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Second form of Hamilton’s Principle

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SECOND FORM OF HAMILTON'S PRINCIPLE

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Expressing Hamilton's principle in the Poincaré formalism, it is shown that the principle is not a variational principle even for holonomic systems. The formalism is then used to study the second form of Hamilton's principle.

1. Basic Relations

We consider a holonomic mechanical system with $n$ degrees of freedom. Let the coordinates $x_1, x_2, \ldots, x_n$ determine the position of the system at any time $t$, and let all the given forces be potential. We assume that the infinitesimal displacements of the system are defined by a transitive group of operators

$$X_p = \xi^p_s(x, t) \partial/\partial x_s,$$

with commutation relations

$$(X_p, X_q) = C^r_{pq} X_r, \quad (\partial/\partial t, X_p) = 0,$$

Throughout, the indices take the values $1, 2, \ldots, n$ and repeated indices denote summation.

According to Poincaré [1], the variation $df(\delta f)$ of an arbitrary function $f(x,t)$ in a real (possible) displacement of the system is determined by

$$df = \left( \frac{\partial f}{\partial t} + \eta_p X_p f \right) dt \quad (\delta f = \omega_p X_p f) \quad (1)$$

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where the Poincaré parameters $\eta_p$ and the parameters $\omega_p$ of possible displacements are independent.

By means of the rule $d\delta f = \delta df$, it is shown in [2] that

$$\delta \eta_p = \frac{d\omega_p}{dt} + C_{qr}^p \eta_q \omega_r.$$  \hspace{1cm} (2)

With $f = x_1$, formula (1) expresses the velocities $\dot{x}_q$ in terms of $\eta_p$'s and the virtual displacements $\delta x_q$ in terms of the parameters $\omega_p$'s. Accordingly, in the sequel, the Lagrangian function $L$ of the system is assumed to be a function of the $x_p$'s, $\eta_p$'s and the time $t$, i.e., $L = L(x_p, \eta_p, t)$.

2. Hamilton's Principle

One of the best known integral principles of mechanics is Hamilton’s principle, according to which the actual motion of a system is such that [4, 5]:

$$\int_{t_0}^{t_1} \delta L \, dt = 0,$$  \hspace{1cm} (3)

where $\delta L$ is a variation of the Lagrangian function in passing from the real trajectory to a varied path which is one of the $\infty^n$ neighbouring paths (compatible with the constraints) and which coincides with real trajectory at the fixed (but arbitrarily chosen) moments of time $t_0$ and $t_1$, so that we have

$$\omega_p(t_0) = 0, \quad \omega_p(t_1) = 0.$$  \hspace{1cm} (4)

According to (1), we have

$$\delta L = \omega_p x_p L + \frac{\partial L}{\partial \eta_p} \delta \eta_p,$$

which, in view of (2), becomes

$$\delta L = \omega_p x_p L + \frac{\partial L}{\partial \eta_p} \left( \frac{d\omega_p}{dt} + C_{qr}^p \eta_q \omega_r \right).$$
Consequently, (3) takes the form

\[
\int_{t_0}^{t_1} \left[ \omega_p X_p L + \frac{\partial L}{\partial \eta_p} \left( \frac{d\omega_p}{dt} + C_{qr}^p \eta_q \omega_r \right) \right] dt = 0. \tag{5}
\]

Integrating by parts, we find, in view of restriction (4) on the choice of varied paths, that

\[
\int_{t_0}^{t_1} \frac{\partial L}{\partial \eta_p} \frac{d\omega_p}{dt} dt = \left[ \frac{\partial L}{\partial \eta_p} \omega_p \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \omega_p \frac{d}{dt} \left( \frac{\partial L}{\partial \eta_p} \right) dt = - \int_{t_0}^{t_1} \omega_p \frac{d}{dt} \left( \frac{\partial L}{\partial \eta_p} \right) dt.
\]

Thus, (5) leads to the relation

\[
\int_{t_0}^{t_1} \left( C_{qr}^p \eta_q \frac{\partial L}{\partial \eta_r} + X_p L - \frac{d}{dt} \frac{\partial L}{\partial \eta_p} \right) \omega_p dt = 0. \tag{6}
\]

This is the form of Hamilton’s principle in the Poincaré formalism.

By virtue of the independence of the parameters \( \omega_p \), we obtain the equations of motion

\[
\frac{d}{dt} \frac{\partial L}{\partial \eta_p} - C_{qr}^p \eta_q \frac{\partial L}{\partial \eta_r} - X_p L = 0. \tag{7}
\]

These are the equations derived by Poincaré [1] in 1901. (There was a mistake in Poincaré’s paper in the indices in \( C_{qr}^p \).) In the case of redundant coordinates, these equations were obtained by Chetaev [1] by using the relations (2). In [2], these equations are derived without using relations (2). Lagrange’s second order equations in generalized coordinates follow as particular cases of equations (7).

