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ON THE STRUCTURE OF TC SEMIGROUPS

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Abstract

An algebra A satisfies TC (the term condition) if $p(a,\tilde{x}) = p(a,\tilde{y})$ iff $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a, b \in A$; $\tilde{x}, \tilde{y} \in A^n$ and any $n+1$–ary term p. TC algebras have been extensively studied. We determine the structure of all TC semigroups. We then specialize this theorem to obtain the structure of reversible TC semigroups $S(aS \cap bS \neq \square$ and $Sa \cap Sb \neq \square$ for all $a, b \in S)$. We build these semigroups from subsemigroups of the direct product $G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup. We then determine quasi–regular TC semigroups S ($a \in S$ implies a^n is a regular element of S for some positive integer n) as certain ideal extensions of $G \times I \times J$ by a zero semigroup. We specialize this result to obtain the structure of periodic TC semigroups. Finally, we show that a TC semigroup S has the congruence extension property (CEP) if and only if S is periodic (a semigroup S has CEP if for every subsemigroup T of S and congruence relation ρ on T, there is a congruence relation $\overline{\rho}$ on S such that $\overline{\rho} \cap (T \times T) = \rho$).

R. McKenzie [9] characterized semigroups of finite exponent obeying the term condition and posed the problem of characterizing all semigroups obeying the term condition. The term condition for algebras (TC) was introduced by McKenzie [8] and algebras obeying TC (also called abelian algebras) have been studied extensively (see, for example, [5] and [4]). An algebra A satisfies TC if $p(a,\tilde{x}) = p(a,\tilde{y})$ if and only if $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a, b \in A$, $\tilde{x}, \tilde{y} \in A^n$ and any $n+1$–ary term p. A semigroup satisfies TC if and only if $(c1) xy = xz$ implies $uy = uz$ $(c2) yx = zx$ implies $yu = zu (c3)\, y_1xy_2 = z_1xz_2$ implies $y_1uy_2 = z_1uz_2$. We first determine the structure of all TC semigroups (Theorem 1.8). We build TC semigroups from subsemigroups of the direct product $G \times I \times J$ where G is an abelian group, I is a left zero semigroup ($xy = x$ for all $x, y \in I$) and J is a right zero semigroup ($xy = y$ for all $x, y \in J$). Following [2], we term a semigroup S right (left) reversible if $Sa \cap Sb \neq \Box$ ($aS \cap bS \neq \Box$) for all $a, b \in S$. S is called reversible if it is both left reversible and right reversible. We may specialize Theorem 1.8 to give the structure of reversible TC semigroups (Theorem 2.1). Precisely, let G be an abelian group. Let V be a subsemigroup of G. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$. If $x \in X_v$, define $m(x) = v$. Let ϕ be a function of $V \times V \to X$ such that $\phi(m, p) \in X_{mp}$ and $\phi(mk, n) = \phi(m, kn)$. Let $[X, V, \phi]$ denote X under the multiplication, $xy = \phi(m(x), m(y))$. We show (Theorem 2.1) that S is a reversible TC semigroup if and only if $S \cong [X, V, \phi]$ for some X, V, and ϕ. A semigroup S is termed quasi-regular, if for every $a \in S$, there exists a positive integer m such that a^m is a regular element of S. Quasi-regular semigroups are a well known class of generalized regular semigroups (see, for example, [11] and reference theorem). We will determine the structure of quasi-regular TC semigroups (Theorem 3.8). Precisely, let G be an abelian group, I, J, and X be sets and ϕ, α, and β be functions of X into G, I, and J respectively. Define a binary relation o on the set $X \cup (G \times I \times J)$ by the following rule. For $a, b \in X$ and $(g, i, j), (h, k, s) \in G \times I \times J$, $aob = (a\phi b\phi, a\alpha, b\beta)$,$ao(g, i, j) = (a\phi g, a\alpha, j)$,$(g, i, j)oa = (g(a\phi), i, a\beta)$, and $(g, i, j)o(h, k, s) = (gh, i, s)$. We show (Theorem 3.8) that S is a quasi-regular TC semigroup if and only if S is isomorphic to some $(X \cup (G \times I \times J), o)$. We also show (Corollary 3.9) that S is a periodic TC semigroup if and only if S is isomorphic to some $(X \cup (G \times I \times J), o)$ with G periodic. A semigroup S has the congruence extension property (CEP) if for every subsemigroup T of S and congruence relation ρ on T, there is a congruence relation $\overline{\rho}$ on S such that $\overline{\rho} \cap (T \times T) = \rho$. The CEP

for algebras has been studied extensively (see, for example, [4]). The CEP for semigroups has been studied most recently by Garcia [3] and Jones [7] (see [3] and [7] for other references). We show (Theorem 4.2) that a TC semigroup S has the congruence extension property (CEP) if and only if S is periodic.

