King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 134

February 1993

Two Computational Problems in Interpolation and Approximation

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INTRODUCTION

We discuss two problems of approximation in this paper. The first problem deals with complex valued functions of the form $f_m(z) = z^m/(z-\rho)$ with $\rho > 1$. For these functions which are analytic in $|z| < \rho$, we establish that the asymptotic distribution of the zeros of their "Taylor sections" and " Lagrange interpolants at the nodes uniformly distributed on a circle $|z| = \sigma$, $\sigma < \rho$" is similar. This notion is illustrated computationally as well. We conjecture that a similar result can be expected for any function analytic in
$|z| < \rho$, $1 < \rho < \infty$.

The second problem is related to L_2-approximation of real valued functions subject to interpolatory constraints. We construct a polynomial $p^*_{m,n}$ of degree m which interpolates a given function $f \in \mathcal{L}^2[a,b]$ at preassigned n distinct nodes and also minimizes the error $p^*_{m,n} - f$ in L_2–sense. It is shown that the error converges to zero with respect to L_2–norm if $f \in C[a,b]$. We also describe a method for computing the optimal solution $p^*_{m,n}$ and apply it to the functions $e^x, |x|, e^{|x|}$ and $1/(1+x^2)$.

NOTATIONS

We list below all the notations to be used in the forthcoming discussion of parts I and II:

I

π_n	:=	Class of all polynomials of degree $\leq n$
A_ρ	:=	Class of all functions analytic in $\|z\| < \rho$ but non analytic on $\|z\| = \rho, \rho > 1$.
$L_n^\sigma(z,f)$	:=	Polynomial of degree n which interpolates $f \in A_\rho$ in the $n+1$ roots of $z^{n+1} - \sigma^{n+1} = 0, 0 < \sigma < \rho$.
$S_n(z,f)$	:=	Taylor's polynomial of degree n for $f \in A_\rho$.
$n(m)$	:=	$n - m + 1$, n and m are integers, $n > m$
$w_{\ell,n(m)}$	:=	$\exp(2\ell\pi i/n(m))$
$z_{\ell,n(m)}$	:=	$\rho \cdot w_{\ell,n(m)}$
$\epsilon_{\ell,n(m)}$	:=	$\frac{1}{2}\min\left(\min_{\substack{1\leq j\leq n(m)-1 \\ \ell\neq j}} \|z_{\ell,n(m)} - z_{j,n(m)}\|, \frac{\rho^2}{\sigma} - \rho, \rho - 1\right)$
$O_{\epsilon_{\ell,n(m)}}(z_{\ell,n(m)})$	:=	$\left\{z \in C : \left\|z - z_{\ell,n(m)}\right\| < \epsilon_{\ell,n(m)}\right\}$
$C_{\epsilon_{\ell,n(m)}}(z_{\ell,n(m)})$	:=	$\left\{z \in C : \left\|z - z_{\ell,n(m)}\right\| = \epsilon_{\ell,n(m)}\right\}$

$w(x)$	:=	Weight function
$\mathcal{L}_w^2[a,b]$	:=	Class of all function $f:[a,b]\rightarrow\Re$ for which $\int_a^b w(x)\lvert f(x)\rvert^2dx<\infty$
Δ_n	:=	$\{x_1,x_2,\cdots,x_n\}\subseteq[a,b]$.
$W_n(x)$	:=	$W_n(x;\Delta_n)=\Pi_{i=1}^n(x-x_i)$
$\Pi_m(\Delta_n)$	:=	Class of all polynomials generated by $x^kW_n(x),\quad k=0,1,2,\cdots m-n$
$L_{n-1}(x,\Delta_n,f)$	:=	Polynomial of degree n-1 which interpolates $f(x)$ at the points of Δ_n.
$< f,g >_w$	:=	$\int_a^b w(x)f(x)g(x)dx$
$\lVert f\rVert_w$	:=	$\sqrt{< f,f >_w}$
$C([a,b],\Delta_n)$	:=	$\{f^*:f^*(x)=f(x)-L_{n-1}(x,\Delta_n,f),f\in C[a,b]\}$.

I On The Zeros of Certain Lagrange Interpolants

The problem related to the distribution of zeros of the sections of power series of an analytic function has a long and respectable history ([4], [7], [10]). We ask a similar question about the Lagrange interpolants to $f\in A_\rho$ at the nodes uniformly distributed on the circle $|z|=\sigma,\ \ 0<\sigma<\rho$. A simple but interesting situtaion in the following example is the basis of our motivation.

Example I Select $f_0(z)=(\rho-z)^{-1}$ with $\rho>\sigma>0$. Then it is easy to see that

$$S_n(z, f_0) = \frac{\rho^{n+1} - z^{n+1}}{\rho^{n+1}(\rho - z)},$$
$$L_n^\sigma(z, f_0) = \frac{\rho^{n+1}}{\rho^{n+1} - \sigma^{n+1}} S_n(z, f_0)$$

and

$$L_n^\sigma(z, f_0) - S_n(z, f_0) = \frac{\sigma^{n+1}}{\rho^{n+1} - \sigma^{n+1}} S_n(z, f_0).$$

This shows that the n zeros of $S_n(z, f_0)$ which are uniformly distributed on the circle $|z| = \rho$ are identical to those of $L_n^\sigma(z, f_0)$ and $L_n^\sigma(z, f_0) - S_n(z, f_0)$.

Instead of $f_0(z)$, if our choice is slightly modified to $f_1(z) = z(\rho - z)^{-1}$,then

$$S_n(z, f_1) = \frac{z(\rho^n - z^n)}{\rho^n(\rho - z)},$$

and

$$L_n^\sigma(z, f_1) = \frac{1}{\rho^{n+1} - \sigma^{n+1}} \left[\rho^{n+1} S_n(z, f_1) + \sigma^{n+1}\right]. \tag{I.1}$$

Contrary to the outcome of the Example I, we note that none of the zeros of of $S_n(z, f_1)$ is identical to a zero of $L_n^\sigma(z, f_1)$. In fact, $S_n(z, f_1)$ possesses one of the zeros at the origin and the rest are uniformly distributed on the circle $|z| = \rho$. On the other hand the location of the zeros of $L_n^\sigma(z, f_1)$ is not obvious from its expression (I.1).

Main Result. Our aim is to investigate the asymptotic behaviour of the zeros of $L_n^\sigma(z, f_m)$, where m is a fixed nonnegative integer and

$$f_m(z) = z^m/(\rho - z). \tag{I.2}$$

In this regard we establish [1]:

Theorem I.1 *Let* $f_m(z) = z^m/(\rho - z)$ *where* m *is a fixed non- negative integer. Let* $\sigma > 0$ *and* $\rho > \max\{1, \sigma\}$. *Then*

(A) *For every sufficiently small neighbourhood* K *of* $z_0 = 0,$, *there exists a positive integer* $N(K)$ *such that* K *contains exactly* m *zeros of* $L_n^\sigma(z, f_m), n \geq N(K)$.

(B) *For all sufficiently large* n *and for all* $\ell = 1, 2, \cdots, n(m) - 1$, *the open sphere* $O_{\epsilon_{\ell,n(m)}}\left(z_{\ell,n(m)}\right)$ *contains exactly one zero of* $L_n^\sigma(z, f_m)$.

The proof of Theorem I.1 is based on the theorems due to Rouche and Hurwitz [6] where we take into account the following observation:

If we set

$$P_n(z, f_m) := L_n^\sigma(z, f_m) - S_n(z, f_m)$$

then for all sufficiently large values of n

$$\left|\frac{P_n(z, f_m)}{S_n(z, f_m)}\right| < 1 \quad \forall z \in C_{\ell,n(m)}, \tag{I.3}$$

$\ell = 1, 2, \cdots, n(m) - 1$.

Remark I.1 The formula

$$S_n(z, f_m) = \frac{z^m}{\rho^{n-m+1}} \cdot \frac{\rho^{n-m+1} - z^{n-m+1}}{\rho - z}, \; m \leq n$$

implies that $S_n(z, f_m)$ has an m-fold zero at the origin and the rest of its zeros are distributed uniformly on the circle $|z| = \rho$. On the other hand, it follows from Theorem I.1 that the asymptotic distribution of the zeros of $S_n(z, f_m)$ and $L_n^\sigma(z, f_m)$ is similar since $\epsilon_{\ell,n(m)} \to 0$ as $n \to \infty$. Thus, m zeros of $L_n^\sigma(z, f)$ will converge to the origin whereas the remaining zeros will be located in the vicinity of the zeros of $S_n(z, f_m)$ lying on the circle $|z| = \rho$.

Numerical Results. To illustrate our main result we have used the ZPOCC program in the IMSL library in order to determine the zeros of the two polynomials $S_n(z, f_m)$ and $L_n^\sigma(z, f_m)$. The zeros are given in the Tables 1-6 corresponding to various values of σ, m and n. The location of these zeros around the circle $|z| = \rho$, where $\rho = 6$, is shown for each case separately in the computer generated figures.

Unlike the case of $L_n^2(z, f_5)$, we observe from the figures that five of the n zeros of $L_n^4(z, f_5)$, $(n = 8, 15, 21)$, move slowly to $z_0 = 0$ as n takes larger values. Our computational results show that the phenomenon of 'slow movement' arises when the values of σ and ρ are close to each other. It may be worthwhile to point out that the region in which the polynomial "$L_n^\sigma(z, f_m) - S_n(z, f_m)$" converges to zero significantly shrinks if we select σ close enough to ρ.

Table 1

$\sigma = 2$, $m = 8$ and $n = 10$

△	○
0.0000 +ι 0.0000	0.5067 +ι 1.2220
0.0000 +ι 0.0000	0.5069 −ι 1.2220
0.0000 +ι 0.0000	−1.2238 +ι 0.5051
0.0000 +ι 0.0000	−1.2238 −ι 0.5051
0.0000 +ι 0.0000	−0.5069 +ι 1.2256
0.0000 +ι 0.0000	−0.5069 −ι 1.2256
0.0000 +ι 0.0000	1.2238 +ι 0.5087
0.0000 +ι 0.0000	1.2238 −ι 0.5087
−3.0000 +ι 5.1961	−2.9998 +ι 5.1958
−2.9999 − ι 5.1961	−3.0000 −ι 5.1958

Table 2

$\sigma = 2$, $m = 8$, and $n = 15$

△	○
0.0000 +ι 0.0000	0.2551 +ι 0.6159
0.0000 +ι 0.0000	0.2551 −ι 0.6159
0.0000 +ι 0.0000	−0.6159 +ι 0.2551
0.0000 +ι 0.0000	−0.6159 −ι 0.2551
0.0000 +ι 0.0000	0.6159 +ι 0.2551
0.0000 +ι 0.0000	0.6159 −ι 0.2551
0.0000 +ι 0.0000	−0.2551 +ι 0.6159
0.0000 + ι 0.0000	−0.2551 −ι 0.6159
−6.0004 +ι 0.0011	−6.0001 +ι 0.0000
4.2426 +ι 4.2426	4.2425 +ι 4.2427
4.2427 −ι 4.2425	4.2426 -ι 4.2424
−4.2423 +ι 4.2427	−4.2429 +ι 4.2426
−4.2430 −ι 4.2428	−4.2425 −ι 4.2425
0.0000 +ι 5.9998	0.0000 +ι 6.0001
0.0000 −ι 6.0010	0.0000 −ι 6.0000

Table 3

$\sigma = 2$, $m = 8$, and $n = 21$

△	○
0.0000 +ι 0.0000	0.1119 +ι 0.2702
0.0000 +ι 0.0000	0.1119 −ι 0.2702
0.0000 +ι 0.0000	−0.2702 +ι 0.1119
0.0000 +ι 0.0000	−0.2702 −ι 0.1119
0.0000 +ι 0.0000	−0.1119 +ι 0.2702
0.0000 +ι 0.0000	−0.1119 −ι 0.2702
0.0000 +ι 0.0000	0.2702 +ι 0.1119
0.0000 + ι 0.0000	0.2702 −ι 0.1119
−5.9998 +ι 0.0000	−6.0000 +ι 0.0000
3.7410 +ι 4.6910	3.7408 +ι 4.6910
3.7416 −ι 4.6913	3.7420 -ι 4.6903
−5.4058 +ι 2.6035	−5.4059 +ι 2.6030
−5.4061 −ι 2.6028	−5.4056 −ι 2.6032
−3.7409 +ι 4.6912	−3.7410 +ι 4.6912
−3.7413 −ι 4.6909	−3.7411 −ι 4.6909
5.4060 +ι 2.6032	5.4058 +ι 2.6032
5.4059 −ι 2.6032	5.4052 -ι 2.6030
1.3351 +ι 5.8459	1.3351 +ι 5.8495
1.3346 −ι 5.8503	1.3350 −ι 5.8506
−1.3347 +ι 5.8496	−1.3351 +ι 5.8495
−1.3356 −ι 5.8496	−1.3353 −ι 5.8494

