King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 138

April 1993

TC Semigroups and Related Semigroups

R.J. Warne

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

TC SEMIGROUPS AND RELATED SEMIGROUPS

R.J. WARNE
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Abstract

An algebra A satisfies TC (the term condition) if $p(a,\tilde{x}) = p(a,\tilde{y})$ iff $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a,b \in A$, $,\tilde{x},\tilde{y} \in A^n$ and any $n+1$–ary term p. TC algebras have been extensively studied. We determine the structure of all TC semigroups and we give a more detailed structure theorem for periodic TC semigroups. We build TC semigroups from subsemigroups of the direct product $G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup. Let S be a semigroup such that $E(S)$, the set of idempotents of S, is a subsemigroup. S is termed a quasi–orthodox semigroup of $a \in S(b \in S)$ implies there exists $e \in E(S)(f \in E(S))$ such that $ax = ay(xb = yb)$ iff $ex = ey(xf = yf)$ for $x,y \in S^1$ (S with appended identity). We determine the structure of quasi-orthodox semigroups which are bands of cancellative monoids.

R. McKenzie [12] characterized semigroups of finite exponent obeying the term condition and posed the problem of characterizing all semigroups obeying the term condition. The term condition for algebras (TC) was introduced by McKenzie [11] and algebras obeying TC (also called abelian algebras) have been studied extensively (see, for example, [8] and [7]). An algebra A satisfies TC if $p(a,\tilde{x}) = p(a,\tilde{y})$ if and only if $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a,b \in A$, $\tilde{x},\tilde{y} \in A^n$ and any $n+1$–ary term p. A semigroup satisfies TC if and only if $(c1) xy = xz$ implies $uy = uz$ $(c2) yx = zx$ implies $yu = zu (c3)\, y_1xy_2 = z_1xz_2$ implies $y_1uy_2 = z_1uz_2$. We first determine the structure of all semigroups obeying the term condition (Theorem 1.8) and we give a more detailed structure theorem for periodic TC semigroups (Theorem 2.9). We build TC semigroups from subsemigroups of the direct product $G \times I \times J$ where G is an abelian group, I is a left zero semigroup ($xy = x$ for all $x, y \in I$) and J is a right zero semigroup ($xy = y$ for all $x, y \in J$). Finally, we determine the structure of a related class of semigroups. Let S be a semigroup such that $E(S)$, the set of idempotents of S, is a subsemigroup. S is termed a quasi-orthodox semigroup if $a \in S(b \in S)$ implies there exists $e \in E(S)(f \in E(S))$ such that $ax = ay(xb = yb)$ iff $ex = ey(xf = yf)$ for $x, y \in S^1$ (S with appended identity). We determine the structure of quasi-orthodox semigroups which are bands of cancellative monoids (Theorem 3.4). Theorem 3.4 generalizes corresponding theorems of Yamada [17] and El-Qallali [5]. For definitions not given here see [1], [2], [9], and [13]. A quasi-orthodox semigroup is termed a quasi-adequate semigroup in [4].

Detailed proofs of the results of section 1 and 2 will be published elsewhere. Section 3 is published here in its entirety.

1. *TC* Semigroups

In this section, we determine the structure of TC semigroups (Theorem 1.4).

The following "normality" condition for TC semigroups discussed by Taylor will be used in the proof of Lemma 1.2, Lemma 1.3, Lemma 1.5, and Lemma 2.5.

Lemma 1.1 (Taylor, [14, Lemma 4]). *A TC semigroup satisfies the identity* $xyzw = xzyw$.

Let R be a semigroup obeying the term condition (TC semigroup).

Define $x \equiv y (x, y \in R)$ iff $xz = yz$ and $zx = zy$ for all $z \in R$. It is easily checked that $\equiv$ is a congruence relation (this fact is valid for any semigroup). If $x \in R$, $\overline{x}$ will denote the $\equiv$ class containing x.

Let $T = R/\equiv$. For $x, y \in T$, define $x\rho y$ if $xT \cap yT \neq \Box$.

Lemma 1.2. *ρ is a congruence relation on T and T/ρ is a left zero semigroup.*

For $x, y \in T$, define $x\lambda y$ if $Tx \cap Ty \neq \Box$.

Lemma 1.3. *λ is a congruence relation on T and T/λ is a right zero semigroup.*

The following lemma will be needed in the proof of Lemmas 1.5 – 1.8.