Detailed proofs of the results of section 1 and of Lemmas 3.2 – 3.5 and Corollary 3.6 will be published elsewhere. The remainder of the paper is published here in its entirety. For definition not given here, see [1], [2], [6], and [10].

1. *TC* Semigroups

In this section, we determine the structure of TC semigroups (Theorem 1.8).

The following "normality" condition for TC semigroups discussed by Taylor will be used in the proof of Lemma 1.2, Lemma 1.3, Lemma 1.5, and Lemma 3.5.

Lemma 1.1 (Taylor, [12, Lemma 4]). *A TC semigroup satisfies the identity $xyzw = xzyw$.*

Let R be a semigroup obeying the term condition (TC semigroup).

Define $x \equiv y (x, y \in R)$ iff $xz = yz$ and $zx = zy$ for all $z \in R$. It is easily checked that $\equiv$ is a congruence relation (this fact is valid for any semigroup). If $x \in R$, $\overline{x}$ will denote the $\equiv$ class containing x.

Let $T = R/\equiv$. For $x, y \in T$, define $x\rho y$ if $xT \cap yT \neq \Box$.

Lemma 1.2. *ρ is a congruence relation on T and T/ρ is a left zero semigroup.*

For $x, y \in T$, define $x\lambda y$ if $Tx \cap Ty \neq \Box$.

Lemma 1.3. *λ is a congruence relation on T and T/λ is a right zero semigroup.*

The following lemma will be needed in the proof of Lemmas 1.5 – 1.8.

Lemma 1.4. *T is a TC semigroup.*

Proof. (Sketch) Use the fact that for $x, y \in R$, $xy = x^2 = y^2$ iff $x \equiv y$ and the term condition.

Let $x_0 \in T$ and let $M = x_0 T x_0$. Define an operation $*$ on M as follows: $x_0 a x_0 * x_0 b x_0 = x_0 a b x_0$.

Lemma 1.5. *$(M, *)$ is a cancellative commutative semigroup.*

Lemma 1.6. *Let $V = \{(x_0 a x_0, a/\rho, a/\lambda) : a \in T\}$. Then, V is a subsemigroup of $(M,*) \times T/\rho \times T/\lambda$ and $a \to (x_0 a x_0, a/\rho, a/\lambda)$ defines an isomorphism of T onto V.*

Let $x\gamma = (x_0 \overline{x} x_0, \overline{x}/\rho, \overline{x}/\lambda)$ for $x \in R$. Then, γ defines a homomorphism of R onto V. Let $m(x) = x_0 \overline{x} x_0$, $i(x) = \overline{x}/\rho$, and $j(x) = \overline{x}/\lambda$.

Define

$$X_{(m,i,j)} = \{x \in R : x\gamma = (m, i, j)\}$$

$$L_{m,i} = \{x \in R : x\gamma = (m, i, j) \text{ for some } j\}$$

$$R_{m,j} = \{x \in R : x\gamma = (m, i, j) \text{ for some } i\}$$

Lemma 1.7. *For m, i, and j, the set $L_{m,i} \cdot R_{n,j}$ consists of precisely one element.*

Define $\phi_{ij}(m, n)$ to be the unique element of $L_{m,i} \cdot R_{n,j}$. Then, for $x, y \in R$,

$$xy = \phi_{i(x)j(y)}(m(x), m(y)).$$

Let

$$M_i = \{m : (m, i, j) \in V \text{ for some } j\}$$

$$N_j = \{m : (m, i, j) \in V \text{ for some } i\}$$

Then

$$\phi_{ij} : M_i \times N_j \to R$$

Construction of *TC* Semigroups

Let G be an abelian group, I be a left zero semigroup, and J be a right zero semigroup. Let V be a subsemigroup of $G \times I \times J$. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$. For $v = (m, i, j)$ and $x \in X_v$, $define$m(x) = m, $i(x) = i$, and $j(x) = j$.

For $(i, j) \in Pr_I V \times Pr_J V$, let ϕ_{ij} be a function from $M_i \times M_j \to X$ such that

1. $\phi_{ij}(m, n) \in X_{(mn,i,j)}$

2. $\phi_{ij}(mk, n) = \phi_{ij}(m, kn)$ for $k \in \cup M_i$

3. $\phi_{ij}(m, kn) = \phi_{ij}(p, kq)$ implies $\phi_{ij}(m, sn) = \phi_{ij}(p, sq)$ for $k, s \in \cup M_i$.

Let (X, V, ϕ) denote X under the multiplication

4. $xy = \phi_{i(x)j(y)}(m(x), m(y))$.