Legend

△ = Zero of $S_n(z, f_m)$

○ = Zero of $L_n^{\sigma}(z, f_m)$

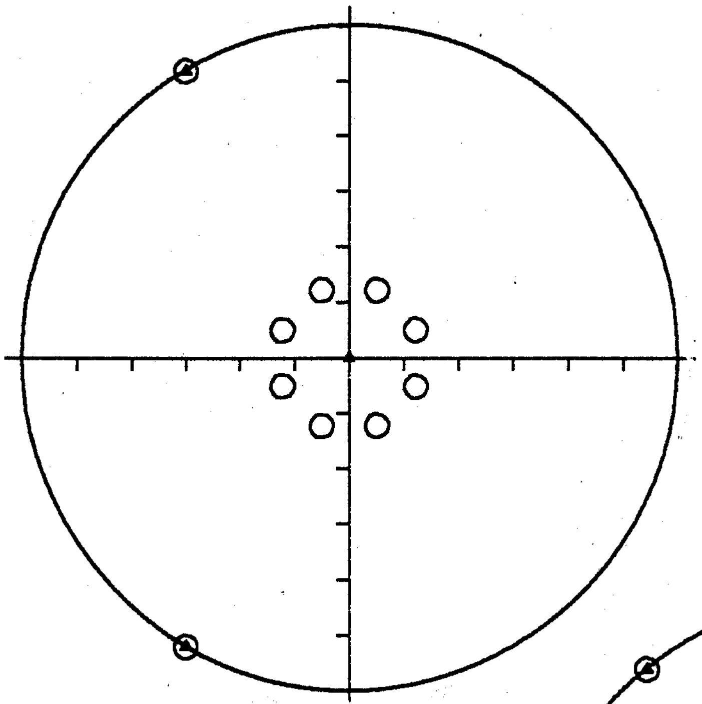


Fig.1 (n=10)

Legend

Δ	=	Zero of $S_n(z, f_m)$
$\bigcirc$	=	Zero of $L_n^{\sigma}(z, f_m)$
(σ, m)	=	(2, 8)
ρ	=	6

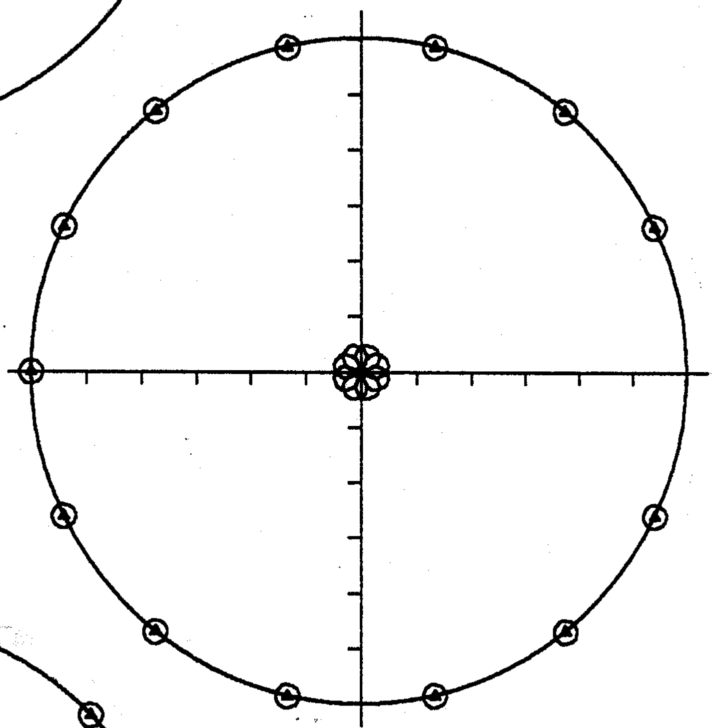


Fig.3 (n=21)

Fig.2 (n=15)

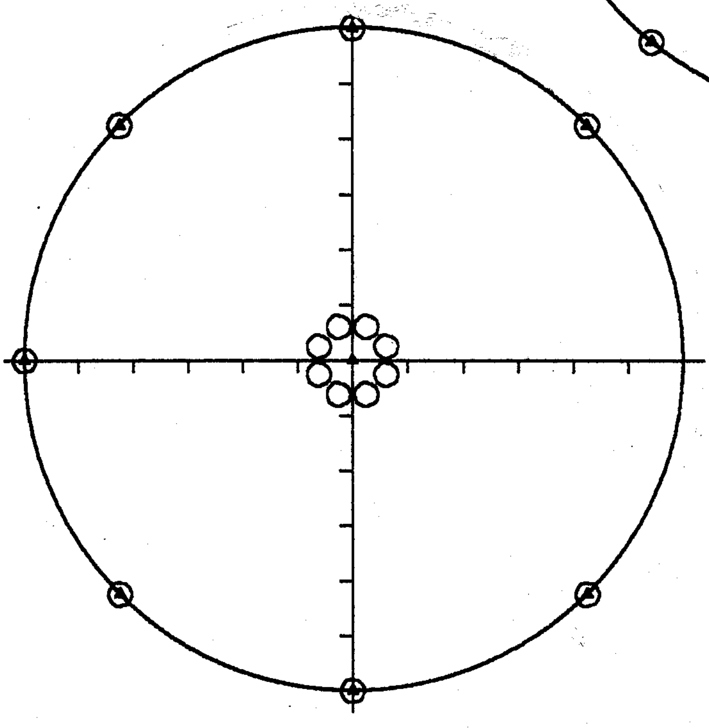


Table 4

$\sigma = 4,\ m = 5,\ n = 8$

△	○
0.0000 +ι 0.0000	2.3210 +ι 1.7075
0.0000 +ι 0.0000	2.3210 −ι 1.7075
0.0000 +ι 0.0000	−0.9318 +ι 2.7622
0.0000 +ι 0.0000	−0.9318 −ι 2.7622
0.0000 +ι 0.0000	−2.9429 +ι 0.0000
0.0000 +ι 6.0000	0.0400 +ι 5.9624
−6.0000 +ι 0.0000	−5.9175 −ι 0.0000
0.0000 − ι 5.9999	0.4098 +ι 5.9624

Table 5

$\sigma = 4,\ m = 5,\ n = 15$

△	○
0.0000 +ι 0.0000	1.3266 +ι 0.9638
0.0000 +ι 0.0000	1.3266 −ι 0.9638
0.0000 +ι 0.0000	−0.5067 +ι 1.5595
0.0000 +ι 0.0000	−0.5067 −ι 1.5596
0.0000 +ι 0.0000	−1.6397 +ι 0.0000
2.4922 +ι 5.4576	2.4924 +ι 5.4575
2.4926 −ι 5.4579	2.4920 −ι 5.4579
−3.9287 + ι 4.5339	−3.9298 +ι 4.5337
−3.9297 −ι 4.5339	−3.9293 −ι 4.5338
5.0474 +ι 3.2439	5.0461 +ι 3.2425
5.0475 −ι 3.2438	5.0462 -ι 3.2427
−0.8546 +ι 5.9385	−0.8527 +ι 5.9382
−0.8538 −ι 5.9393	−0.8530 −ι 5.9376
−5.7568 +ι 1.6905	−5.7558 +ι 1.6909
−5.7559 −ι 1.6897	−5.7560 −ι 1.6909

Table 6

$\sigma = 4,\ m = 5,\ n = 21$

△	○
0.0000 +ι 0.0000	0.8153 +ι 0.5923
0.0000 +ι 0.0000	0.8153 −ι 0.5923
0.0000 +ι 0.0000	−0.3114 +ι 0.9584
0.0000 +ι 0.0000	−0.3113 −ι 0.9584
0.0000 +ι 0.0000	−1.0077 +ι 0.0000
4.4342 +ι 4.0418	4.4338 +ι 4.0404
4.4337 −ι 4.0399	4.4339 −ι 4.0419
5.5950 +ι 2.1675	5.5939 +ι 2.1673
5.5941 −ι 2.1668	5.5947 −ι 2.1673
2.6756 +ι 5.3701	2.6739 +ι 5.3703
2.6765 −ι 5.3673	2.6746 -ι 5.3715
−5.9100 +ι 1.1062	−5.8977 +ι 1.1025
−5.9114 −ι 1.1068	−5.8982 −ι 1.1017
−1.6383 +ι 5.7741	−1.6408 +ι 5.7711
−1.6559 −ι 5.7667	−1.6419 −ι 5.7708
0.5572 +ι 5.9753	0.5539 +ι 5.9746
0.5602 −ι 5.9783	0.5532 -ι 5.9748
−3.6164 +ι 4.7986	−3.6145 +ι 4.7895
−3.5823 −ι 4.7748	−3.6159 −ι 4.7886
−5.1078 +ι 3.1622	−5.1004 +ι 3.1595
−5.1044 −ι 3.1948	−5.1027 −ι 3.1585

Legend

△ = Zero of $S_n(z, f_m)$

○ = Zero of $L_n^{\sigma}(z, f_m)$

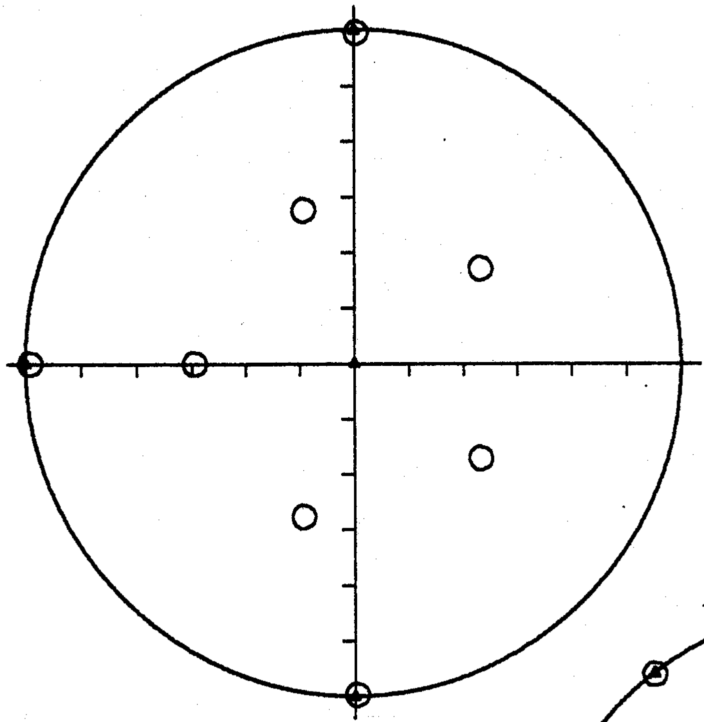


Fig.4 (n=8)

Legend

Δ	=	Zero of $S_n(z, f_m)$
$\bigcirc$	=	Zero of $L_n^{\sigma}(z, f_m)$
(σ, m)	=	(4, 5)
ρ	=	6

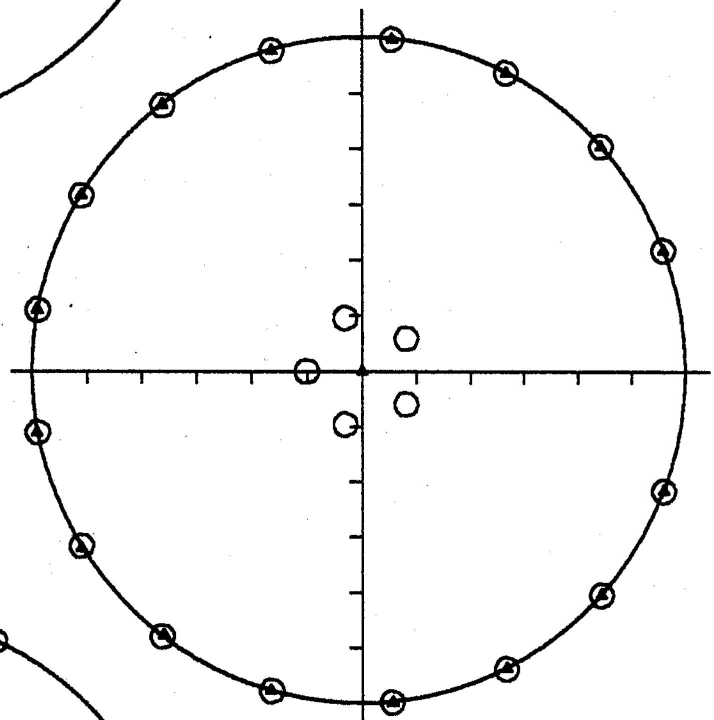
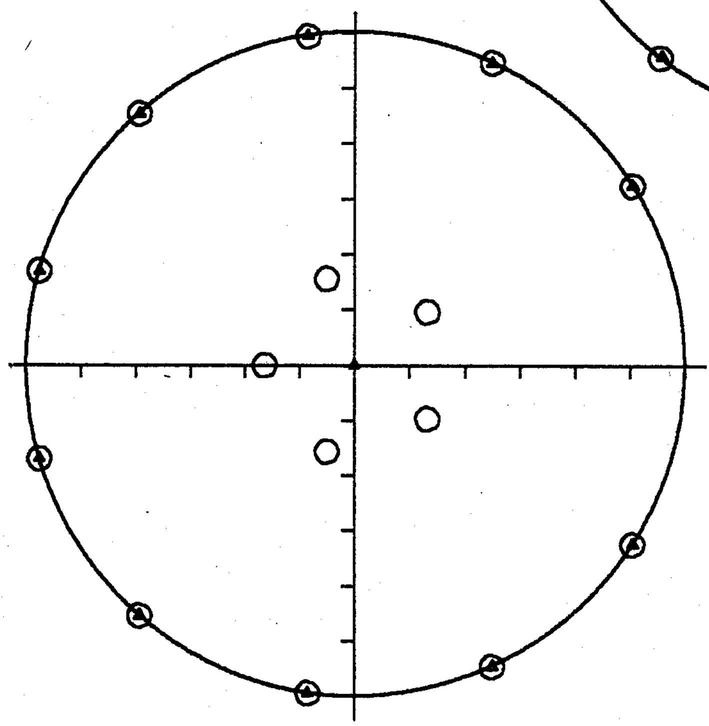