Lemma 1.4. *T is a TC semigroup.*

Proof. (Sketch) Use the fact that for $x, y \in R$, $xy = x^2 = y^2$ iff $x \equiv y$ and the term condition.

Let $x_0 \in T$ and let $M = x_0 T x_0$. Define an operation $*$ on M as follows: $x_0 a x_0 * x_0 b x_0 = x_0 a b x_0$.

Lemma 1.5. *$(M, *)$ is a cancellative commutative semigroup.*

Lemma 1.6. *Let $V = \{(x_0 a x_0, a/\rho, a/\lambda) : a \in T\}$. Then, V is a subsemigroup of $(M, *) \times T/\rho \times T/\lambda$ and $a \to (x_0 a x_0, a/\rho, a/\lambda)$ defines an isomorphism of T onto V.*

Let $x\gamma = (x_0 \overline{x} x_0, \overline{x}/\rho, \overline{x}/\lambda)$ for $x \in R$. Then, γ defines a homomorphism of R onto V. Let $m(x) = x_0 \overline{x} x_0$, $i(x) = \overline{x}/\rho$, and $j(x) = \overline{x}/\lambda$.

Define

$$X_{(m,i,j)} = \{x \in R : x\gamma = (m, i, j)\}$$

$$L_{m,i} = \{x \in R : x\gamma = (m, i, j) \text{ for some } j\}$$

$$R_{m,j} = \{x \in R : x\gamma = (m, i, j) \text{ for some } i\}$$

Lemma 1.7. *For m, i, and j, the set $L_{m,i} \cdot R_{n,j}$ consists of precisely one element.*

Define $\phi_{ij}(m,n)$ to be the unique element of $L_{m,i} \cdot R_{n,j}$. Then, for $x, y \in R$

$$xy = \phi_{i(x)j(y)}(m(x), m(y)).$$

Let

$$M_i = \{m : (m,i,j) \in V \text{ for some } j\}$$

$$N_j = \{m : (m,i,j) \in V \text{ for some } i\}$$

Then

$$\phi_{ij} : M_i \times N_j \to R$$

Construction of TC Semigroups

Let G be an abelian group, I be a left zero semigroup, and J be a right zero semigroup. Let V be a subsemigroup of $G \times I \times J$. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$. For $v = (m,i,j)$ and $x \in X_v$, define $m(x) = m$, $i(x) = i$, and $j(x) = j$.

For $(i,j) \in Pr_I V \times Pr_J V$, let ϕ_{ij} be a function from $M_i \times M_j \to X$ such that

1. $\phi_{ij}(m,n) \in X_{(mn,i,j)}$

2. $\phi_{ij}(mk,n) = \phi_{ij}(m,kn)$ for $k \in \cup M_i$

3. $\phi_{ij}(m,kn) = \phi_{ij}(p,kq)$ implies $\phi_{ij}(m,sn) = \phi_{ij}(p,sq)$ for $k, s \in \cup M_i$.

Let (X, V, ϕ) denote X under the multiplication

4. $xy = \phi_{i(x)j(y)}(m(x), m(y))$.

Theorem 1.8. *R is a TC semigroup if and only if $R \cong (X, V, \phi)$ for some X, V, and ϕ.*

Proof (Sketch). Let R be a TC semigroup. Let $X = \cup(X_{(m,i,j)} : (m,i,j) \in V)$. Note that $(M, *)$ (Lemma 1.5) may be embedded in an abelian group G. Apply Lemma 1.2 (to obtain I), Lemma 1.3 (to obtain J), Lemma 1.5, Lemma 1.6 and the comments below Lemma 1.6, Lemma 1.7 and the comments below Lemma 1.7, the associative law (used for (2)), and ($c3$) of the term condition (used for (3)). Conversely, using (1), (2), and (4), (X, V, ϕ) is a semigroup. Use (4) and (1) to establish ($c1$) and ($c2$) of the term condition and use (4) and (3) to establish ($c3$).

2. Periodic *TC* Semigroups

In this section, we determine the structure of periodic TC semigroups (Theorem 2.9).

Let S be a semigroup. An element $a \in S$ is said to be of finite order if there exists a positive integer $n(a)$ such that $a^{n(a)}$ is an idempotent. S is termed periodic if every element of S is of finite order. Let S_{Reg} denote the set of regular elements of S. Let $E(S)$ denote the set of idempotents of S.

Let S denote a TC semigroup.

Lemma 2.1. *If $x, y \in S$ and $e \in E(S)$, then $xey = xy$.*

Proof. Apply TC to $e(ey) = ey$.