Theorem 1.8. *R is a TC semigroup if and only if $R \cong (X, V, \phi)$ for some X, V, and ϕ.*

Proof (Sketch). Let R be a TC semigroup. Let $X = \cup(X_{(m,i,j)} : (m, i, j) \in V)$. Note that $(M, *)$ (Lemma 1.5) may be embedded in an abelian group G. Apply Lemma 1.2 (to obtain I), Lemma 1.3 (to obtain J), Lemma 1.5, Lemma 1.6 and the comments below Lemma 1.6, Lemma 1.7 and the comments below Lemma 1.7, the associative law (used for (2)), and (c3) of the term condition (used for (3)). Conversely, using (1), (2), and (4), (X, V, ϕ) is a semigroup. Use (4) and (1) to establish (c1) and ($c2$) of the term condition and use (4) and (3) to establish ($c3$).

2. Reversible TC Semigroups

In this section, we determine the structures of reversible TC semigroups (Theorem 2.1)

Construction of Reversible TC Semigroups

Let G be an abelian group. Let V be a subsemigroup of G. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$.

If $x \in X_v$, define $m(x) = v$. Let ϕ be a function of $V \times V \to X$ such that

$$\phi(m,p) \in X_{mp} \tag{2.1}$$

$$\phi(mk,n) = \phi(m,kn) \tag{2.2}$$

Let $[X,V,\phi]$ denote X under the multiplication

$$xy = \phi(m(x),m(y)) \tag{2.3}$$

Theorem 2.1. *S is a reversible TC semigroup if and only if $S \cong [X,V,\phi]$ for some X,V, and ϕ.*