Fig.6 (n = 21)

Fig.5 (n=15)



Concluding Remarks. In general, if $f \in A_\rho$, $\sigma > 0$, and $\rho > \max\{1, \sigma\}$ then for every $z \in C$ with $|z| < \rho$, it is well known that

$$\left.\begin{array}{rl} & \lim_{n\to\infty} S_n(z,f) = f(z) \\ \text{and} & \\ & \lim_{n\to\infty} L_n^\sigma(z,f) = f(z) \end{array}\right\}. \qquad (I.4)$$

Since the convergence in (I.4) is uniform on any compact subset of the region $D_\rho = \{z \in C : |z| < \rho\}$, we have the following result which is an immediate consequence of Hurwitz theorem [6]:

Theorem I.2 *If z_0 is an m-fold zero of $f \in A_\rho$ in the region D_ρ then for all sufficiently small $\epsilon > 0$, the disk $|z - z_0| < \epsilon$ contains exactly m zeros (counted with their multiplicities) of each of the polynomials $S_n(z,f)$ and $L_n^\sigma(z,f)$, $n \geq N(\epsilon)$.*

Remark I.2 If $f \in A_\rho$ has r zeros, say $z_j, (j = 1, \cdots, r)$, (counted with their multiplicities) in D_ρ and if $\xi_{j,n}, (j = 1, \cdots, r)$, are the respective zeros of $S_n(z,f)$ and $L_n^\sigma(z,f)$ lying in the neighbourhood of z_j as narrated in Theorem I.2, then

$$\lim_{n\to\infty} |\xi_{j,n} - \eta_{j,n}| = 0, \quad j = 1, 2, \cdots, r.$$

Besides the r zeros described in the above remark, we are also interested in discussing the asymptotic behaviour of the remaining zeros of $S_n(z,f)$ and $L_n^\sigma(z,f)$ lying in the region D_ρ. For this, first we note that [11]

$$P_n(z,f) : = L_n^\sigma(z,f) - S_n(z,f)$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\cdot\frac{\sigma^{n+1}(t^{n+1}-z^{n+1})}{t^{n+1}(t^{n+1}-\sigma^{n+1})}dt$$

$$= O\left(\left(\frac{|z|\sigma}{\rho^2}\right)^n\right) \tag{I.5}$$

and

$$S_n(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\cdot\frac{t^{n+1}-z^{n+1}}{t^{n+1}}dt$$

$$= O\left(\max\left(1,\left(\frac{|z|}{\rho}\right)\right)^n\right) \tag{I.6}$$

where Γ is the circle $|t| = R, \;\; \sigma < R < \rho$.

Next assume that $z_{1,n}, z_{2,n}, \cdots, z_{\ell,n}$ are the only distinct zeros of $S_n(z,f)$ in D_ρ such that

$$z_{k,n} \neq \xi_{j,n}, \quad k = 1,\cdots,\ell \quad \text{and} \quad j = 1,\cdots,r,$$

where $\xi_{j,n}, (j = 1,\cdots,r)$, are the zeros of $S_n(z,f)$ considered in the Remark 4.1. Let

$$\epsilon_{k,n} = \frac{1}{2}\min\left\{ \min_{\substack{1\le j\le \ell \\ j\neq k}} |z_{j,n} - z_{k,n}|, \rho - 1, \frac{\rho^2}{\sigma} - \rho \right\}. \tag{I.7}$$

In the light of the inequality (I.3) and the estimates (I.5) and (I.6), it seems appropriate to suggest the following:

Conjecture: Let $f \in A_\rho$, $\sigma > 0$ and $\rho > \max\{1, \sigma\}$. If $C_{k,n}$ denotes the boundary of the disk $|z - z_{k,n}| < \epsilon_{k,n}$, $k = 1, 2, \cdots, \ell$, then for all sufficiently large values of n

$$\left|\frac{P_n(z,f)}{S_n(z,f)}\right| < 1, \qquad \forall z \in C_{k,n}, \tag{I.8}$$

$k = 1, 2, \cdots, \ell$. Moreover, $\epsilon_{k,n} \to 0$ as $n \to \infty$.

Note that $S_n(z,f)$ has exactly one zero inside the circle $C_{k,n}$. Therefore, due to Rouche's theorem [6], $L_n^\sigma(z,f)$ will have exactly one zero, say $\tilde{z}_{k,n}$ on or within $C_{k,n}$, $(k = 1, \cdots, \ell)$, if the above conjecture is valid. This will lead us to conclude that the asymptotic distribution of the zeros of $S_n(z,f)$ and $L_n^\sigma(z,f)$ is similar.

II Simultaneous Interpolation And L_2 Approximation

It is known [3] that the polynomial $p \in \pi_m$ which approximates a given function $f \in C[a,b]$ in the sense of least squares over [a,b] also interpolates $f(x)$ at $m+1$ distinct points of [a,b]. The location of these nodes, however, emerges from the process of L_2 - approximation and is not available at hand. If we pre-assign the course of the approximating polynomial through n given points we come across a constrained minimization problem described below.

Given n distinct points $x_1, \cdots, x_n \in [a,b]$ and $f \in \mathcal{L}_w^2[a,b]$, determine

$p^*_{m,n} \in \pi_m (m > n)$ which solves

$$(P1) \qquad \min_{\substack{p \in \pi_m \\ p(x_i) = f(x_i) \\ i = 1, \cdots, n}} \int_a^b w(x) \left| p(x) - f(x) \right|^2 dx$$

Main Result. In order to solve the problem (P1) we revive an approach due to J.L. Walsh [11] which transforms (P1) to a minimization problem without interpolatory constraints. More precisely, (P1) is equivalnt to

$$(P2) \qquad \min_{q \in \pi_m(\Delta_n)} \int_a^b w(x) \left| f^*(x) - q(x) \right|^2 dx$$

where

$$f^*(x) = f(x) - L_{n-1}(x, \Delta_n, f).$$

The solution of (P2) relies on the orthonormal basis of $\Pi_m(\Delta_n)$ which is an $m - n + 1$ dimensional space and is spanned by $\left\{ x^j W_n(x) \right\}_{j=0}^{m-n}$. We describe our main result of this part as follows [2]:

Theorem II.1 *Let* $\Delta_n = \{x_2, \cdots, x_n\} \subseteq [a, b]$ *and let* $f \in \mathcal{L}^2_w[a, b]$. *If* $\phi_0, \phi_1, \cdots, \phi_{m-n}$ *is an orthonormal basis of* $\Pi_m(\Delta_n)$ *then*

$$q^*(x) = \sum_{i=0}^{m-n} < \phi_i, f^* >_w \phi_i(x)$$

is the optimal solution of the problem (P2). Moreover,

$$p^*_{m,n}(x) := L_{n-1}(x, \Delta_n, f) + q^*(x) \qquad \text{(II.9)}$$

is the optimal solution of the problem (P1).

Convergence. It can be shown that the class of polynomials

$$\Pi(\Delta_n) := < x^j W_n : \quad j = 0, 1, 2, \cdots >$$

is uniformly dense in $C([a, b], \Delta_n)$. This observation leads us to establish

Theorem II.2. *Let n be a fixed nonnegative integer and let* $\Delta_n = \{x_1, \cdots, x_n\} \subseteq [a, b]$. *If* $f \in C[a, b]$ *then*

$$\lim_{m \to \infty} ||f - p^*_{m,n}||_w = 0$$

where $p^*_{m,n}$ *is the optimal solution of (P1) given by (II.1).*

Construction of Orthogonal Basis. As noted in Theorem II.1, the solution of (P2) and, consequently, that of (P1) depends on the orthogonal basis $\{\phi_i\}_{i=0}^{m-n}$ of $\Pi_m(\Delta_n)$. One can use the 3-term recurrance relation [3] to determine $\{\phi_i\}_{i=0}^{m-n}$ according to the following scheme:

$$\begin{aligned}
\psi_0(x) &= W_n(x) \\
\psi_1(x) &= (x - \alpha_0)\psi_0(x) \\
\psi_j(x) &= (x - \alpha_j)\psi_j(x) - \beta_j \psi_{j-1}(x), \quad j = 1, 2, 3 \cdots
\end{aligned}$$

where

$$\begin{aligned}
\alpha_j &= < \psi_j, x\psi_j >_w / ||\psi_j||_w^2, \quad j = 0, 1, 2, \cdots \\
\beta_j &= ||\psi_j||_w^2 / ||\psi_{j-1}||_w^2, \quad j = 1, 2, \cdots
\end{aligned}$$

Now define

$$\phi_j(x) = \psi_j(x)/||\psi_j||_w, \quad j = 0, 1, 2, \cdots$$

Like the situation of the Legendre polynomials, we do observe that $\alpha_j = 0, \ \forall_j$, in the above recurrance relation provided that $[a, b] = [-r, r], r > 0, w(x)$ is an even function and the nodes $x_k's$ are symmetric with respect to the origin.

Computational Procedure. In order to develop an algorithm for computing the optimal solution of (P1) we consider an explicit representation of $W_n^2(x)$ which is a polynomial of degree $2n$. We note that

$$W_n^2(x) = \sum_{j=0}^{2n} d_{n,j} x^j \tag{II.10}$$

where $d_{n,j}$ satisfies the recurrance relation

$$d_{n,j} = \sum_{\substack{0 \le k \le 2(n-1) \\ 0 \le \ell \le 2 \\ k+\ell=j}} d_{n-1,k} \cdot a_{n,\ell} \ , \quad j = 0, 1, 2, \cdots, 2n$$

with

$$d_{0,0} = x_1^2, \ d_{0,1} = -2x_1, \ d_{0,2} = 1$$

$$a_{i,0} = x_i^2, \ a_{i,1} = -2x_i, \ a_{i,2} = 1$$

$$(i = 1, 2, \cdots, n)$$

The relation (II.2) enables us to compute the integrals

$$\int_a^b w(x) x^v W_n^2(x) dx \quad v = 0, 1, 2, \cdots$$

which are essential components of ϕ_i's.

Other expressions, like $< L_{n-1}(x, \Delta_n, f), \phi_k >_w, < f, \phi_k >_w$ etc. involved in $p^*_{m,n}(x)$ are computed with a similar consideration.

Numerical Results

In this section we discuss application of the Simultaneous Interpolation and Least Squares Method to the functions $(e^x, [-1,1]), (|x|, [-1,1])$, $\left(1/(1+x^2), [-5,5]\right)$ and $\left(e^{-|x|}, [-1,1]\right)$. We consider equally spaced nodes and the weight function $w(x) \equiv 1$ in our examples. The maximum absolute error and the least squares error due to our approximating polynomial $p^*_{m,n}$ are compared with the corresponding errors due to

(i) the Lagrange interpolating polynomial

(ii) the best L_2-approximation polynomial

(iii) the natural cubic splines.

Our numerical observations are presented in the Tables II.1 - II.3 where we have used the following abbreviations:

S.I.L.P.	:=	Simultaneous interpolation and least squares polynomial
L.I.P.	:=	Lagrange interpolation polynomial
B.L.P.	:=	Best L_2-approximating polynomial
N.C.S.	:=	Natural cubic spline
Func./Int.	:=	Function to be approximated and the interval
A.P.	:=	Approximating Process
N.E.N.	:=	Number of equally - spaced nodes
N.O.P.	:=	Number of orthogonal polynomials used for L_2-minimization
L_2-ER	:=	Least squares error between function and the approximating polynomial
L_∞-ER	:=	Absolute maximum error between function and the approximating polynomial

At the end we have included the graphs of the functions considered above and their approximating polynomials. The figures are self-explanatory.