Lemma 2.2. *If $E(S) \neq \Box$, $E(S)$ is a rectangular band and $E(S) \cong E(S)e \times eE(S)$ for any fixed $e \in E(S)$. Moreover, $E(S)e$ is a left zero semigroup and $eE(S)$ is a right zero semigroup.*

Proof (Sketch). Use Lemma 2.1. The required isomorphism is given by $x\phi =$

(xe, ex).

Lemma 2.3. *Every regular element of S is contained in a subgroup of S.*

Proof (Sketch). If $axa = a$, let $g = axxa$. Then, $a \in H_g$, the group of units of gSg.

Lemma 2.4. *S_{Reg} is a regular semigroup or $S_{Reg} = \Box$.*

Lemma 2.5. *$S_{Reg} \cong G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup or $S_{Reg} = \Box$.*

Proof (Sketch). Fix $e \in E(S)$, let $I = E(S)e$, $J = eE(S)$, and $G = H_e$. Every element of S_{Reg} may be uniquely expressed in the form igj $(i \in I, g \in G, j \in J)$. The required isomorphism is $igj \to (i, g, j)$. G is abelian by Lemma 1.1. Lemmas 2.1–2.4 are utilized.

Corollary 2.6. *A regular semigroup S obeys the term condition (TC) if and only if $S \cong G \times I \times J$ where G is an abelian group, I is a left zero semigroup, and J is a right zero semigroup.*

Let F denote the set of elements of S of finite order.

Lemma 2.7. *F is a subsemigroup of S or $F = \Box$.*

Proof (Sketch). Use Lemma 1.1 and Lemma 2.2

Lemma 2.8. *If $e \in E(S)$ and $a \in F$, then both ae and $ea \in S_{Reg}$.*

Proof (Sketch). By Lemma 2.7, $(ea)^r \in E(S)$ for some positive integer r. Using Lemma 2.1, $(ea)^{r+1} = ea$.

Let S be a periodic TC semigroup. Let $X = S - S_{\text{Reg}}$. Let $a, b \in X$ and $(e, i, j) \in E(S)$ (then $e^2 = e$ is the identity of G). Using Lemma 2.5 and Lemma

2.8,

$$a(e,i,j) = (a',i',j')(a' \in G, i' \in I, j' \in J)$$

$$a(e,i,j) = (a\phi_i, i\rho_a, j)$$

where $\phi_i : X \to G$ (depending on i) and $\rho_a : I \to I$.

Similarly,

$$(e,i,j)a = (a\psi_j, i, j\lambda_a)$$

$$\psi_j : X \to G \text{ and } \lambda_a : J \to J.$$

Using $((e,i,j)a)(e,k,s) = (e,i,j)(a(e,k,s))$, we may show that $\psi_j = \phi_k$ for every $j \in J$ and $k \in K$. Put $\phi = \psi_j = \phi_k$.

By Lemma 2.1, $aa = (a(e,i,j))a$ which implies ρ_a and λ_a are constant mappings. Let $a\alpha = i\rho_a$ for all $i \in I$ and $a\beta = j\lambda_a$ for all $j \in J$.

Note $ab = (a(e,i,j))b$.

Construction of Periodic TC Semigroups

Let G be an abelian group, I, J, and X be sets and ϕ, α, and β functions of X into G, I, and J respectively. Define a binary operation o on the set $(G \times I \times J) \cup X$ by the following rule.

For $a, b \in X$ and $(g,i,j), (h,k,s) \in G \times I \times J$,

$$\begin{aligned} aob &= (a\phi b\phi, a\alpha, b\beta) \\ ao(g,i,j) &= (a\phi g, a\alpha, j) \\ (g,i,j)oa &= (g(a\phi), i, a\beta) \\ (g,i,j)o(h,k,s) &= (gh, i, s) \end{aligned}$$

Theorem 2.9 *S is a periodic TC semigroup if and only if S is isomorphic to some $(X \cup (G \times I \times J), o)$ with G periodic.*

Proof (Sketch). To prove the converse, note that all products may be written as products of elements of $G \times I \times J$. For example, $aob = (a\phi, a\alpha, a\beta)(b\phi, b\alpha, b\beta)$ and $ao(g,i,j) = (a\phi, a\alpha, a\beta)(g,i,j)$. So since $G \times I \times J$ obeys TC, so does $(X \cup (G \times I \times J), o)$.

