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SECOND FORM OF HAMILTON'S PRINCIPLE

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Expressing Hamilton's principle in the Poincaré formalism, it is shown that the principle is not a variational principle even for holonomic systems. The formalism is then used to study the second form of Hamilton's principle.

1. Basic Relations

We consider a holonomic mechanical system with n degrees of freedom. Let the coordinates $x_1, x_2, \ldots, x_n$ determine the position of the system at any time t, and let all the given forces be potential. We assume that the infinitesimal displacements of the system are defined by a transitive group of operators

$$X_p = \xi_s^p(x,t)\partial/\partial x_s$$

with commutation relations

$$(X_p, X_q) = C_{pq}^r X_r, \quad (\partial/\partial t,\, X_p) = 0,$$

Throughout, the indices take the values $1, 2, \ldots, n$ and repeated indices denote summation.

According to Poincaré [1], the variation $df\ (\delta f)$ of an arbitrary function $f(x,t)$ in a real (possible) displacement of the system is determined by

$$df = \left(\frac{\partial f}{\partial t} + \eta_p X_p f\right) dt \quad (\boldsymbol{\delta} f = \omega_p X_p f) \tag{1}$$

where the Poincaré parameters η_p and the parameters ω_p of possible displacements are independent.

By means of the rule $d\delta f = \delta df$, it is shown in [2] that

$$\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{qr}\eta_q\omega_r. \tag{2}$$

With $f = x_1$, formula (1) expresses the velocities $\dot{x}_q$ in terms of η_p's and the virtual displacements δx_q in terms of the parameters ω_p's. Accordingly, in the sequel, the Lagrangian function L of the system is assumed to be a function of the x_p's, η_p's and the time t, i.e., $L = L(x_p, \eta_p, t)$.

2. Hamilton's Principle

One of the best known integral principles of mechanics is Hamilton's principle, according to which the actual motion of a system is such that [4, 5]:

$$\int_{t_0}^{t_1} \delta L\, dt = 0, \tag{3}$$

where δL is a variation of the Lagrangian function in passing from the real trajectory to a varied path which is one of the ∞^n neighbouring paths (compatible with the constraints) and which coincides with real trajectory at the fixed (but arbitrarily chosen) moments of time t_0 and t_1, so that we have

$$\omega_p(t_0) = 0, \quad \omega_p(t_1) = 0. \tag{4}$$

According to (1), we have

$$\delta L = \omega_p X_p L + \frac{\partial L}{\partial \eta_p}\delta\eta_p,$$

which, in view of (2), becomes

$$\delta L = \omega_p X_p L + \frac{\partial L}{\partial \eta_p}\left(\frac{d\omega_p}{dt} + C^p_{qr}\eta_q\omega_r\right).$$

Consequently, (3) takes the form

$$\int_{t_0}^{t_1} \left[\omega_p X_p L + \frac{\partial L}{\partial \eta_p} \left(\frac{d\omega_p}{dt} + C_{qr}^p \eta_q \omega_r \right) \right] dt = 0. \tag{5}$$

Integrating by parts, we find, in view of restriction (4) on the choice of varied paths, that

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \eta_p} \frac{d\omega_p}{dt} dt = \frac{\partial L}{\partial \eta_p} \omega_p \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \omega_p \frac{d}{dt} \frac{\partial L}{\partial \eta_p} dt = - \int_{t_0}^{t_1} \omega_p \frac{d}{dt} \frac{\partial L}{\partial \eta_p} dt.$$

Thus, (5) leads to the relation

$$\int_{t_0}^{t_1} \left(C_{qp}^r \eta_q \frac{\partial L}{\partial \eta_r} + X_p L - \frac{d}{dt} \frac{\partial L}{\partial \eta_p} \right) \omega_p dt = 0. \tag{6}$$

This is the form of Hamilton's principle in the Poincaré formalism.

By virtue of the independence of the parameters ω_p, we obtain the equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \eta_p} - C_{qp}^r \eta_q \frac{\partial L}{\partial \eta_r} - X_p L = 0. \tag{7}$$

These are the equations derived by Poincaré [1] in 1901. (There was a mistake in Poincaré's paper in the indices in C_{qr}^p.) In the case of redundant coordinates, these equations were obtained by Chetaev [1] by using the relations (2). In [2], these equations are derived without using relations (2). Lagrange's second order equations in generalized coordinates follow as particular cases of equations (7).