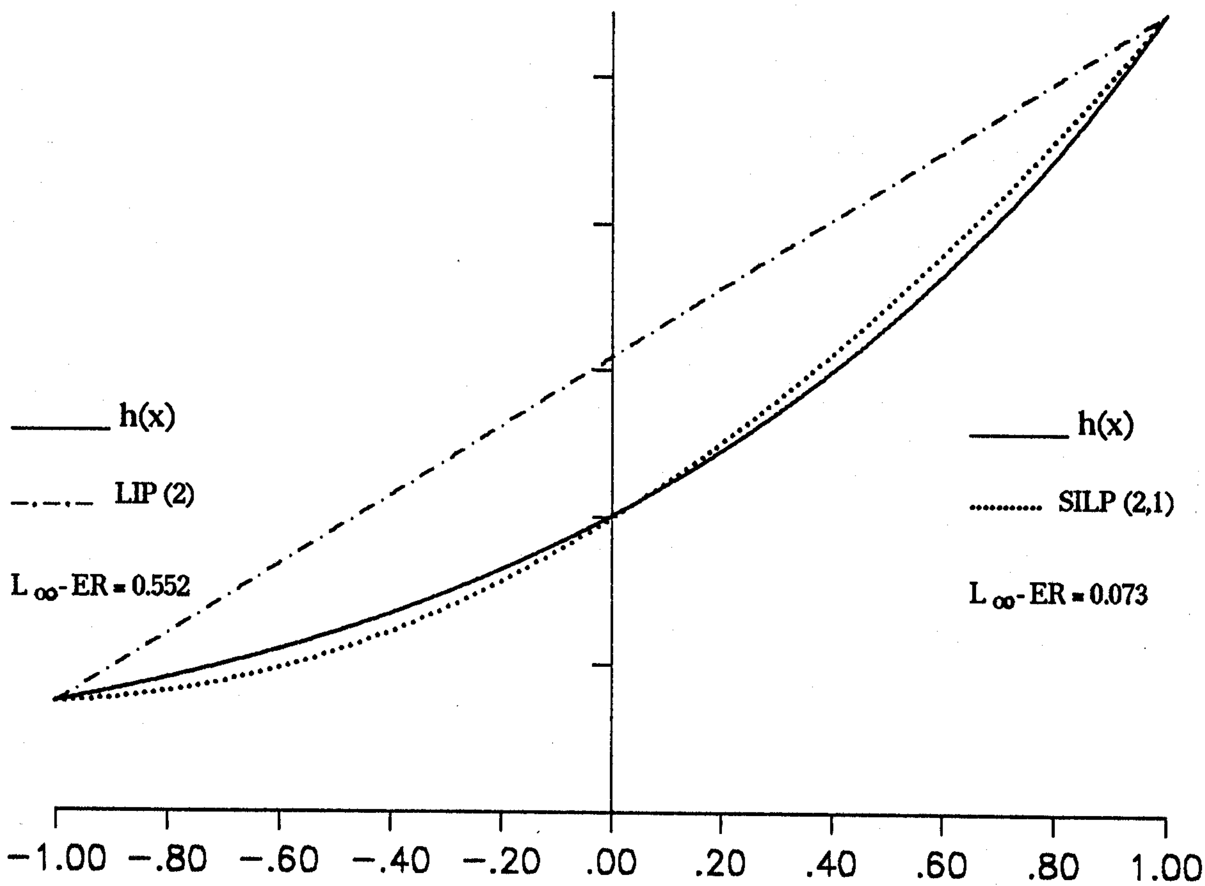
Table I S. I. L. P. Vs. L. I. P.

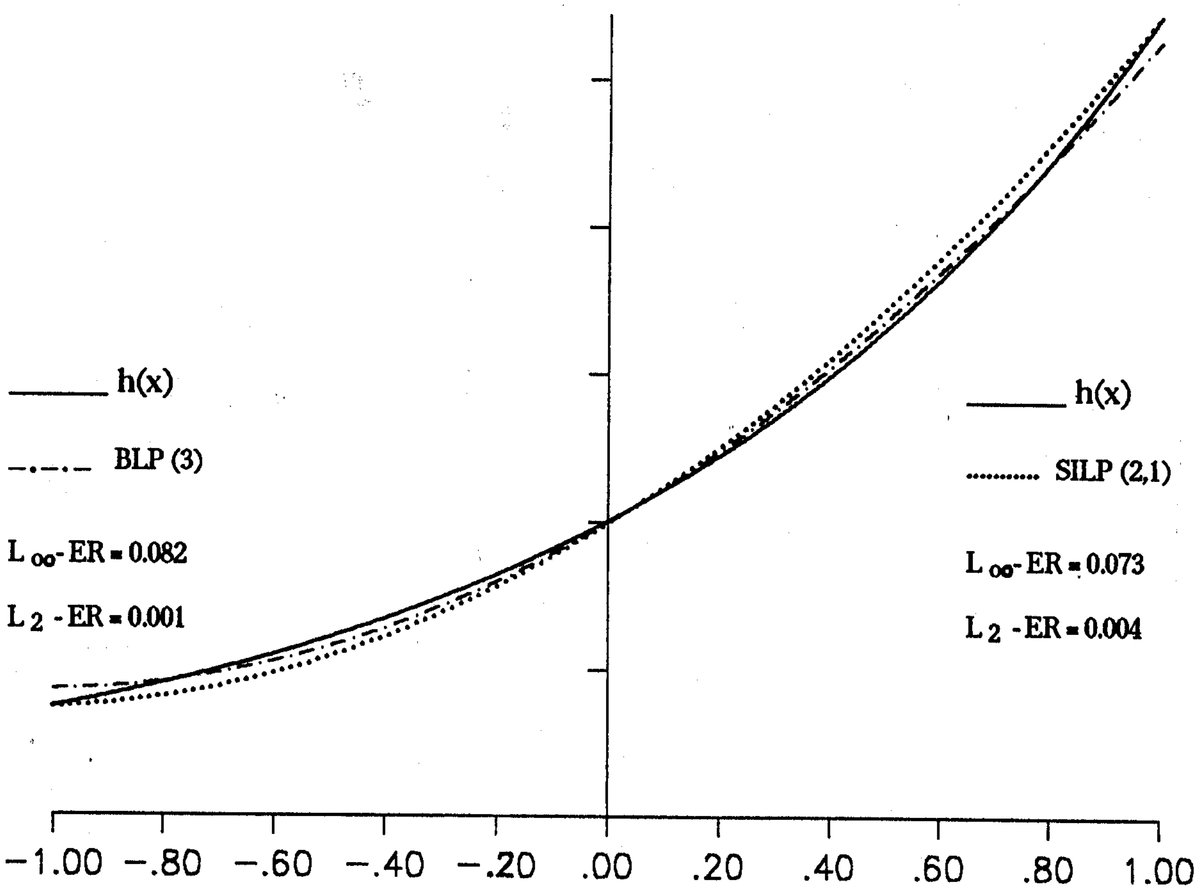
Func/Int.	A.P.	N.E.N	N.O.P	L_∞- ER
e^x [-1,1]	LIP	2		0.552
	SILP	2	1	0.073
Abs(x) [-1,1]	LIP	9		0.316
	SILP	9	1	0.315
		9	3	0.049
$1/(1+x^2)$ [-5,5]	LIP	9		1.045
	SILP	9	1	1.031
		9	3	0.152
$e^{-Abs(x)}$ [-1,1]	LIP	9		0.314
	SILP	9	3	0.050

Table II S. I. L. P. Vs. B. L. P.

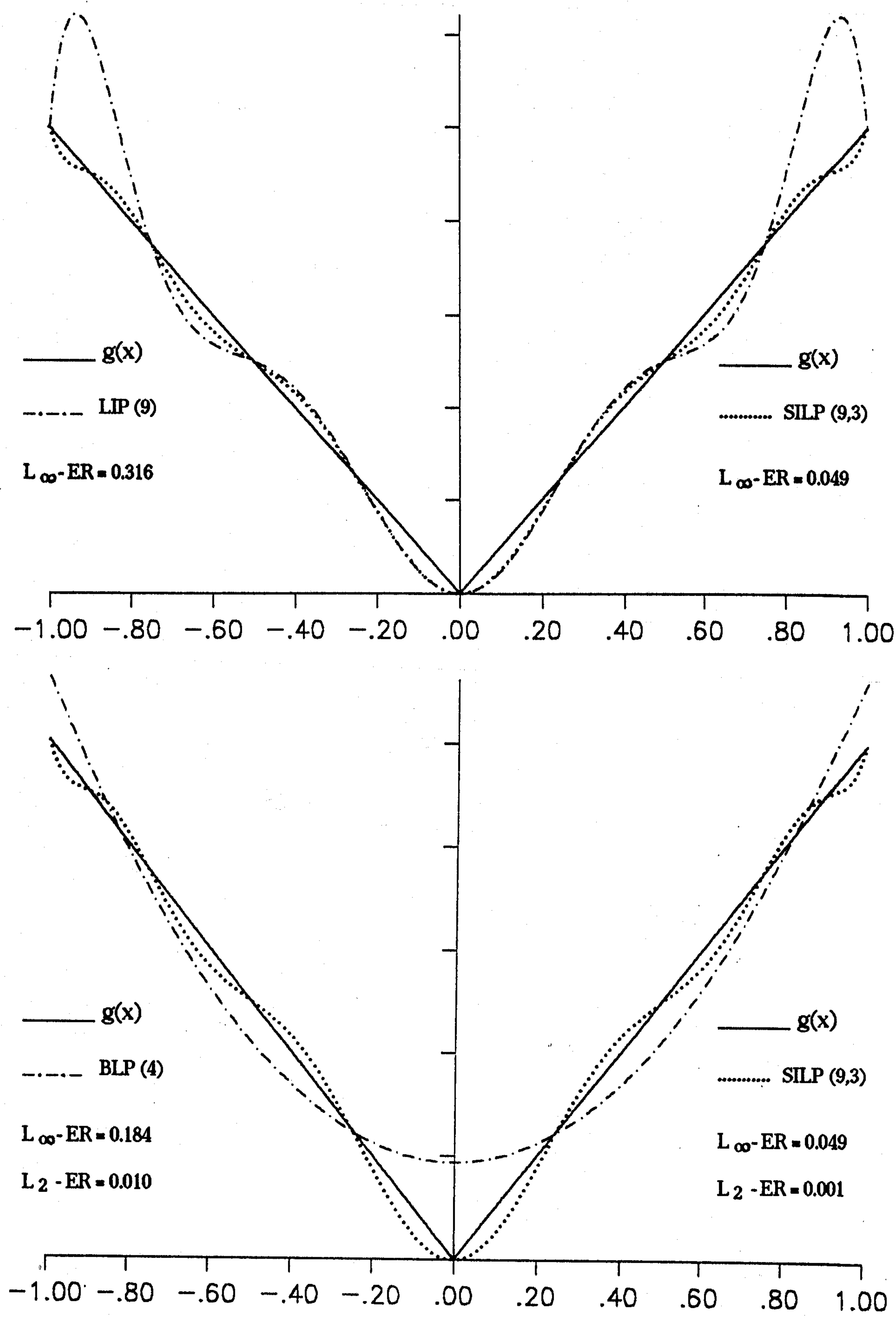
Func/Int.	A.P.	N.E.N	N.O.P	L_∞-ER	L_2-ER
Exp(x) [-1,1]	BLP		3	0.083	0.001
	SILP	2	1	0.073	0.004
Abs(x) [-1,1]	BLP		4	0.184	0.010
	SILP	9	1	0.315	0.025
		9	3	0.049	0.001
$1/(1+x^2)$ [-5,5]	BLP		8	0.222	0.073
	SILP	9	1	1.031	1.378
		9	3	0.152	0.050
$e^{-Abs(x)}$ [-1,1]	BLP		8	0.082	0.001
	SILP	9	3	0.050	0.001

COMPARISON OF SILP vs LIP & BLP
h(x) = Exp (x)

