, ..., q_{n_r}^r, q_1^c, q_2^c, ..., q_{2n_c}^c\right]$ where $q_l^r \in \mathbf{R}_n$, $l = 1, 2, ..., n_r$ and $q_k^c \in \mathbf{R}_n$, $k = 1, 2, ..., 2n_c$. Let $co : \mathbf{R}^{N\times n} \to \mathbf{R}^{nN}$ be the isomorphism which transforms an $N \times n$ matrix into an $nN \times 1$ column vector by stacking the rows of Q and transposing it. Following Davis(1979), $co(ABC) = (A\otimes C')co(B)$ where $\otimes$ denotes the Kronecker product. Applying this isomorphism to (3.2) yields

$$\left[J \otimes I_n - \int_{-r}^{0} (e^{J\theta}J) \otimes d\mu'(\theta) - \int_{-r}^{0} e^{J\theta} \otimes d\alpha'(\theta)\right] coQ = 0 \qquad (3.4)$$

Making use of the definition of J and basic Kronecker product algebra, we obtain

$$\begin{aligned} J \otimes I_n &= [(\oplus_{l=1}^{n_r} D_l^r) \oplus (\oplus_{k=1}^{n_c} D_k^c)] \otimes I_n \\ &= [\oplus_{l=1}^{n_r} (D_l^r \otimes I_n)] \oplus [\oplus_{k=1}^{n_c} (D_k^c \otimes I_n)], \end{aligned}$$

$$\left(e^{J\theta}J\right) \otimes d\mu'(\theta) = \left[\oplus_{l=1}^{n_r} \left(e^{D_l^r\theta} D_l^r \otimes d\mu'(\theta)\right)\right] \oplus \left[\oplus_{k=1}^{n_c} \left(e^{D_k^c\theta} D_k^c \otimes d\mu'(\theta)\right)\right]$$

and

$$e^{J\theta} \otimes d\alpha'(\theta) = \left[\oplus_{l=1}^{n_r} \left(e^{D_l^r\theta} \otimes d\alpha'(\theta)\right)\right] \oplus \left[\oplus_{k=1}^{n_c} \left(e^{D_k^c\theta} \otimes d\alpha'(\theta)\right)\right].$$

Substituting these in (3.4) yields

$$q_l^r \left[(D_l^r)' \otimes I_n - \int_{-r}^{0} \left(e^{D_l^r\theta} D_l^r\right)' \otimes d\mu(\theta) - \int_{-r}^{0} \left(e^{D_l^r\theta}\right)' \otimes d\alpha(\theta)\right] = 0 \quad (3.5)$$

for $l = 1,2,...,n_r$ and

$$\left(q^c_{2k-1}, q^c_{2k}\right)\left[(D^c_k)' \otimes I_n - \int_{-r}^{0}\left(e^{D^c_k\theta}D^c_k\right)' \otimes d\mu(\theta) - \int_{-r}^{0}\left(e^{D^c_k\theta}\right)' \otimes d\alpha(\theta)\right] = 0 \tag{3.6}$$

for $k = 1,2,...,n_c$. Recalling that $D^r_l = s_l \in \mathbf{R}$, it is evident from (3.5) that $q^r_l, l = 1,2,...,n_r$ are the real left eigenvectors corresponding to the the real eigenvalues. To reach a similar conclusion from (3.6) for the complex eigenvalues, define the map $\psi : \mathbf{R}_{2n} \to \mathbf{C}_n$ by $\psi(v_1, v_2) = (v_1 + iv_2)$ and observe that ψ is an isomorphism. For brevity, let

$$A_{11} \stackrel{def}{=} \sigma_k I_n - \int_{-r}^{0} e^{\sigma_k\theta}\left(\sigma_k \cos\omega_k\theta - \omega_k \sin\omega_k\theta\right) d\mu(\theta) - \int_{-r}^{0} e^{\sigma_k\theta} \cos\omega_k\theta d\alpha(\theta) \tag{3.7}$$

$$A_{12} \stackrel{def}{=} \omega_k I_n - \int_{-r}^{0} e^{\sigma_k\theta}\left(\sigma_k \sin\omega_k\theta + \omega_k \cos\omega_k\theta\right) d\mu(\theta) - \int_{-r}^{0} e^{\sigma_k\theta} \sin\omega_k\theta d\alpha(\theta) \tag{3.8}$$

Applying ψ to (3.6) yields

$$\begin{aligned} 0 &= \psi\Bigg(\left(q^c_{2k-1}, q^c_{2k}\right)\Bigg[(D^c_k)' \otimes I_n - \int_{-r}^{0}\left(e^{D^c_k\theta}D^c_k\right)' \otimes d\mu(\theta) \\ &\qquad - \int_{-r}^{0}\left(e^{D^c_k\theta}\right)' \otimes d\alpha(\theta)\Bigg]\Bigg) \\ &= \psi\left[\left(q^c_{2k-1}, q^c_{2k}\right)\right](A_{11} + iA_{12}) \\ &= \left(q^c_{2k-1}+, q^c_{2k}\right)\Delta(s_k) \end{aligned}$$

This implies that $(q^c_{2k-1} + iq^c_{2k})$ is a left eigenvector corresponding to the complex eigenvalue s_k, $k = 1,2,...,n_c$. Sufficiency follows from the isomorphism of *co* and ψ. The non-emptiness of the solution set is a consequence of the fact that an eigenvector always exists.