In the presence of nonpotential forces, there exists a relation analogous to Hamilton's principle, which can be formulated as [4, 5]:

$$\int_{t_0}^{t_1} (\delta L + \delta' W) dt = 0, \tag{8}$$

where $\delta' W$is the virtual work of the nonpotential forces and the prime (′) indicates that $\delta' W$ is not necessarily the differential of the function W. If $U(x_p, t)$ denotes the force function, we have

$$\delta' W = \omega_p X_p U.$$

Thus, in the Poincaré formalism, the relation (8) becomes

$$\int_{t_0}^{t_1} \left[C^r_{qp} \eta_q \frac{\partial L}{\partial \eta_r} + X_p(L+U) - \frac{d}{dt}\frac{\partial L}{\partial \eta_p} \right] \omega_p dt = 0. \tag{9}$$

The corresponding equations of motion are [2]:

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_p} - C^r_{qp}\eta_q \frac{\partial L}{\partial \eta_r} - X_p(L+U) = 0. \tag{10}$$

In order to see whether Hamilton's principle in the form (6) or (9) is a variational principle, let us consider a functional:

$$J = \int_{t_0}^{t_1} F\left(y_1(t), \ldots, y_n(t), \dot{y}_1(t), \ldots, \dot{y}_n(t), t\right) dt.$$

We say that the set of functions $y_s(t)$ confer a stationary value upon this functional if the variation of this functional, with restrictions on the variations δy_s of the functions $y_s(t)$ and calculated up to first order terms in δy_s, is zero. To wit,

$$\delta J = 0, \quad y_s(t_0) = 0, \quad y_s(t_1) = 0.$$

In the case of equation (6), we only assert the vanishing of quantity

$$\delta' R = \int_{t_0}^{t_1} \left(C^r_{qp}\eta_q \frac{\partial L}{\partial \eta_r} + X_p L - \frac{d}{dt}\frac{\partial L}{\partial \eta_p} \right) \omega_p dt$$

but, in view of the presence of the parameters ω_p, there is no functional R because there does not exist a quantity whose variation is equal to $\delta' R$. Thus, Hamilton's principle in the form (6) does not formulate a problem in the calculus of variations. Similar remarks hold for the relation (9). Thus, Hamilton's principle when expressed in the Poincaré parameters, is not a variational principle even in the case of holonomic systems.

3. Second Form of Hamilton's Principle

The second form of Hamilton's principle is obtained by replacing the Lagrangian L in relation (3) by the Hamiltonian function H defined as

$$H(x_p, y_p, t) = y_p\eta_p - L(x_p, \eta_p, t),$$

where the new variables y_p are defined by

$$y_p = \frac{\partial L}{\partial \eta_p}.$$

Thus, relation (3) becomes

$$\int_{t_0}^{t_1} \delta(y_p\eta_p - H)dt = 0,$$

where the ω_p satisfy conditions (4).

We can write (11) in the form

$$\int_{t_0}^{t_1} \left[\eta_p\delta y_p + y_p\delta\eta_p - \omega_p X_p H - \frac{\partial H}{\partial y_p}\delta y_p\right] dt = 0. \tag{12}$$

In the terms $y_p\delta\eta_p$ we substitute for $\delta\eta_p$ from (2), integrate by parts the terms $y_p\frac{d\omega_p}{dt}$ and use conditions (4) on ω_p. Then (12) becomes

$$\int_{t_0}^{t_{11}} \left[\left(\eta_p - \frac{\partial H}{\partial y_p}\right)\delta y_p - \left(\dot{y}_p + X_p H - C^r_{qp}\frac{\partial H}{\partial y_q}y_r\right)\omega_p\right] dt = 0. \tag{13}$$

In order that (13) be satisfied for arbitrary ω_p and δy_p, it is necessary that the terms in each parenthesis be separately zero. This leads to the equations of motion in the canonical form [3]:

$$\eta_p = \frac{\partial H}{\partial y_p}, \dot{y}_p = -X_p H + C^r_{qp}\frac{\partial H}{\partial y_q}y_r. \tag{14}$$

The presence of ω_p's in the relation (13) implies that, in general, the second form of Hamilton's principle is also not a variational principle even for holonomic problems.

Finally, we remark that Hamilton's principle in the forms (6) and (13) is not exactly equivalent because in (6) only the ω's are arbitrary and the $\delta\eta$'s are determined from relations (2) whereas in (13) not only the ω's but the δy's are also chosen arbitrarily.

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