Proof. Assume R is a reversible TC semigroup. We first show $\theta : a \to x_0ax_0$ defines an isomorphism of T onto $(M,*)$. Clearly, θ defines a mapping of T onto M. Suppose $a\theta = b\theta$, i.e., $x_0ax_0 = x_0bx_0$. Let $x_0 = \overline{y}_0$, $a = \overline{a}_1$, and $b = \overline{b}_1$ for $y_0, a_1, b_1 \in R$. Since R is reversible there exists t_0, t_1, t_2, and $t_3 \in R$ such that $t_0a_1 = t_1b_1$ and $a_1t_2 = b_1t_3$. Hence, $\overline{t}_0\overline{a}_1 = \overline{t}_1\overline{b}_1$ and $\overline{a}_1\overline{t}_2 = \overline{b}_1\overline{t}_3$. Furthermore, $\overline{y}_0\overline{a}_1\overline{y}_0 = \overline{y}_0\overline{b}_1\overline{y}_0$. Using Lemma 1.4, $\overline{t}_0\overline{a}_1\overline{t}_0 = \overline{t}_0\overline{b}_1\overline{t}_0$. Hence, $\overline{t}_1\overline{b}_1\overline{t}_0 = \overline{t}_0\overline{b}_1\overline{t}_0$. Thus, again using Lemma 1.4, $\overline{t}_1\overline{a}_1 = \overline{t}_0\overline{a}_1$. Hence, $\overline{t}_1\overline{a}_1 = \overline{t_1}\overline{b}_1$. Likewise, $\overline{a}_1\overline{t}_2 = \overline{b}_1\overline{t}_2$. Consequently, $mt_1a_1 = mt_1b_1$ and $a_1t_2m = b_1t_2m$ for all $m \in R$. Thus, using the term condition, $\overline{a}_1 = \overline{b}_1$, that is, $a = b$. Hence, θ is injective. Let $a, b \in T$. Thus, $a\theta * b\theta = x_0ax_0 * x_0bx_0 = x_0abx_0 = (ab)\theta$. So, θ is an isomorphism. Alternatively, we could have used the fact that R is reversible to show $|T/\rho| =, |T/\lambda| = 1$ and then used Lemma 1.6 to show θ defined an isomorphism of T onto $(M, *)$. Let $m(x) = x_0\overline{x}x$ for $x \in R$. Then, m defines a homomorphism of R onto $(M, *)$. Define $X_n = \{x \in R : m(x) = n\}$. Using Lemma 1.7, X_nX_p consists of a single element for any $n, p \in M$. To prove this result directly, let $a, c \in X_n$ and $b, d \in X_p$. Thus, $m(ab) = m(ad)$. So, $x_0\overline{ab}x_0 = x_0\overline{ad}x_0$. So, $\theta(\overline{ab}) = \theta(\overline{ad})$. Hence, $\overline{ab} = \overline{ad}$. Thus, $mab = mad$ for all $m \in R$. Hence, using the term condition, $ab = ad$. Similarly, $ad = cd$. Thus, $ab = cd$. We define $\phi(n, p)$ to be the unique element of $X_n \cdot X_p$. Let $V = (M, *)$ and let G be an abelian group in which $(M, *)$ is embedded. Thus, ϕ is a function of $V \times V$ into $X = \cup(X_n : n \in V)$. For $x, y \in R$, we have $xy = \phi(m(x), m(y))$. Since $X_nX_p \subseteq X_{np}, \phi(n,p) \in X_{np}$. Let $p, k, n \in V$. Let $x \in X_p, y \in X_k$, and $z \in X_n$. Thus, $\phi(pk, n) = \phi(m(x)m(y), m(z)) = \phi(m(xy), m(z)) = (xy)z = x(yz) = \phi(m(x), m(y)m(z)) = \phi(p, kn)$. Conversely, we show that $[X, V, \phi]$ is a reversible TC semigroup. Using (2.1) and (2.3), $[X, V, \phi]$ is a groupoid. Let $x \in X_n$ and $y \in X_k$. Using (2.3) and (2.1), $xy = \phi(m(x), m(y)) = \phi(n, k) \in X_{nk}$. So, $X_nX_k \subseteq X_{nk}$. Thus, $m(xy) = nk = m(x)m(y)$. So, m is a homomorphism of $[X, V, \phi]$ onto V. Let $x, y, z \in X$. Thus, using (2.3) and (2.2), $(xy)z = \phi(m(xy), m(z)) = \phi(m(x)m(y), m(z)) = \phi(m(x), m(y)m(z)) = x(yz)$. Hence, $[X, V, \phi]$ is a semigroup. We next establish the term condition. Let $xy = xz(x, y, z \in X)$. We will show $uy = uz$ $(u \in X)$. Using (2.3), $\phi(m(x), m(y)) = xy = xz = \phi(m(x), m(z))$. Thus, by (2.1), $m(x)m(y) = m(x)m(z)$. Hence, $m(y) = m(z)$. Thus, $uy = \phi(m(u), m(y)) = \phi(m(u), m(z)) = uz$. Similarly, $yx = zx$ implies $yu = zu$. Finally, we show $y_1xy_2 = z_1xz_2$ implies $y_1uy_2 = z_1uz_2$. Since $y_1(xy_2) = z_1(xz_2)$, $\phi(m(y_1), m(x)m(y_2)) = \phi(m(z_1), m(x)m(z_2))$. Thus, using (2.1), $m(y_1)m(x)m(y_2) = m((z_1)m(x)m(z_2)$. So, $m(y_1)m(y_2) = m(z_1)m(z_2)$. Thus, using (2.3) and (2.2), $y_1(uy_2) = \phi(m(y_1), m(uy_2)) = \phi(m(y_1), m(u)m(y_2)) = \phi(m(y_1), m(y_2)m(u)) = \phi(m(y_1)m(y_2), m(u)) = \phi(m(z_1)m(z_2), m(u)) = \phi(m(z_1), m(z_2)m(u)) = \phi(m(z_1), m(u)m(z_2)) = z_1(uz_2)$. Next, we show that $[X, V, \phi]$ is a reversible semigroup. Let $x, y, t \in X$. Then, $x(yt) = (xy)t = \phi(m(x)m(y), m(t)) = \phi(m(y)m(x), m(t)) = (yx)t = y(xt)$. Hence, $[X, V, \phi]$ is left reversible. Similarly, $[X, V, \phi]$ is right reversible.

3. Quasi-regular TC Semigroups

In this section, we determine the structure of quasi–regular TC semigroups (Theorem 3.8) and, as a corollary, we determine the structure of periodic TC semigroups (Corollary 3.9).

Let S be a semigroup. S is called a quasi–regular semigroup, if for every $a \in S$, there is a positive integer m such that a^m is a regular element of S ($a \in aSa$). An element $a \in S$ is said to be of finite order if there exists a positive integer $n(a)$ such that $a^{n(a)}$ is an idempotent. S is termed periodic if every element of S is of finite order. Let S_{Reg} denote the set for regular element of S. Let $E(S)$ denote the set of idempotents of S.

Let S denote a TC semigroup.

Lemma 3.1. *If $x, y \in S$ and $e \in E(S)$, then $xey = xy$.*

Proof. Apply TC to $e(ey) = ey$.

Lemma 3.2. *If $E(S) \neq \Box$, $E(S)$ is a rectangular band and $E(S) \cong E(S)e \times eE(S)$ for any fixed $e \in E(S)$. Moreover, $E(S)e$ is a left zero semigroup and $eE(S)$ is a right zero semigroup.*

Proof (Sketch). Use Lemma 3.1. The required isomorphism is given by $x\phi =$

(xe, ex).

Lemma 3.3. *Every regular element of S is contained in a subgroup of S.*

Proof (Sketch). If $axa = a$, let $g = axxa$. Then, $a \in H_g$, the group of units of gSg.