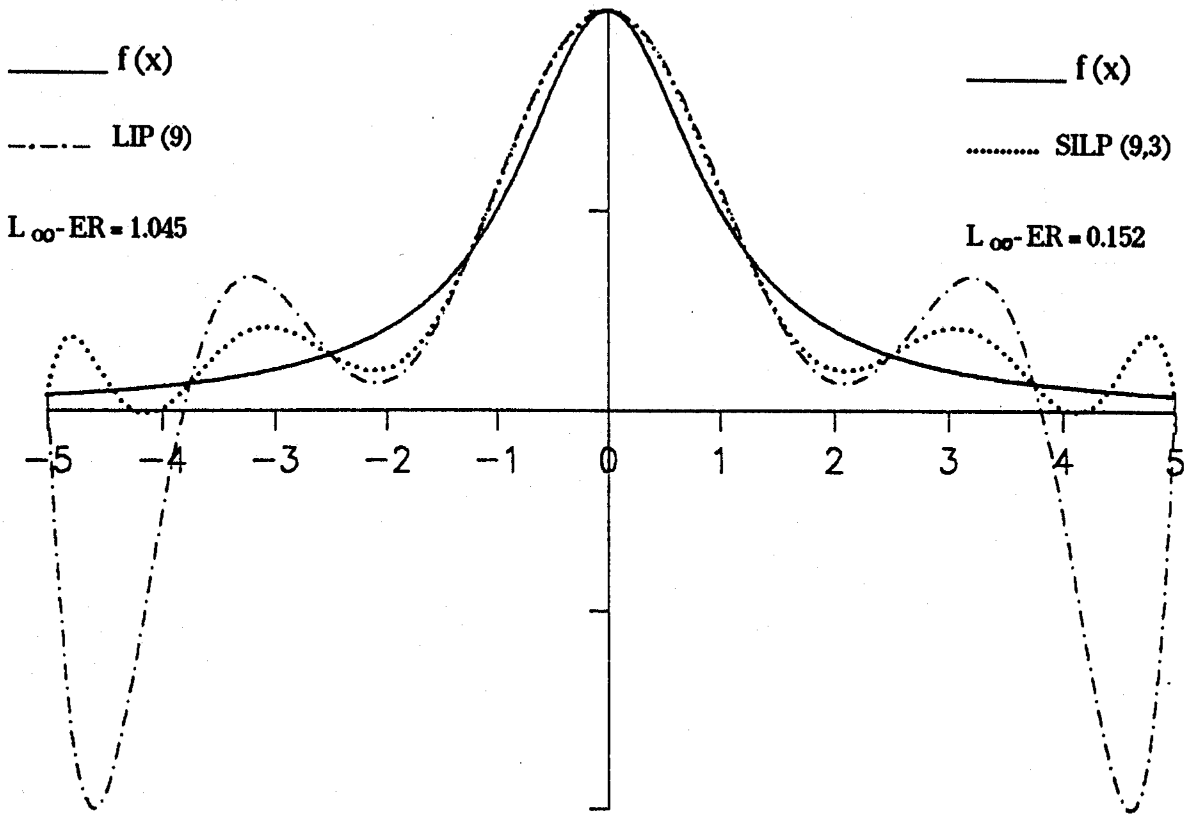


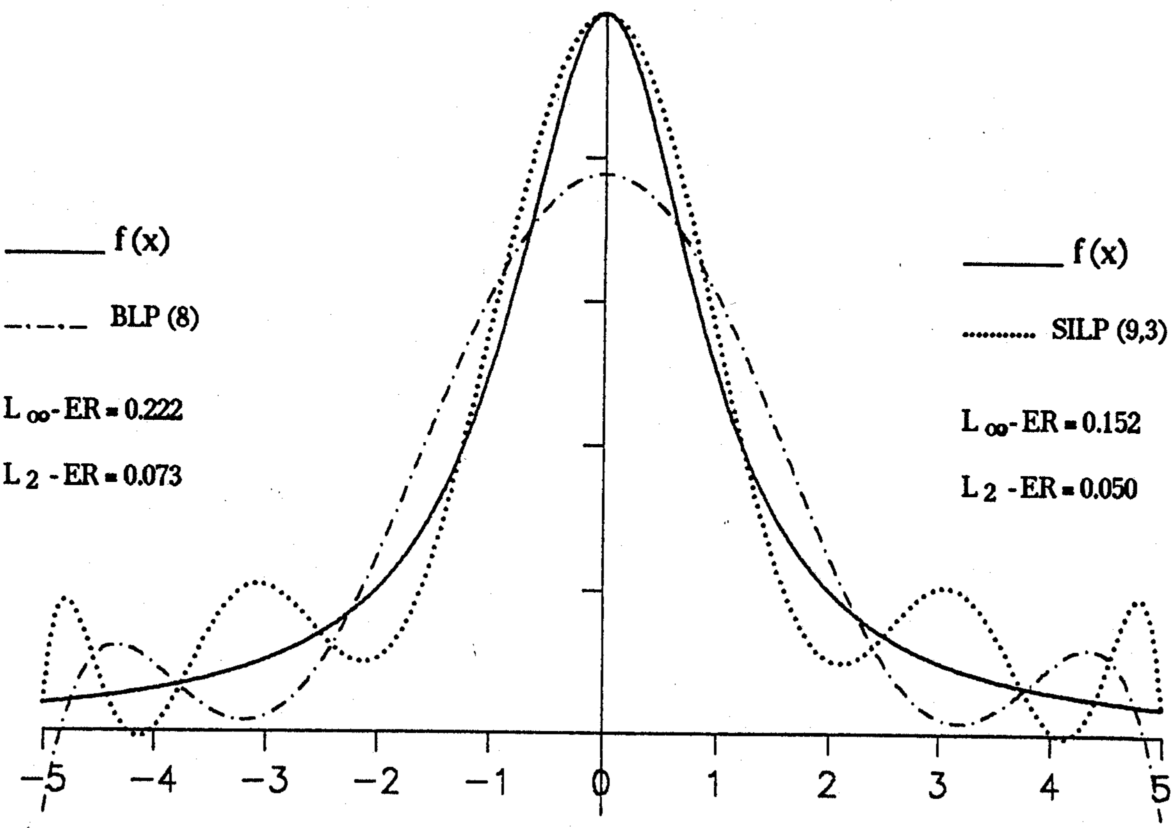


COMPARISON OF SILP vs LIP & BLP
g(x) = Abs (x)



COMPARISON OF SILP vs LIP & BLP
$f(x) = 1/(1+x^2)$





COMPARISON OF SILP vs LIP & BLP
k(x) = Exp (-abs (x))

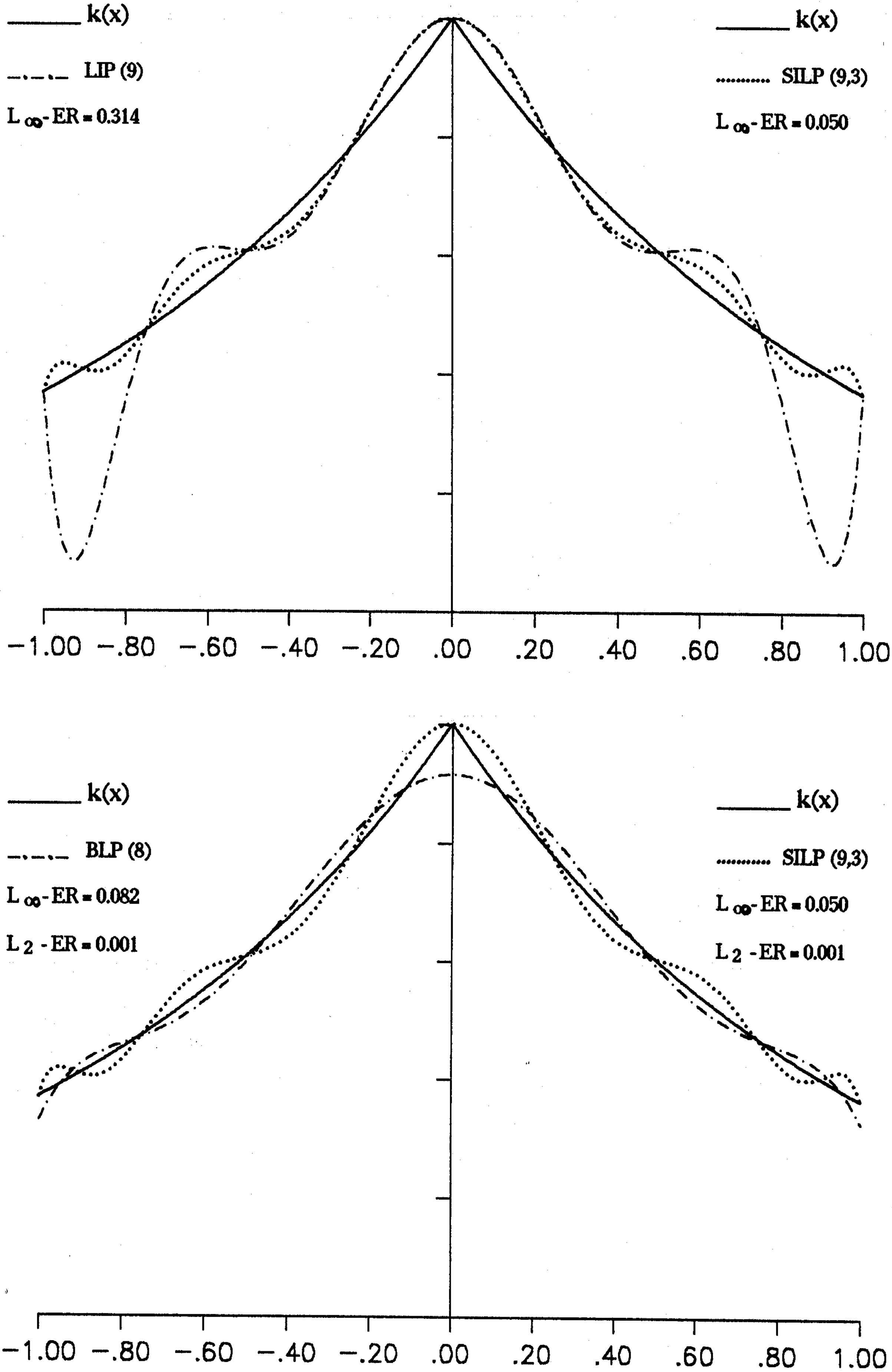
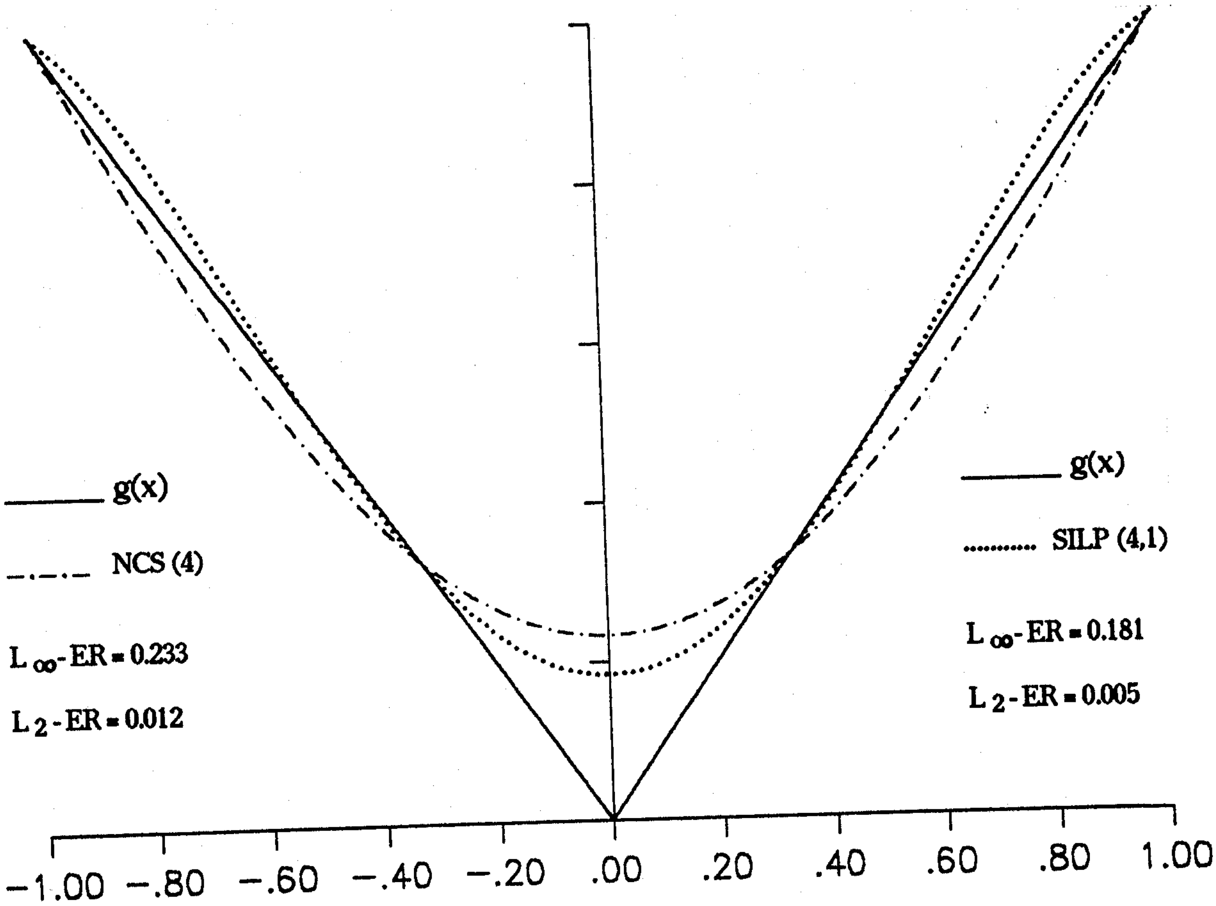
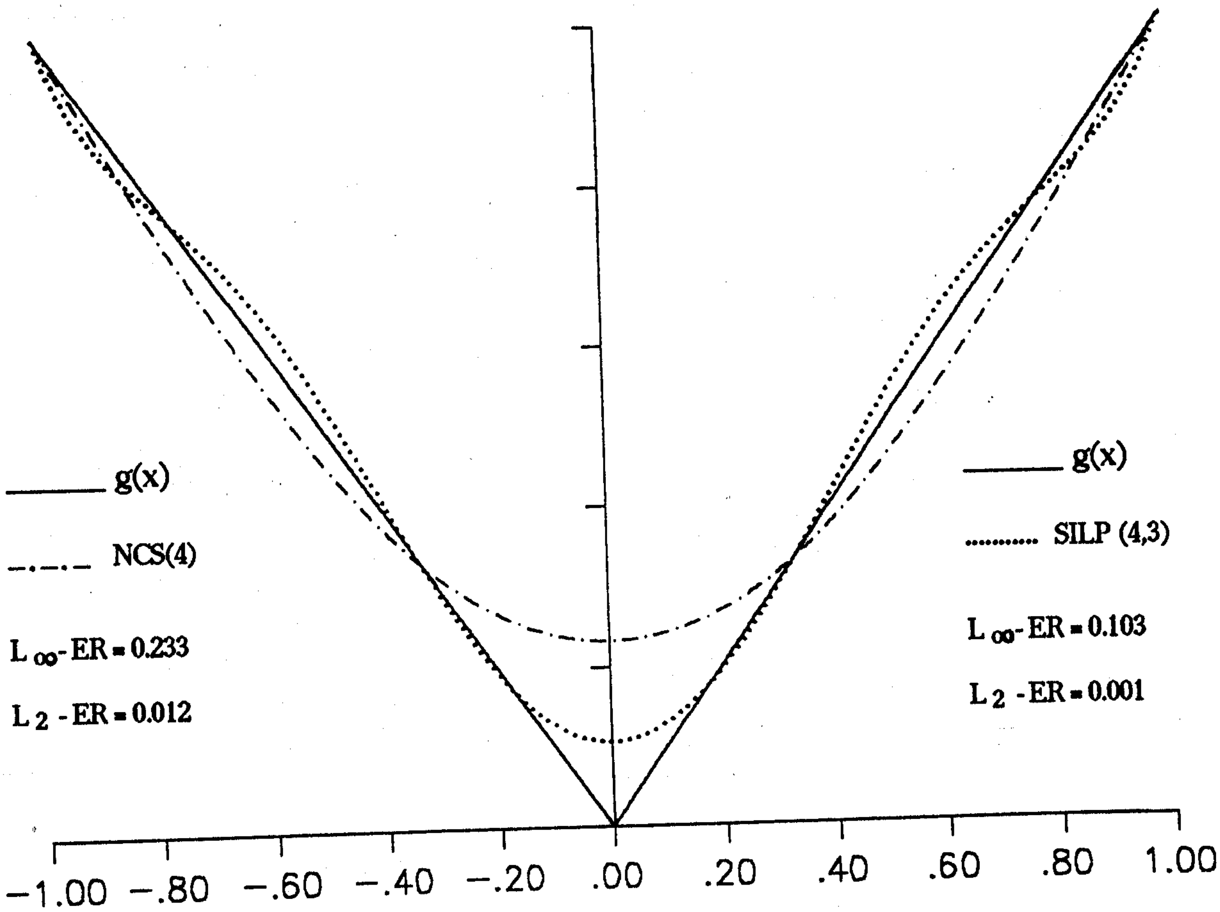