4 The Principal Result

4.1 The reducing transformation

The results of Sections 2 and 3 are exploited as follows. Assume that the Lebesgue decomposition of μ does not contain a singular function so that $\mathcal{S}_n$ is amenable to (2.2). Then from (2.4) a ν_0 is determined so that $\mathbf{C}^+_{-\nu_0}$ is free of a neutral chain. The finitely many eigenvalues in $\mathbf{C}^+_{-\nu_0}$ are then identified with the modes of an ordinary differential equation by the following transformation.

Lemma 4.1 *Assume that the Lebesgue decomposition of μ does not contain a singular function so that $\mathcal{S}_n$ is amenable to (2.2). Let*

$$\begin{aligned} z(t) &= Q[x(t) - \int_{-r}^{0} d\mu(\theta)x(t+\theta)] + \int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}JQd\mu(\theta)x(\tau)d\tau \\ &+ \int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Qd\alpha(\theta)x(\tau)d\tau + \int_{-h}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Qd\beta(\theta)u(\tau)d\tau \end{aligned} \tag{4.1}$$

where $u \in \mathbf{L}_1^{loc}((0,\infty);\mathbf{R}^m)$, $x(\cdot) \in \mathcal{C}((0,\infty);\mathbf{R}^n)$ satisfies (1.1), $J \in \mathbf{R}^{N\times N}$ is a Jordan matrix defined by (3.1) such that $\sigma(J) = \sigma_u(\mathcal{S}_n)$ and $Q \in \mathbf{R}^{N\times n}$ is a solution of the glcme, (3.2). Then for $t >$ max$\{r,h\}$, z is absolutely continuous and satisfies(a.e)

$$\mathcal{S}_o \qquad \dot{z}(t) = Jz(t) + Bu(t) \tag{4.2}$$

where

$$B = \int_{-h}^{0} e^{J\theta}Qd\beta(\theta). \tag{4.3}$$

Proof. The solution of (1.1) presupposes that $\mathcal{D}x_t \in C^1((0,\infty);\mathbf{R}^n)$ and the solution $x(\cdot)$ being continuous, it follows that the first three terms of (4.1) are differentiable on (r,∞). The term $\int_{-h}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Qd\beta(\theta)u(\tau)d\tau$ is absolutely continuous because $u \in \mathbf{L}_1^{loc}((0,\infty);\mathbf{R}^m)$. See Manitius and Olbrot(1979, pg.542). Thus $z(\cdot)$ is a.e. differentiable. The rest follows by differentiation under the integral.

Lemma 4.2 *Let $K \in \mathbf{R}^{m\times N}$ be a constant gain. Then for $t > \max\{r,h\}$,*

$$u(t) = Kz(t) \tag{4.4}$$

is equivalent to the retarded functional differential equation(rfde)

$$\frac{du}{dt}(t) = KBu(t) + KJz(t), \qquad u(0) = Kz(0) \tag{4.5}$$

Proof. Equation (4.5) follows from premultiplying (4.2) by K. From the definition of z in (4.1), it follows that (4.5) is a functional differential equation. Thus, the pair (1.1) and (4.4) constitute an nfde. The equivalence of (4.4) and (4.5) then follows from the uniqueness of the solution of (4.5). See Manitius and Olbrot(1979, pg.542). The technique is summarized as an important device on page 597 of Kailath(1980).

Lemma 4.3 *If S_n is ν_0 spectrally stabilizable, i.e., rank $\left[\Delta(s) \,|\, \int_{-h}^{0} e^{s\theta}d\beta(\theta)\right]$ $= n \quad \forall s \in \mathbf{C}^+_{-\nu_0}$, then the pair $(J,B) \in \mathbf{R}^{N\times N} \times \mathbf{R}^{N\times m}$ is completely controllable.*

Proof. The proof is omitted but reader may consult the proof of the analogous statement for rfde's in Fiagbedzi and Pearson(1990, pg. 806).

The principal result now follows.