Lemma 3.4. *S_{Reg} is a regular semigroup or $S_{Reg} = \Box$.*

Lemma 3.5. *$S_{Reg} \cong G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup or $S_{Reg} = \Box$.*

Proof (Sketch). Fix $e \in E(S)$, let $I = E(S)e$, $J = eE(S)$, and $G = H_e$. Every element of S_{Reg} may be uniquely expressed in the form igj $(i \in I, g \in G, j \in J)$. The required isomorphism is $igj \to (i, g, j)$. G is abelian by Lemma 1.1. Lemmas 3.1–3.4 are utilized.

Corollary 3.6. *A regular semigroup S obeys the term condition (TC) if and only if $S \cong G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup.*

Lemma 3.7. *Let S be a quasi-regular TC semigroup. Then if $f \in E(S)$ and $a \in S$, $fa, af \in S_{Reg}$.*

Proof. There exists a positive integer r such that $(fa)^r \in S_{\text{Reg}}$. So, $(fa)^r = (g, i, j) \in G \times I \times J$ by Lemma 3.5. Using Lemma 3.1, $(fa)^r = fa^r$. This, $fa^r = (g, i, j)$. Let $f = (e, k, \ell)$ where e is the identity of G. Hence, $fa^r(g^{-1}, k, \ell) = f(fa^r)(g^{-1}, k, \ell) = (e, k, \ell)(g, i, j)(g^{-1}, k, \ell) = (e, k, \ell)(e, i, \ell) = (e, k, \ell)$. So, $fa^r(g^{-1}, k, \ell) = f$. Thus, $fa(a^{r-1}(g^{-1}, k, \ell))fa = f(fa) = fa$. Hence, $fa \in S_{\text{Reg}}$. (Note, if $r = 1$, $fa = (g, i, j) \in S_{\text{Reg}}$). Similarly, $af \in S_{\text{Reg}}$.

Let S be a quasi-regular TC semigroup. Let $X = S - S_{\text{Reg}}$. Let $a, b \in X$ and

$(e,i,j) \in E(S)$. Using Lemma 3.5 and Lemma 3.7,

$$a(e,i,j) = (a',i',j')(a' \in G,\ i' \in I,\ j' \in J)$$

Hence,

$$a(e,i,j) = (a(e,i,j))(e,i,j)) = (a',i',j')(e,i,j) = (a',i',j)$$

Suppose

$$a(e,i,s) = (a'',i'',s).$$

Then,

$$a(e,i,j) = (a(e,i,s))(e,i,j) = (a'',i'',s)(e,i,j) = (a'',i'',j)$$

Hence,

$$a' = a'' \quad \text{and} \quad i' = i''.$$

Thus, we may write

$$a(e,i,j) = (a\phi_i, i\rho_a, j)$$

where ϕ_i is a mapping of X into G (depending on i) and ρ_a is a mapping of I into I.

Similarly,

$$(e,i,j)a = (a\psi_j, i, j\lambda_a)$$

where ψ_j is a mapping of X into G and λ_a is a mapping of J into J. Hence,

$$\begin{aligned} a(g,i,j) &= (a(e,i,j))(g,i,j) \\ &= (a\phi_i, i\rho_a, j)(g,i,j) \\ &= (a\phi_i g, i\rho_a, j). \end{aligned}$$

Similarly,

$$(g,i,j)a = (g(a\psi_j),\ i, j\lambda_a).$$

However,

$$((e,i,j)a)(e,k,s) = (a\psi_j, i, j\lambda_a)(e,k,s) = (a\psi_j, i, s)$$

and

$$(e,i,j)(a(e,k,s)) = (e,i,j)(a\phi_k, k\rho_a, s) = (a\phi_k, i, s)$$

So, $\psi_j = \phi_k$ for every $j \in J$ and $k \in I$. Put $\phi = \psi_j = \phi_k$. Using Lemma 3.1.,

$$aa = (a(e,i,j))a = (a\phi, i\rho_a, j)a = (a\phi a\phi, i\rho_a, j\lambda_a).$$

Thus, ρ_a and λ_a are constant mappings: Let $a\alpha = i\rho_a$ and $a\beta = j\lambda_a$ for $a \in X$, $i \in I$, and $j \in J$. Hence, α and β define functions of X into I and J respectively.

Furthermore,

$$ab = (a(e,i,j))b = (a\phi, a\alpha, j)b = (a\phi b\phi, a\alpha, b\beta).$$

Construction of Quasi–regular TC Semigroups

Let G be an abelian group, I, J, and X be sets and ϕ, α, and β functions of X into G, I, and J respectively. Define a binary operation o on the set $(G \times I \times J) \cup X$ by the following rule.