Table III S. I. L. P. Vs. N. C. S.

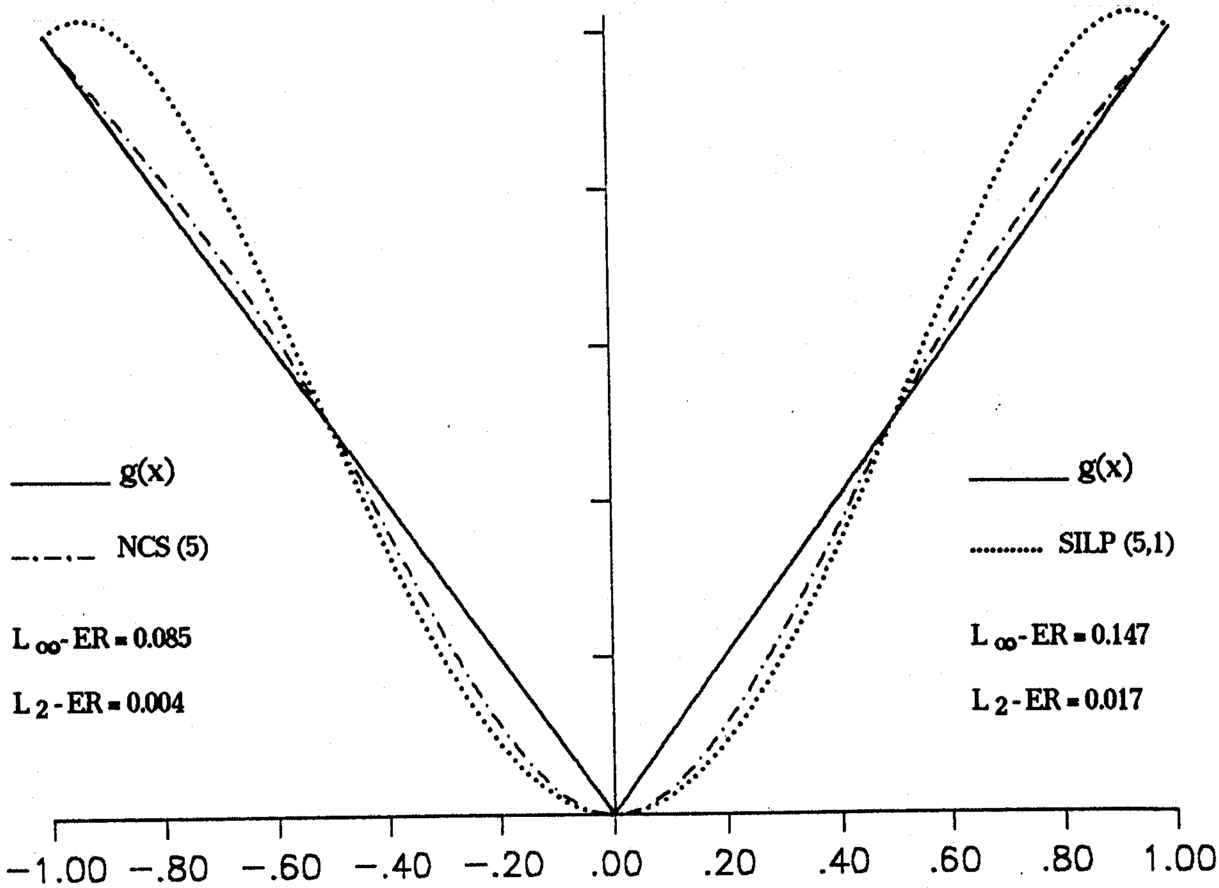
Func/Int.	A.P.	N.E.N	N.O.P	L_∞-ER	L_2-ER
Abs(x) [-1,1]	NCS	4		0.233	0.015
		5		0.085	0.004
	SILP	4	1	0.181	0.005
		4	3	0.103	0.001
		5	1	0.147	0.017
		5	3	0.094	0.004
$1/(1+x^2)$ [-5,5]	NCS	5		0.279	0.200
		7		0.129	0.030
	SILP	5	3	0.300	0.222
		5	5	0.217	0.111
		7	3	0.169	0.083
		7	5	0.116	0.040
$e^{-Abs(x)}$ [-1,1]	NCS	5		0.085	0.004
		7		0.057	0.001
	SILP	5	3	0.094	0.005
		5	5	0.074	0.003
		7	3	0.066	0.002
		7	5	0.060	0.001

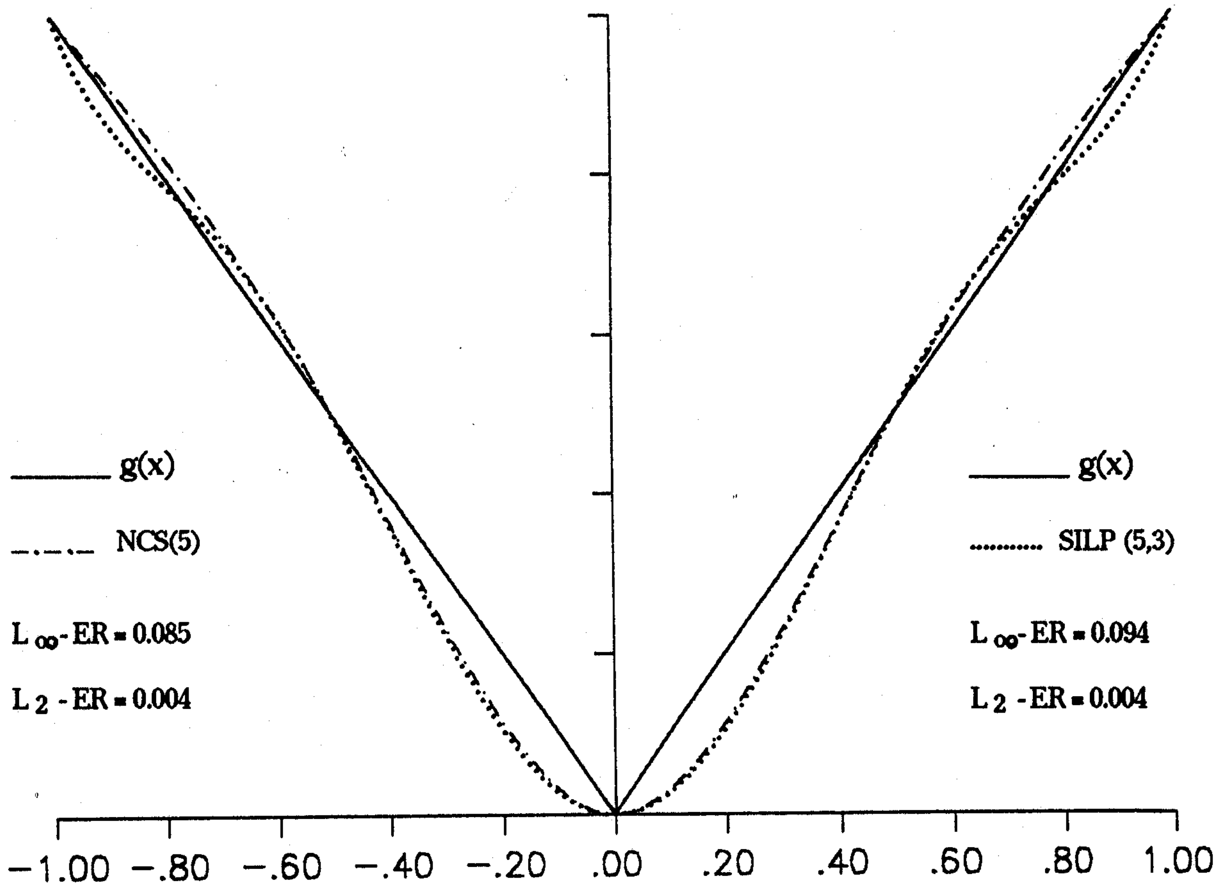
COMPARISON OF SILP vs NCS (4)
g(x) = Abs (x)