In the presence of nonpotential forces, there exists a relation analogous to Hamilton’s principle, which can be formulated as [4, 5]:

\[
\int_{t_0}^{t_1} (\delta L + \delta' W) dt = 0, \tag{8}
\]

where \( \delta' W \) is the virtual work of the nonpotential forces and the prime (') indicates that \( \delta' W \) is not necessarily the differential of the function \( W \). If \( U(x_p, t) \) denotes the
force function, we have

\[ \delta'W = \omega_p X_p U. \]

Thus, in the Poincaré formalism, the relation (8) becomes

\[ \int_{t_0}^{t_1} \left[ C'_{q'p} \eta_q \frac{\partial L}{\partial \eta_r} + X_p (L + U) - \frac{d}{dt} \frac{\partial L}{\partial \eta_p} \right] \omega_p dt = 0. \]  \hspace{1cm} (9)

The corresponding equations of motion are [2]:

\[ \frac{d}{dt} \frac{\partial L}{\partial \eta_p} - C'_{q'p} \eta_q \frac{\partial L}{\partial \eta_r} - X_p (L + U) = 0. \]  \hspace{1cm} (10)

In order to see whether Hamilton’s principle in the form (6) or (9) is a variational principle, let us consider a functional:

\[ J = \int_{t_0}^{t_1} F (y_1(t), \ldots, y_n(t), \dot{y}_1(t), \ldots, \dot{y}_n(t), t) dt. \]

We say that the set of functions \( y_s(t) \) confer a stationary value upon this functional if the variation of this functional, with restrictions on the variations \( \delta y_s \) of the functions \( y_s(t) \) and calculated up to first order terms in \( \delta y_s \), is zero. To wit,

\[ \delta J = 0, \quad y_s(t_0) = 0, \quad y_s(t_1) = 0. \]

In the case of equation (6), we only assert the vanishing of quantity

\[ \delta' R = \int_{t_0}^{t_1} \left( C'_{q'p} \eta_q \frac{\partial L}{\partial \eta_r} + X_p L - \frac{d}{dt} \frac{\partial L}{\partial \eta_p} \right) \omega_p dt \]

but, in view of the presence of the parameters \( \omega_p \), there is no functional \( R \) because there does not exist a quantity whose variation is equal to \( \delta' R \). Thus, Hamilton’s principle in the form (6) does not formulate a problem in the calculus of variations. Similar remarks hold for the relation (9). Thus, Hamilton’s principle when expressed
in the Poincaré parameters, is not a variational principle even in the case of holonomic systems.

3. Second Form of Hamilton's Principle

The second form of Hamilton's principle is obtained by replacing the Lagrangian $L$ in relation (3) by the Hamiltonian function $H$ defined as

$$H(x_p, y_p, t) = y_p \eta_p - L(x_p, \eta_p, t),$$

where the new variables $y_p$ are defined by

$$y_p = \frac{\partial L}{\partial \eta_p}.$$

Thus, relation (3) becomes

$$\int_{t_0}^{t_1} \delta(y_p \eta_p - H) dt = 0,$$

where the $\omega_p$ satisfy conditions (4).

We can write (11) in the form

$$\int_{t_0}^{t_1} \left[ \eta_p \delta y_p + y_p \delta \eta_p - \omega_p X_p H - \frac{\partial H}{\partial y_p} \delta y_p \right] dt = 0. \quad (12)$$

In the terms $y_p \delta \eta_p$ we substitute for $\delta \eta_p$ from (2), integrate by parts the terms $y_p \frac{d \omega_p}{dt}$ and use conditions (4) on $\omega_p$. Then (12) becomes

$$\int_{t_0}^{t_1} \left[ \left( \eta_p - \frac{\partial H}{\partial y_p} \right) \delta y_p - \left( \dot{y}_p + X_p H - C_{qp} \frac{\partial H}{\partial y_q} y_r \right) \omega_p \right] dt = 0. \quad (13)$$

In order that (13) be satisfied for arbitrary $\omega_p$ and $\delta y_p$, it is necessary that the terms in each parenthesis be separately zero. This leads to the equations of motion in the canonical form [3]:

$$\eta_p = \frac{\partial H}{\partial y_p}, \dot{y}_p = -X_p H + C_{qp} \frac{\partial H}{\partial y_q} y_r. \quad (14)$$
The presence of $\omega_p$'s in the relation (13) implies that, in general, the second form of Hamilton's principle is also not a variational principle even for holonomic problems.

Finally, we remark that Hamilton's principle in the forms (6) and (13) is not exactly equivalent because in (6) only the $\omega$'s are arbitrary and the $\delta \eta$'s are determined from relations (2) whereas in (13) not only the $\omega$'s but the $\delta y$'s are also chosen arbitrarily.

References