For $a, b \in X$ and $(g,i,j), (h,k,s) \in G \times I \times J$,

$$\begin{aligned} aob &= (a\phi b\phi, a\alpha, b\beta) \\ ao(g,i,j) &= (a\phi g, a\alpha, j) \\ (g,i,j)oa &= (g(a\phi), i, a\beta) \\ (g,i,j)o(h,k,s) &= (gh, i, s) \end{aligned}$$

Theorem 3.8 *S is a quasi–regular TC semigroup if and only if S is isomorphic to some $(X \cup (G \times I \times J), o)$.*

Proof. To prove the converse, let $S = (X \cup (G \times I \times J), o)$. By a routine calculation, S is a semigroup. We next establish the term condition. Note that $aob = (a\phi, a\alpha, a\beta)o(b\phi, b\alpha, b\beta)$;$ao(g,i,j) = (a\phi, a\alpha, a\beta)o(g,i,j)$; and $(g,i,j)oa = (g,i,j)o(a\phi, a\alpha, a\beta)$. If $a \in X$, let $\overline{a} = (a\phi, a\alpha, a\beta)$. If $(g,i,j) \in G \times I \times J$, let $\overline{(g,i,j)} = (g,i,j)$. Then, $p(u, x_1, x_2, \ldots, x_n) = p(\overline{u}, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ for every term p. Since (as is easily seen), G, I, and J obey TC, $G \times I \times J$ obeys TC. Thus, S obeys TC.

Corollary 3.9. *S is a periodic TC semigroup if and only if S is isomorphic to some $(X \cup (G \times I \times J), o)$ with G periodic.*

Proof. Let S be a periodic TC semigroup. Hence, S is a quasi-regular TC semigroup. Thus, S is isomorphic to some $(X \cup (G \times I \times J), o)$ by Theorem 3.8. Since $g \rightarrow (g, i_0, j_0)\,(i_0 \in I, j_0 \in J$, fixed) defines an isomorphism of G onto $G \times \{i_0\} \times \{j_0\}$, G is a periodic group. Conversely, suppose S is isomorphic to some $(X \cup (G \times I \times J), o)$ with G periodic. So, S is a TC semigroup by Theorem 3.8. Since $a^k = ((a\phi)^k, a\alpha, a\beta)$ and $(g,i,j)^k = (g^k, i, j)$ for any positive integer $k \geq 2$ and G is a periodic group, S is a periodic semigroup.

4. *TC* Semigroups and the Congruence Extension Property

In this section, we show that a TC semigroup has the congruence extension property if and only if it is periodic (Theorem 4.2). We will first need the following lemma.

Lemma 4.1 *Let $S = G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup. Let ρ be a congruence relation on S. Then, there exists a subgroup N_ρ of G, an equivalence relation ρ_I on I, and an equivalence relation ρ_J on J such that $(g,i,j)\rho(h,r,s)$ iff $gh^{-1} \in N_\rho, (i,r) \in \rho_I$, and*

$(j,s) \in \rho_J$. *Conversely, let* N *be a subgroup of* G*, let* δ *be an equivalence relation on* I*, and let* λ *be an equivalence relation on* J*. Then,* $\rho = (((g,i,j),(h,r,s)) \in S \times S : gh^{-1} \in N, (i,r) \in \delta$*, and* $(j,s) \in \lambda)$ *is a congruence relation on* S*. Furthermore,* $N = N_\rho, \delta = \rho_I$*, and* $\lambda = \rho_J$*.*