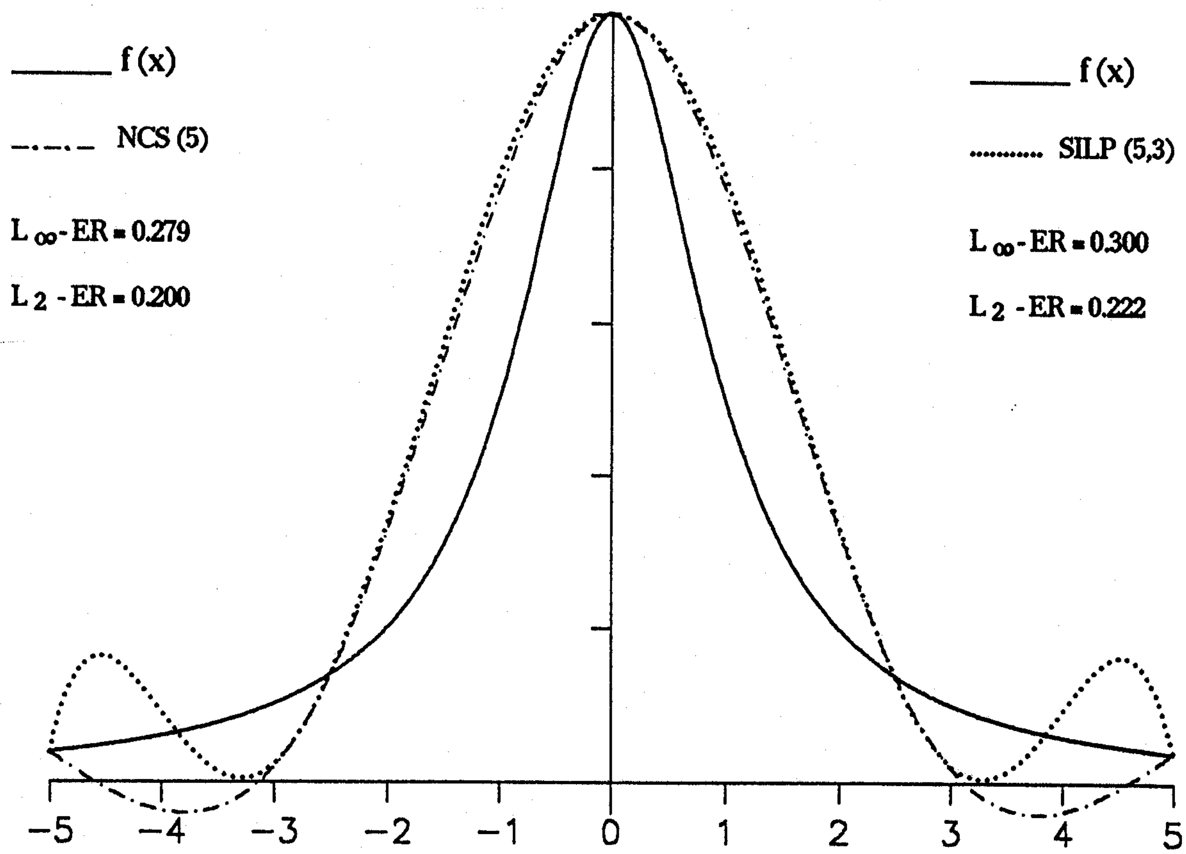


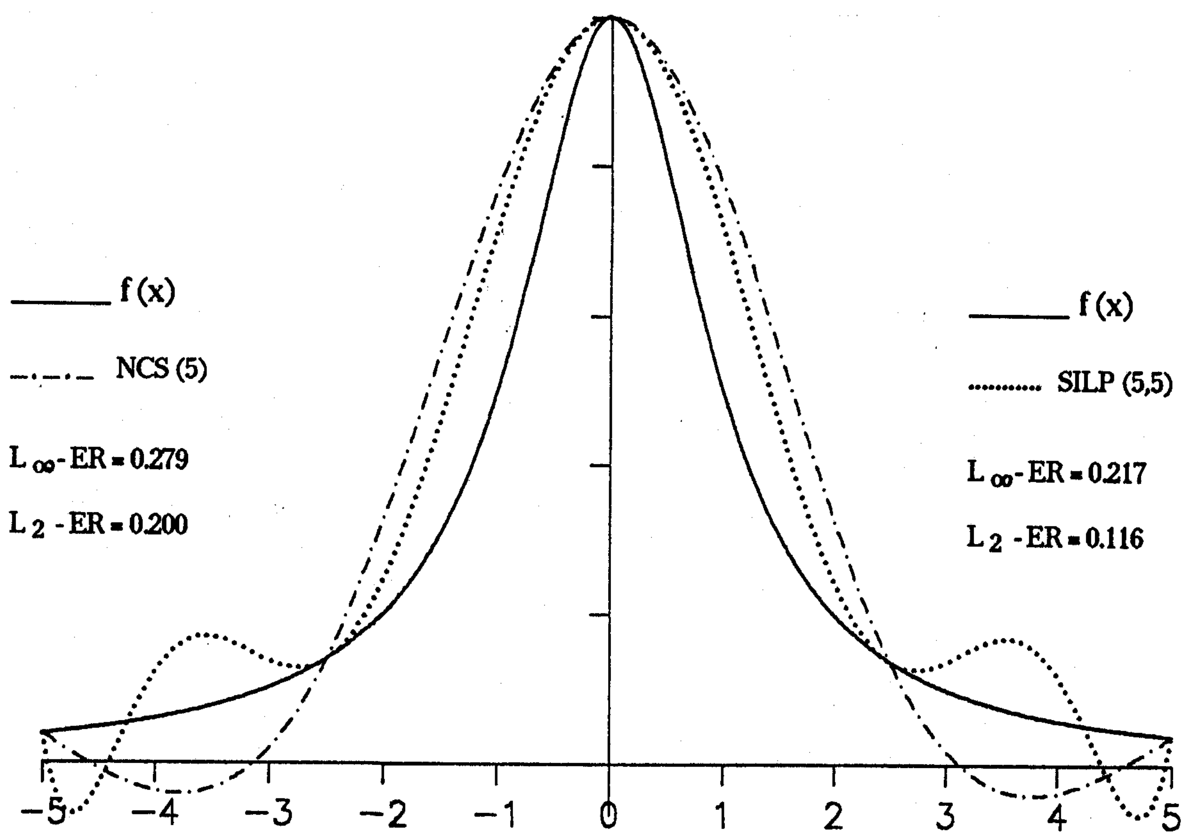
COMPARISON OF SILP vs NCS (5)
g(x) = Abs (x)



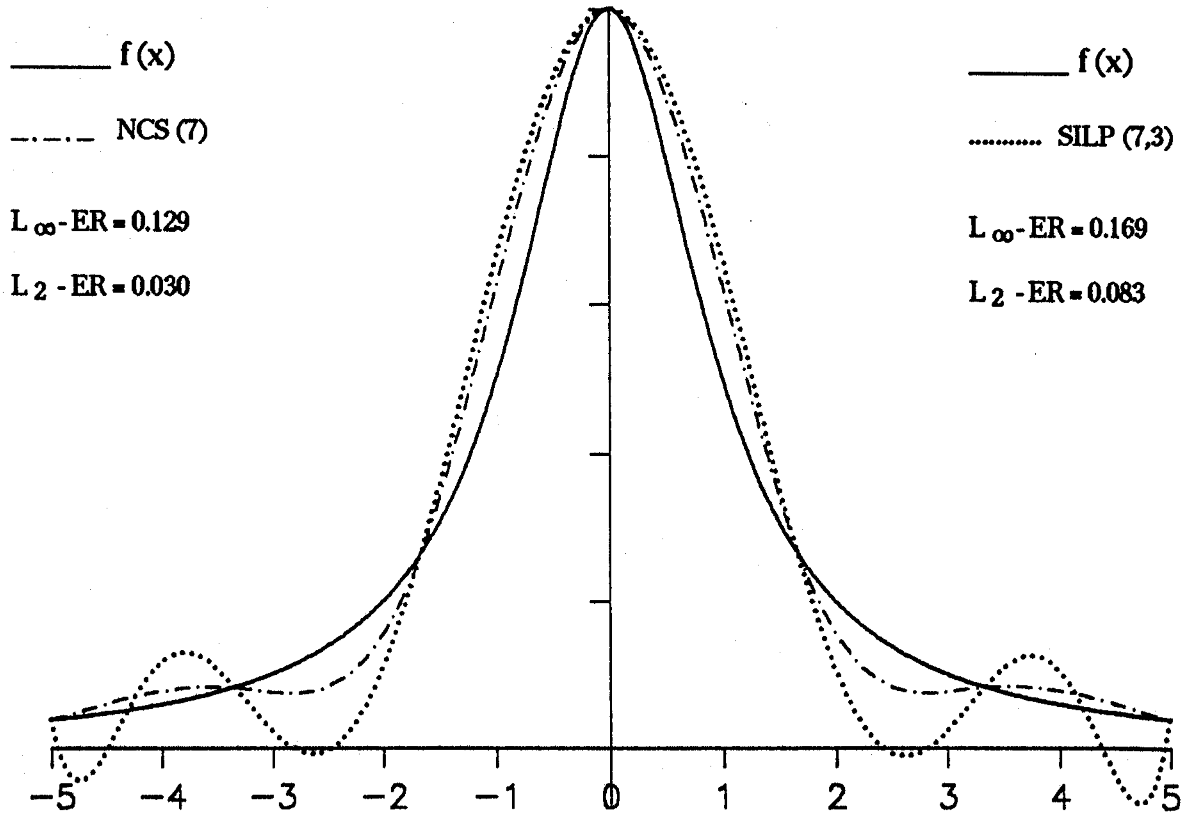


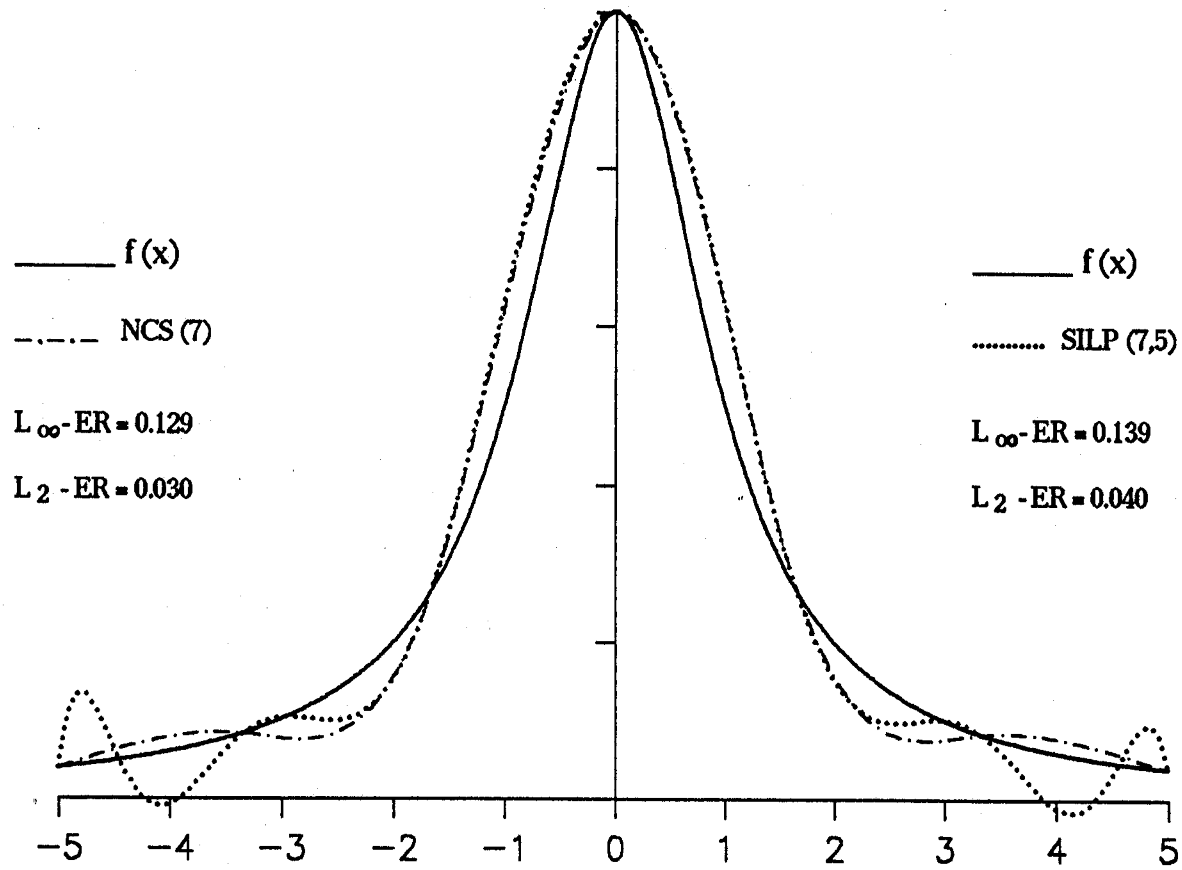
COMPARISON OF SILP vs NCS (5)
$f(x) = 1/(1+x^2)$