3. Quasi-Orthodox Semigroups Which Are Bands of Cancellative Monoids

In this section, we determine the structure of quasi–orthodox semigroups which are bands of cancellative monoids (Theorem 3.4).

A semigroup S is a band of cancellative monoids if $S = \cup(S_\alpha : \alpha \in B)$ where S_α is a cancellative monoid, $S_\alpha \cap S_\beta = \Box$ if $\alpha \neq \beta$, B is a band (idempotent semigroup), and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

Define $(a,b) \in L^*$ iff, for all $x, y \in S^1$, $ax = ay$ iff $bx = by$. R^* is defined dually. Let $H^* = R^* \cap L^*$.

We may also equivalently define $(a,b) \in L^*$ iff $(a,b) \in S \times S$ and $(a,b) \in \mathcal{L}$ (Green's relation) in some oversemigroup of S. [6, 10].

We will term a semigroup S a quasi-orthogroup if $E(S)$ is a subsemigroup and each H^*-class of S contains an idempotent. A structure theorem for quasi-orthogroups is given in [16]. To prove Theorem 3.4, we will first need the "gross" structure of quasi-orthogroups (Lemma 3.1). The following terminology will be used in the proof of Lemma 3.1. Let S be a semigroup and let I and J be sets and let $P : J \times I \to S$ with $(j,i)P = p_{ji}$. Let $M(S,I,J,P)$ denote $S \times I \times J$ under the multiplication $(a,i,j)(b,r,s) = (ap_{jr}b, i, s)$. We term $M(S,I,J,P)$ a Rees matrix semigroup over S with entries in P. We also need the following notation to state Lemma 3.1. Let S be a semigroup. For $a \in S$, $L^*_a(S)$ will denote the L^*-class of S containing a. See [6] for definition of J^*. If S is a regular semigroup, $J = J^*$.

Lemma 3.1. *A semigroup S is a quasi-orthogroup if and only if S is a semilattice*

$Y = S/J^*$ of semigroups $(S_y : y \in Y)$ where $S_y = T_y \times E(S_y)$ where T_y is a cancellative monoid and $E(S_y)$ is a rectangular band, $L_a^*(S) = L_a^*(S_y)$ and $R_a^*(S) = R_a^*(S_y)$ for $y \in Y$ and $a \in S_y$ and $E(S)$ is a semilattice Y of rectangular bands $(E(S_y) : y \in Y)$.

Proof. Utilizing [6, Theorem 6.8 and its proof and Corollary 5.2], we obtain the above theorem (except the statement about $E(S)$) with $S_y = M(T_y, I_y, J_y, P_y)$, a Rees matrix semigroup over a cancellative monoid T_y where the entries of P_y are units U of T_y. As is easily shown, [2, Lemma 3.6] is valid for the above matrix semigroups if we require the mappings to have range U. Using this lemma we may "normalize" P_y such that all the elements in a given row and a given column are the identity e of T_y. Then using the assumption that $E(S)$ is a subsemigroup, we may show $p_{ji} = e$, the identity of T_y, for all $j \in J_y$ and $i \in I_y$. Hence $M(T_y, I_y, J_y, P_y) = T_y \times E(S_y)$ where $E(S_y)$ is a rectangular band.

In the proof of Theorem 3.4, we will need a quasi-orthogroup analogue to the minimum inverse semigroup congruence of an orthogroup (an orthodox union of groups) (Proposition 3.3).

For $(g,i,j), (h,r,s) \in S$, a quasi-orthogroup, define $(g,i,j)\delta(h,r,s)$ if (g,i,j), $(h,r,s) \in S_y$, say, and $g = h$.

We show (Proposition 3.3) that δ is the smallest good congruence on S (aL^*b implies $a\delta L^* b\delta$ and aR^*b implies $a\delta R^* b\delta$) such that $E(S/\delta)$ is a semilattice and, furthermore, that S/δ is a strong semilattice Y of the cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 3.1). To show δ is a congruence relation, we will need the following lemma.

Lemma 3.2. *Let $S_y = T_y \times E_y$ and $S_z = T_z \times I_z \times J_z$ where T_y and T_z are cancellative monoids, E_y is a rectangular band, I_z is a left zero semigroup and J_z is* *a right zero semigroup. Assume there exist*

a) A left representation $A \to \lambda_A$ of S_y by transformations of I_z.

b) A right representation $A \to \rho_A$ of S_y by transformations of J_z.

c) A homomorphism ϕ of T_y into T_z.