Proof. Let ρ be a congruence relation on S. Fix $i_0 \in I$ and $j_0 \in J$. Let $N_\rho = \{a \in G : (a, i_0, j_0)\rho(e, i_0, j_0)$ where e is the identity of $G\}$. Then, N_ρ is a subgroup of G. Define $(i,j) \in \rho_I$ iff $(e,i,t)\rho(e,j,t)$ for all $t \in J$ and $(k,r) \in \rho_J$ iff $(e,s,k)\rho(e,s,r)$ for all $s \in I$. Then, ρ_I and ρ_J are equivalence relations on I and J respectively. Let $(g,i,j)\rho(h,r,s.)$ Thus, $(g,i_0,j_0) = (e,i_0,j_0)(g,i,j)(e,i_0,j_0)\rho(e,i_0,j_0)(h,r,s,)(e,i_0, j_0) = (h,i_0,j_0)$. Thus $(gh^{-1},i_0,j_0)\rho(e,i_0,j_0)$. Hence, $gh^{-1} \in N_\rho$. Since $h^{-1}(g^{-1})^{-1} \in N_\rho$, $(h^{-1}(g^{-1})^{-1}, i_0, j_0)\ \rho(e,i_0,j_0)$ and, thus, $(g^{-1},i_0,j_0)\rho(h^{-1},i_0,j_0)$. Thus, (e,i,j_0) $\rho(e,r,j_0)$ and, hence, $(e,i,t)\rho(e,r,t)$ for all $t \in J$ or $(i,r) \in \rho_I$. Similarly, $(j,s) \in \rho_J$. Suppose $gh^{-1} \in N_\rho, (i,r) \in \rho_I$, and $(j,s) \in \rho_J$. Thus, $(g,i,j) = (e,i,j_0)(g,i_0,j_0)$ $(e,i_0,j)\rho(e,r,j_0)\,(h,i_0,j_0)(e,i_0,s) = (h,r,s)$. Conversely, let $\rho = ((g,i,j),(h,r,s)) \in S \times S : gh^{-1} \in N, (i,r) \in \delta$, and $(j,s) \in \lambda)$. It is routine to verify that ρ is a congruence relation. Furthermore, $a \in N_\rho$ iff $(a,i_0,j_0)\rho(e,i_0,j_0)$ iff $a \in N$. Thus, $N = N_\rho$. In addition, $(i,r) \in \rho_I$ iff $(e,i,t)\rho(e,r,t)$ for all $t \in J$ iff $(i,r) \in \delta$. Thus, $\delta = \rho_I$. Similarly, $\lambda = \rho_J$.

Theorem 4.2. *Let* S *be a* TC *semigroup. Then,* S *has the congruence extension property* (CEP) *if and only if* S *is periodic.*

Proof. Let S be a periodic TC semigroup. The structure of S is given by Corollary 3.9. We will use this corollary and its notation without explicit mention. Let T be a subsemigroup of S. Let $X' = T \cap X$ and $R = T \cap (G \times I \times J)$. If $a \in T \cap X$, then $a^2 \in T \cap (G \times I \times J)$. Thus, $R \neq \phi$, and, hence, R is a subsemigroup of $G \times I \times J$. Hence, using [3, Lemma 6.2], $R = H \times A \times B$ where H is a subgroup of $G, A \subseteq I$, and $B \subseteq J$. Let ρ be a congruence relation on T. Let $\rho^* = \rho \cap (R \times R)$.

Then, ρ^* is a congruence relation on R. Thus, using Lemma 4.1 and its notation, $\rho^* = ((g,i,j),(h,k,\ell)) \in R \times R : gh^{-1} \in N_{\rho^*}, (i,k) \in \rho_A^*$, and $(j,\ell) \in \rho_\beta^*$ where N_{ρ^*} is a subgroup of H, ρ_A^* is an equivalence relation on A, and ρ_B^* is an equivalence relation on B). Define $\rho' = ((g,i,j),(h,k,\ell)) \in (G \times I \times J) \times G \times I \times J : gh^{-1} \in N_{\rho^*}, (i,k) \in \rho_A^* \cup \Delta_I$, and $(j,\ell) \in \rho_B^* \cup \Delta_J)$. Thus, by Lemma 4.1, ρ' is a congruence relation on $G \times I \times J$. It is easily seen that $\rho' \cap (R \times R) = \rho^*$. Let $\overline{\rho} = \rho' \cup \rho \cup \Delta_{X-X'}$. We will show that $\overline{\rho}$ is a congruence relation on S and $\overline{\rho} \cap (T \times T) = \rho$. We first show $\overline{\rho}$ is an equivalence relation. It is easily checked that $\overline{\rho}$ is reflexive and symmetric. We next establish transitivity. Suppose $(u,v) \in \rho'$ and $(v,w) \in \rho$. Let $u = (g,i,j)$ and $v = (h,k,\ell)$. Thus, $gh^{-1} \in N_{\rho^*} \subseteq H$ implies $g \in H$, $k \in A$ implies $i \in A$, and $\ell \in B$ implies $j \in B$. Hence, $(g,i,j) \in H \times A \times B$. Thus, since $\rho' \cap (R \times R) = \rho^* \subseteq \rho$,$((g,i,j),(h,k,\ell)) \in \rho$ or $(u,v) \in \rho$. Hence, $(u,w) \in \rho \subseteq \overline{\rho}$. The other cases are obvious or dual to the above. We will now show that $\overline{\rho}$ is compatible. Let $((g,i,j),(h,k,\ell)) \in \rho'$ and let $a \in X$. Thus, $a(g,i,j) = (a\phi g, a\alpha, j)$ and $a(h,k,\ell) = (a\phi h, a\alpha, \ell)$. Since $a\phi gh^{-1}(a\phi)^{-1} = gh^{-1} \in N_{\rho^*}$,$(a\alpha, a\alpha) \in \rho_A^* \cup \Delta_I$,$(j,\ell) \in \rho_B^* \cup \Delta_J$,$(a(g,i,j), a(h,k,\ell)) \in \rho' \subseteq \overline{\rho}$. Similarly, $((g,i,j)a,(h,k,\ell)a) \in \rho' \subseteq \overline{\rho}$. Let $(u,v) \in \rho \cap (X' \times X')$. Thus, if $(e,i,j) \in H \times A \times B$ where e is the identity of G, $((e,i,j)u,(e,i,j)v) \in \rho$. Hence, $((u\phi, i, u\beta),(v\phi, i, v\beta)) \in \rho^*$. Thus, $u\phi(v\phi)^{-1} \in N_{\rho^*}$, and $(u\beta, v\beta) \in \rho_B^*$. Similarly, $(u(e,i,j), v(e,i,j)) \in \rho$ implies $(u\alpha, v\alpha) \in \rho_A^*$. Let $a \in X$. Thus, $au = (a\phi u\phi, a\alpha, u\beta)$ and $av = (a\phi v\phi, a\alpha, v\beta)$. However, $a\phi u\phi(v\phi)^{-1}(a\phi)^{-1} = u\phi(v\phi)^{-1} \in N_{\rho^*}$, $(a\alpha, a\alpha) \in \rho_A^* \cup \Delta_I$ and $(u\beta, v\beta) \in \rho_B^* \cup \Delta_J$. Thus, $(au, av) \in \rho' \subseteq \overline{\rho}$. Similarly, $(ua, va) \in \rho' \subseteq \overline{\rho}$, and,, for $(g,i,j) \in G \times I \times J$,$((g,i,j)u,(g,i,j)v) \in \rho' \subseteq \overline{\rho}$, and $(u(g,i,j), v(g,i,j)) \in \rho' \subseteq \overline{\rho}$. Next, suppose $(u,v) \in \rho \cap (R \times R)$. Thus, $(u,v) \in \rho^* \subseteq \rho'$. Thus, $(tu, tv) \in \rho' \subseteq \overline{\rho}$ and $(ut, vt) \in \rho' \subseteq \overline{\rho}$ for all $t \in S$. Now, suppose $(a,(g,i,j)) \in \rho$ where $a \in X'$ and $(g,i,j) \in H \times A \times B$. Let $(e,i,j) \in H \times A \times B$ where e is the identity of G.