Theorem 4.1 *Assume that the Lebesgue decomposition of μ does not contain a singular part. Also, assume that $\mathcal{S}_n$ is ν_0 spectrally stabilizable where ν_0 satisfies (2.4) . Then the state feedback,*

$$\begin{aligned} u(t) &= -K\Big\{Q[x(t) - \int_{-r}^{0} d\mu(\theta)x(t+\theta)] + \int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}JQd\mu(\theta)x(\tau)d\tau \\ &+ \int_{-r}^{0}\int_{t+\theta}^{t} +e^{J(t+\theta-\tau)}Qd\alpha(\theta)x(\tau)d\tau + \int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Qd\beta(\theta)u(\tau)d\tau\Big\} \end{aligned} \tag{4.6}$$

exponentially stabilizes $\mathcal{S}_n$ with a margin of stability equal to ν_0 if $\sigma(J) = \sigma_u(\mathcal{S}_n)$ and $K \in \mathbf{R}^{m\times N}$ is such that $\sigma(J - BK) \subset \mathbf{C}^-_{-\nu_0}$.

Proof. The controller $u : \mathcal{S}_n \to \overline{\mathcal{S}}_n$ where the closed loop system, $\overline{\mathcal{S}}_n$, is identified with the pair of equations, (1.1) and (4.4). In view of Lemma 4.2, $\overline{\mathcal{S}}_n$ is equivalent to a system of nfde defined by (1.1) and (4.5). Therefore, its ν_0 exponential stability is assured if all the eigenvalues of $\overline{\mathcal{S}}_n$ are located in $\mathbf{C}^-_{-\nu_0}$ where, for a fixed $\delta > 0$, $\nu_0 \geq \delta$. Laplace transformation of (4.6) or equivalently (4.4) gives

$$K(J-sI)^{-1}Q\Delta(s)X(s) - \Big(I - K(J-sI)^{-1}[B - \int_{-h}^{0} Qe^{s\theta}d\beta(\theta)]\Big)U(s) = IC$$

on taking advantage of the glcme while Laplace transformation of (1.1) yields

$$\Delta(s)X(s) - \int_{-h}^{0} e^{s\theta}d\beta(\theta)U(s) = IC.$$

In either case, IC denotes functions of the system initial conditions. Thus, the characteristic matrix of $\overline{\mathcal{S}}_n$ is given by

$$\overline{\Delta}(s) = \begin{pmatrix} \Delta(s) & -\int_{-h}^{0} e^{s\theta}d\beta(\theta) \\ K(J-sI)^{-1}Q\Delta(s) & -I + K(J-sI)^{-1}[B - \int_{-h}^{0} Qe^{s\theta}d\beta(\theta)] \end{pmatrix}.$$

By elementary properties of determinants, the closed loop system characteristic function, $\det \overline{\Delta}(s)$, can then be written as

$$\begin{aligned} \det \overline{\Delta}(s) &= \det \begin{pmatrix} I_n & 0 \\ -K(J-sI)^{-1}Q & I_m \end{pmatrix} \overline{\Delta}(s) \\ &= \det \begin{pmatrix} \Delta(s) & -\int_{-h}^{0} e^{s\theta} d\beta(\theta) \\ 0 & -I_m + K(J-sI)^{-1}B \end{pmatrix} \\ &= (-1)^m \frac{\det \Delta(s)}{\det(sI-J)} \det(sI - J + BK) \end{aligned} \tag{4.7}$$

By Lemma 4.3, (J, B) is a completely controllable pair which allows the determination of K so that $\sigma(J - BK) \subset \mathbf{C}^-_{-\nu_0}$. Also, by construction, $\det(sI - J) | \det \Delta(s)$. Therefore, by (4.7) ,

$$\sigma(\overline{\mathcal{S}}_n) = \{\sigma(\mathcal{S}_n) \setminus \sigma(J)\} \cup \sigma(J - BK)$$

and $\sigma(\overline{\mathcal{S}}_n) \subset \mathbf{C}^-_{-\nu_0}$ from which the exponential stability follows.

5 Concluding Remarks

The mathematical model considered in this work embraces autonomous, linear, neutral systems which exhibit point and distributed delay in the state as well as control variables. Unlike earlier works which have given existence results for this class of systems, the method of transformation has yielded an implementable feedback controller under the standard assumption that there is no unstable neutral root chain. An outstanding issue is the need for an automatic code to compute the (finite) system unstable eigenvalues. With regard to the question of robustness, it is known(see, for example, Hale(1977,pg.289)) that a neutral system response does not necessarily depend continuously on the delay. It is therefore of interest to identify a subclass of neutral systems whose response varies continuously with the delay. Some results in this direction can be found in Hale(1977, pg. 289), Datko(1978). It appears however that our criterion which verifies the assumption of stable neutral root chains could form the basis for such an identification.

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