Define a binary operation on $S_y \cup S_z$ extending the given ones on S_y and S_z by defining products of $A = (a, e) \in S_y$ and $(b, i, j) \in S_z$ as follows.

$$(a, e)(b, i, j) = (a\phi b,\ \lambda_A i, j)$$

$$(b, i, j)(a, e) = (b(a\phi), i, j\rho_A).$$

Then $S_y \cup S_z$ becomes a semigroup with S_z as an ideal.

Conversely, every possible binary associative operation on $S_y \cup S_z$ extending the given ones on S_y and S_z, and such that S_z is an ideal, can be constructed in the above manner.

Proof. Lemma 3.2 has been established by Clifford [3, Lemma 2.5] in the case T_y and T_z are groups. Clifford's proof is easily seen to be valid when T_y and T_z are just cancellative monoids.

Proposition 3.3. *Let S be a quasi-orthogroup. Then, δ is the smallest good congruence on S such that $E(S/\delta)$ is a semilattice. Furthermore, S/δ is isomorphic to a semigroup $T = \cup(T_y : y \in Y)$ which is a strong semilattice Y of the cancellative monoids $(T_y : y \in Y)$ under the isomorphism $(g, i, j)\delta\tau = g$.*

Proof. We first show that δ is a congruence relation on S. Let $\bar{\delta}$ denote the smallest congruence relation on S containing δ. Suppose $a\bar{\delta}b$. Then, there exists $a = a_1, a_2, \ldots, a_n = b \in S$ such that $a_i = x_i u_i y_i$, $a_{i+1} = x_i v_i y_i$ where $x_i, y_i \in S^1$ and $(u_i, v_i) \in \delta$ for $1 \le i \le n-1$. Suppose $x_i = (w, p, q)_y \in S_y, y_i = (h, c, d)_z \in S_z$,

$u_i = (g,m,n)_t \in S_t$ and $v_i = (g,r,s)_t \in S_t$. Let $\theta = yzt$. Hence, $a_i = (A,a,b)_\theta \in S_\theta$ and $a_{i+1} = (B,e,f)_\theta \in S_\theta$, say. So,

$$(A,a,b)_\theta = (w,p,q)_y(g,m,n)_t(h,c,d)_z$$

$$(B,e,f)_\theta = (w,p,q)_y(g,r,s)_t(h,c,d)_z.$$

If we multiply both of the above equations on the left and the right by $(e,a,b)_\theta$ where e is the identity of T_θ, we obtain

$$\left.\begin{aligned}(A,a,b)_\theta &= (\bar{w},\bar{p},\bar{q})_\theta(g,m,n)_t(\bar{h},\bar{c},\bar{d})_\theta \\ (B,a,b)_\theta &= (\bar{w},\bar{p},\bar{q})_\theta(g,r,s)_t(\bar{h},\bar{c},\bar{d})_\theta.\end{aligned}\right\}. \tag{1}$$

Using Lemma 3.2, we obtain

$$\left.\begin{aligned}(\bar{w},\bar{p},\bar{q})_\theta(g,m,n)_t &= (\bar{w}(g\phi_{t,\theta}),\, \bar{p}, \bar{q}\rho(g,m,n)_t)_\theta \\ (\bar{w},\bar{p},\bar{q})_\theta(g,r,s)_t &= (\bar{w}(g\phi_{t,\theta}),\, \bar{p}, \bar{q}\rho(g,r,s)_t)_\theta\end{aligned}\right\} \tag{2}$$

where $\phi_{t,\theta}$ is a homomorphism of T_t into T_θ and $A \to \rho_A$ is a right representation of S_t by transformations of J_θ. Combining (1) and (2), we obtain $A = B$. Thus, $a_i \delta a_{i+1}$ for $1 \le i \le n-1$. Thus, $a\delta b$. Hence, $\delta = \bar{\delta}$, and thus δ is a congruence relation on S. Since $(g,i,j)\delta\tau = g$ defines a one-to-one mapping of S/δ onto $T = \cup(T_y : y \in Y)$, it is easily checked that T becomes a semilattice Y of cancellative monoids $(T_y : y \in Y)$ under the multiplication $ab = (a\tau^{-1}b\tau^{-1})\tau$ and τ defines an isomorphism of S/δ onto T. For $a \in T_z$ and $z \ge y$ define $aC_{z,y} = ae_y$ where e_y is the identity of T_y. It is routine to verify that $C_{z,y}$ is a homomorphism of T_z into T_y; $C_{y,y}$ is the identity automorphism of T_y; and for $a \in T_q$, $b \in T_r$, $ab = aC_{q,qr}bC_{r,qr}$. Since $e_y e_z = e_{yz}$ for all $y,z \in Y$, $C_{y,z}C_{z,w} = C_{y,w}$ for $y \ge z \ge w$. So, T is a strong semilattice Y of the T_y. Suppose aR^*b. Then, $a,b \in S_y$, say. Hence, $a\delta$, $b\delta \in T_y$. Thus, using the fact T is a strong semilattice Y of the T_y, $a\delta R^* b\delta$. Similarly, aL^*b implies $a\delta L^* b\delta$. So, δ is a good congruence. (So, each H^*-class of S/δ contains an idempotent). Clearly,