Hence, $(a(e,i,j),(g,i,j)) \in \rho$. Thus, $((a\phi, a\alpha, j),(g,i,j)) \in \rho \cap (R \times R) = \rho^*$. So, $a\phi g^{-1} \in N_{\rho^*}$ and $(a\alpha, i) \in \rho_A^*$. Similarly, $((e,i,j)a,(g,i,j)) \in \rho$ implies $(a\beta, j) \in \rho_B^*$. Let $b \in X$. Thus, $ba = (b\phi a\phi, b\alpha, a\beta)$ and $b(g,i,j) = (b\phi g, b\alpha, j)$. However, $b\phi a\phi g^{-1}(b\phi)^{-1} = a\phi g^{-1} \in N_{\rho^*}, (b\alpha, b\alpha) \in \rho_A^* \cup \Delta_I$, and $(a\beta, j) \in \rho_B^* \cup \Delta_J$. Thus, $(ba, b(g,i,j)) \in \rho' \subseteq \overline{\rho}$. Similarly, $(ab,(g,i,j)b) \in \rho' \subseteq \overline{\rho}$, and, for $(h,k,\ell) \in G \times I \times J$, $((h,k,\ell)a,(h,k,\ell)(g,i,j)) \in \rho' \subseteq \overline{\rho}$ and $(a(h,k,\ell),(g,i,j)(h,k,\ell)) \in \rho' \subseteq \overline{\rho}$. The case $((g,i,j),a) \in \rho$ where $(g,i,j) \in H \times A \times B$ and $a \in X'$ is valid by symmetry. The case $(u,v) \in \Delta_{X-X'}$ is valid by reflexivity of $\overline{\rho}$. Finally, we show $\overline{\rho} \cap (T \times T) = \rho$. Let $(u,v) \in \overline{\rho} \cap (T \times T)$. If $(u,v) \in \rho'$, then $(u,v) \in \rho' \cap ((H \times A \times B) \times (H \times A \times B)) = \rho^* \subseteq \rho$. Thus, it easily follows that $\overline{\rho} \cap (T \times T) \subseteq \rho$. Clearly, $\rho \subseteq \overline{\rho} \cap (T \times T)$ and, hence, $\overline{\rho} \cap (T \times T) = \rho$. Thus, S has the congruence extension property (CEP). Conversely, suppose a TC semigroup S has CEP. Then, S is periodic by [3, Corollary 3.2].

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