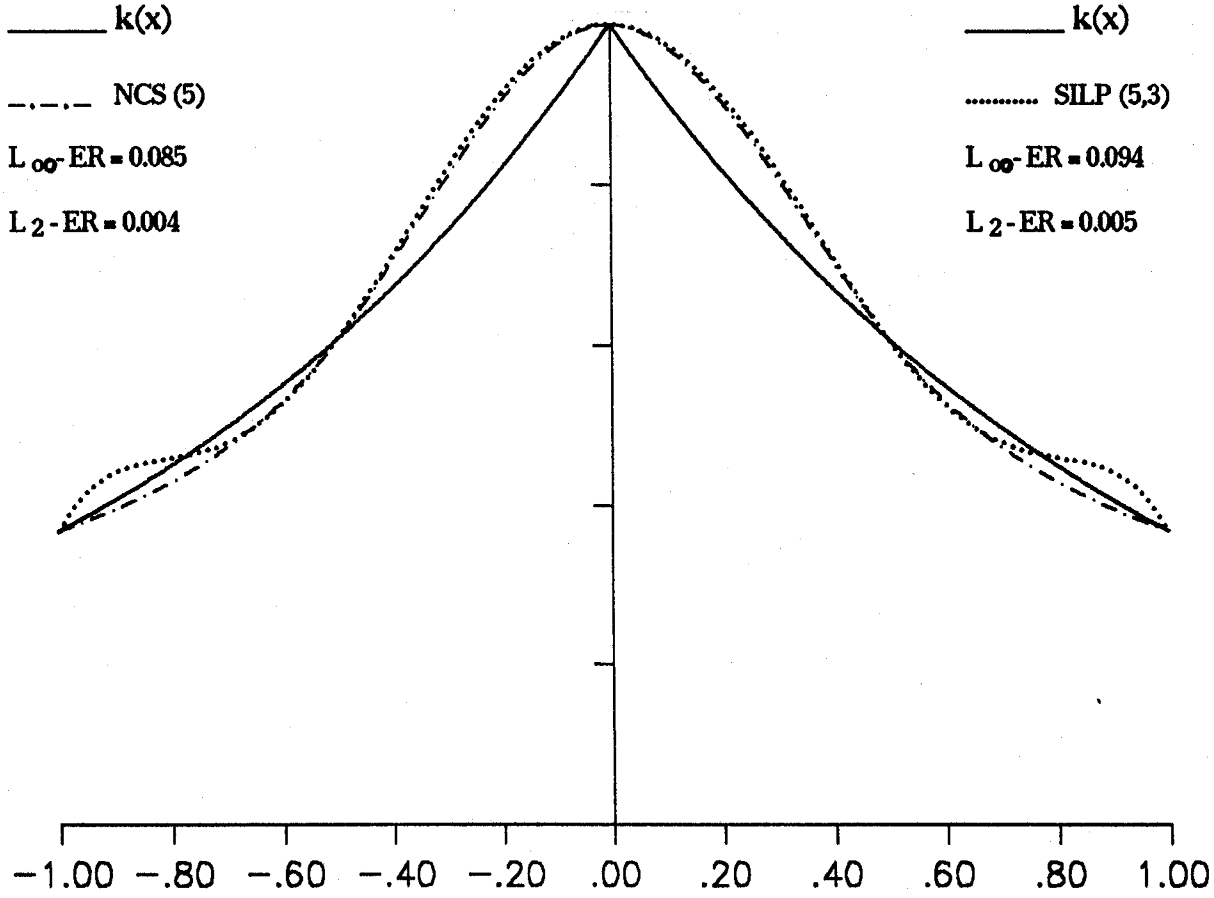


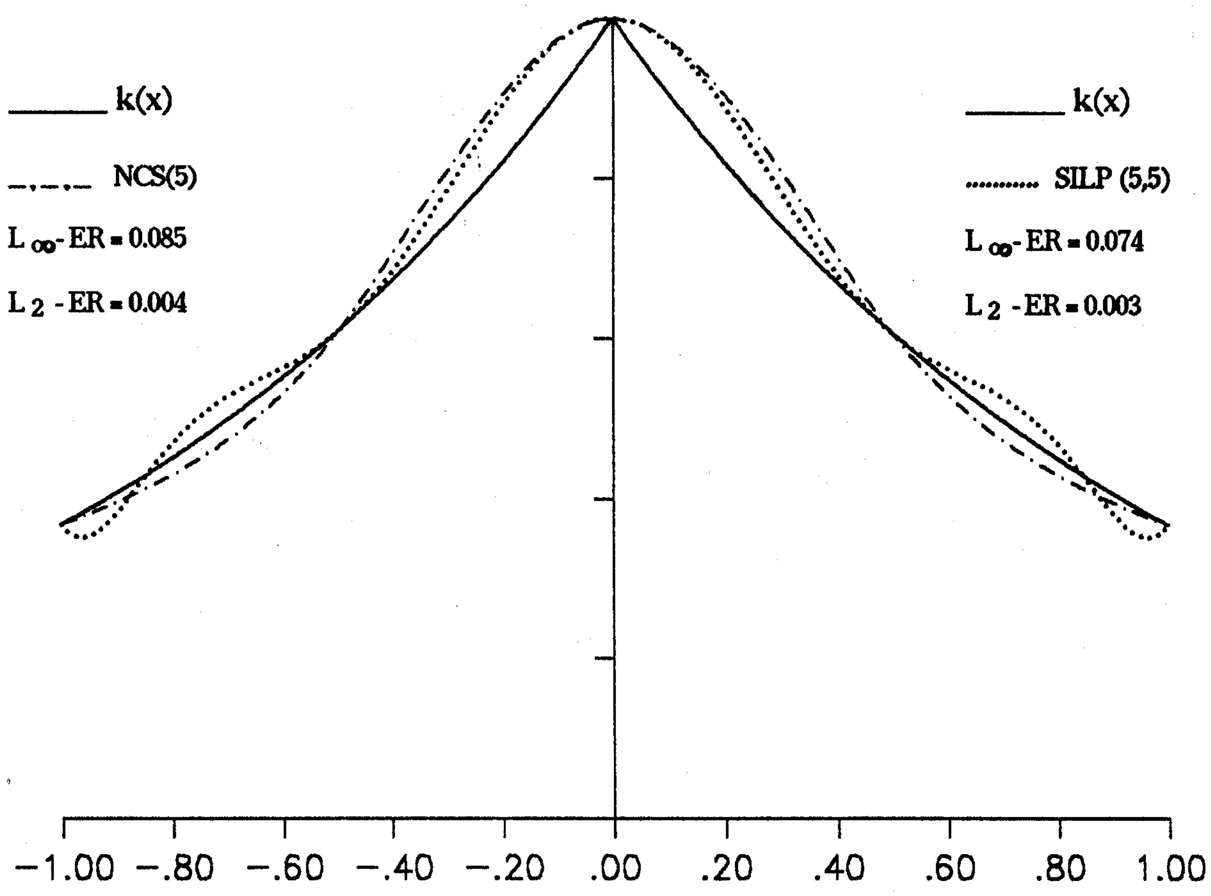
COMPARISON OF SILP vs NCS (7)
$f(x) = 1/(1+x^2)$



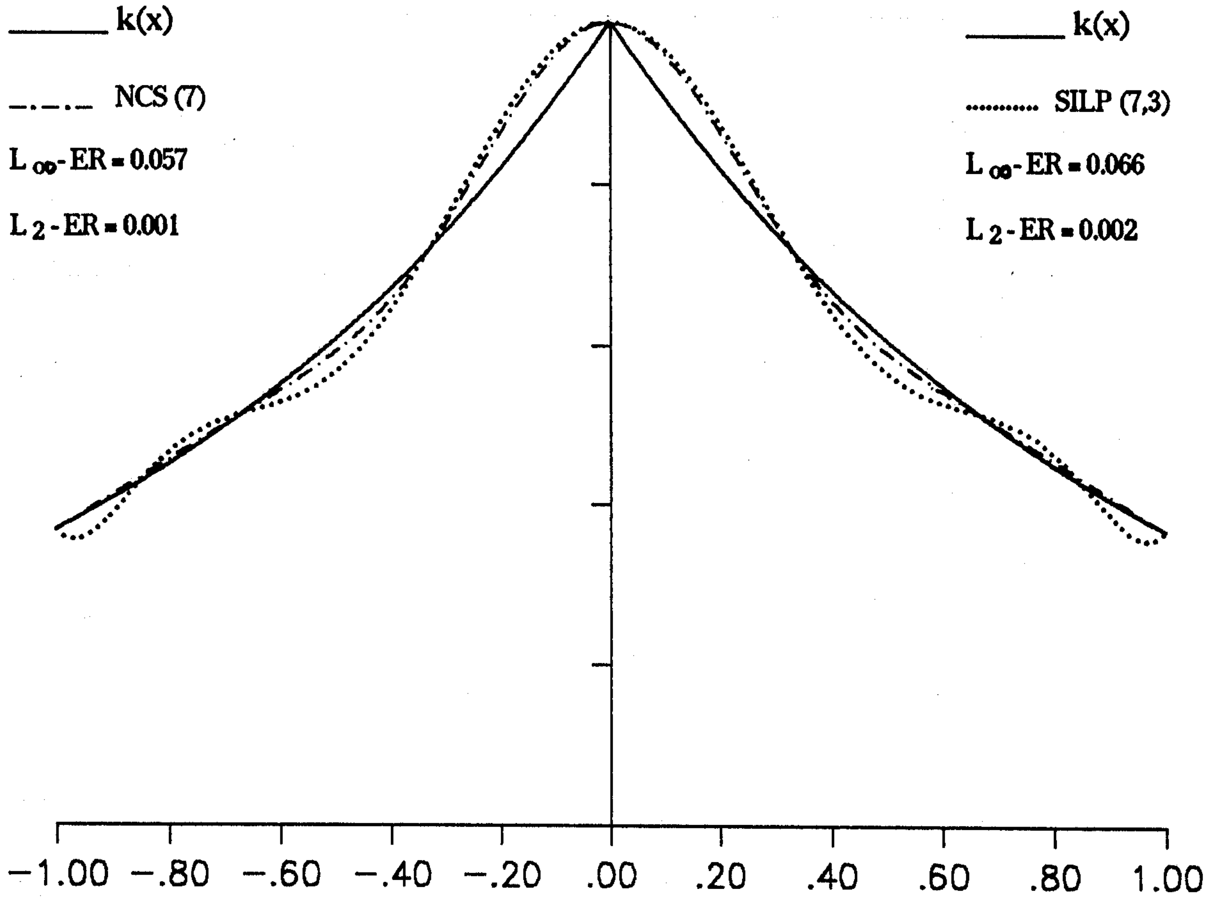


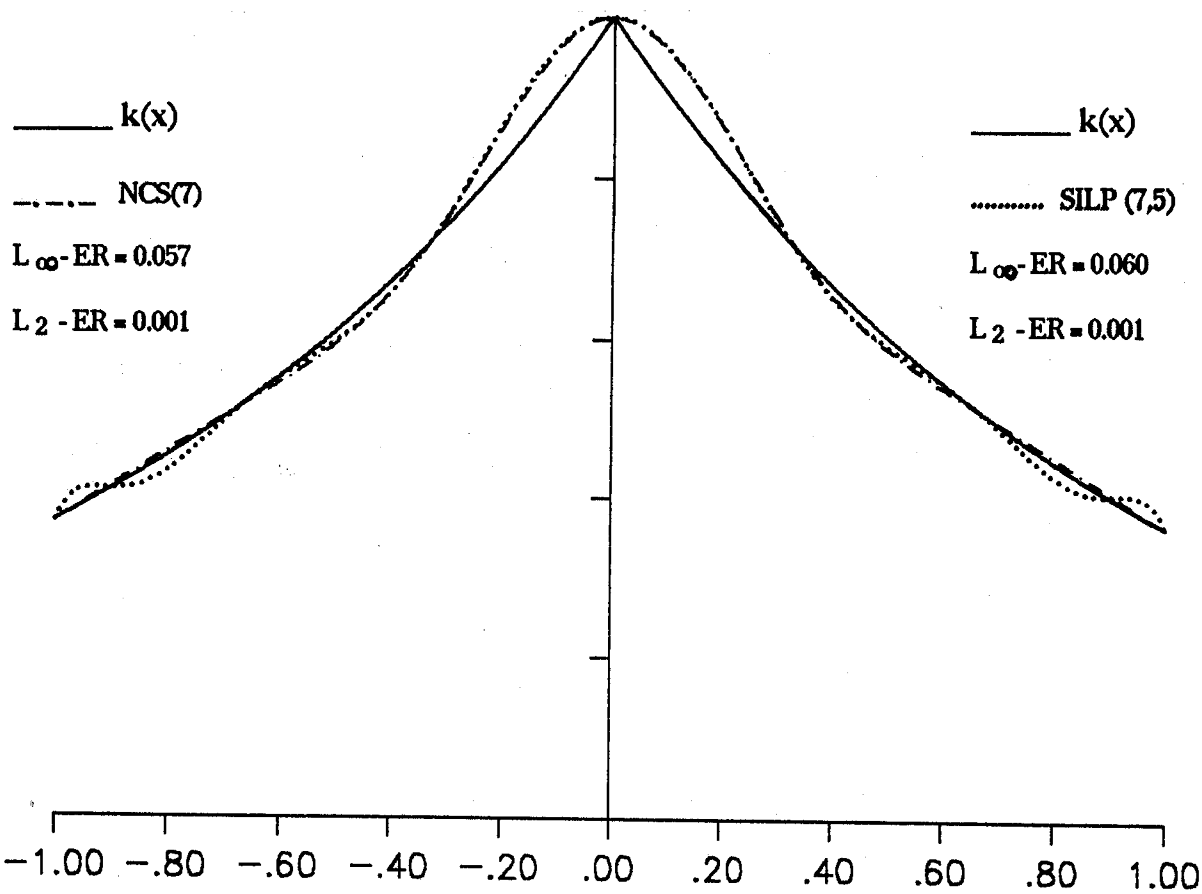
COMPARISON OF SILP vs NCS (5)
k(x) = Exp (-abs (x))





COMPARISON OF SILP vs NCS (7)
k(x) = Exp (-abs (x))





Acknowledgement. The authors acknowledge the excellent research and computer facilities availed at King Fahd University of Petroleum and Minerals during the preparation of this paper.

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