$E(S/\delta)$ is a semilattice. If λ is another such congruence, use the fact $E(S_y)\lambda$ is a singleton to show $\delta \leq \lambda$.

Remark The statement and proof of Proposition 3 is contained in the proof of [15, Theorem 6].

We are now in a position to establish Theorem 3.4.

Theorem 3.4. *Let E be a band and $E = \cup(E_\alpha : \alpha \in Y)$ be its maximal semilattice decomposition. To each $\alpha \in Y$ assign a cancellative monoid M_α such that $M_\alpha \cap M_\beta = \Box$ if $\alpha \neq \beta$. Furthermore, suppose that for $\alpha > \beta$ there exists a homomorphism*

$$\pi_{\alpha,\beta} : M_\alpha \to M_\beta$$

such that if $\alpha > \beta > \nu$ then $\pi_{\alpha,\nu} = \pi_{\alpha,\beta}\pi_{\beta,\nu}$. Set $\pi_{\alpha,\alpha}$ equal to the identity automorphism on M_α. Let $S = \cup((E_\alpha \times M_\alpha) : \alpha \in Y)$ and define a multiplication on S by $(e,x)(f,y) = (ef, x\pi_{\alpha,\alpha\beta}y\pi_{\beta,\alpha\beta})$ for any $(e,x) \in E_\alpha \times M_\alpha$, $(f,y) \in E_\beta \times M_\beta$. Then, S is a quasi-orthodox semigroup which is a band of cancellative monoids. Conversely, any quasi-orthodox semigroup which is a band of cancellative monoids can be constructed in this manner.

Proof. Let S be a quasi-orthodox semigroup which is a band of cancellative monoids. Then, using [5, Lemma 4.1], each H^*-class of S contains an idempotent and H^* is a congruence relation on S. Thus, S is a quasi-orthogroup on which H^* is a congruence relation. By Proposition 3.3, S/δ is isomorphic to $T = \cup(T_y : y \in Y)$, a strong semilattice Y of cancellative monoids $(T_y : y \in Y)$ (notation of Lemma 3.1). Thus, $T_y \cap T_z = \Box$ if $y \neq z$ and for $y > z$, there exists a homomorphism $\pi_{y,z} : T_y \to T_z$ such that for $y > z > w$, $\pi_{y,z}\pi_{z,w} = \pi_{y,w}$ and $\pi_{y,y}$ is the identity automorphism on T_y for $y \in Y$. Furthermore, $ab = a\pi_{y,yz}b\pi_{z,yz}$ for $a \in T_y$ and $b \in T_z$. Let $P = \cup(E(S_y) \times T_y) : y \in Y)$ under the multiplication $(e,x)(f,q) = (ef, xq)$ where $(e,x) \in$

$E(S_y) \times T_y$ and $(f,q) \in E(S_z) \times T_z$. We will show that $(g,i,j)_y\lambda = ((e_y,i,j)_y,g)$, where $(g,i,j)_y \in S_y$ and e_y is the identity of T_y, defines an isomorphism of S onto P. Clearly, λ defines a one-to- one mapping of S onto P. We will next show that λ defines a homomorphism of S onto P. Let $(g,i,j)_y \in S_y$ and $(h,r,s)_z \in S_z$. Thus, $(g,i,j)_y H^*(e_u,r,s)_u$, say. Hence, it is easily checked that $u = y$. Using the fact that H^* is a congruence relation, $(g,i,j)_y H^*(e_y,i,j)_y$. Similarly, $(h,r,s)_z H^*(e_z,r,s)_z$. Suppose $(g,i,j)_y(h,r,s)_z = (t,k,p)_{yz}$. Hence, $(t,k,p)_{yz}H^*(e_y,i,j)_y(e_z,r,s)_z$. Thus, $(e_y,i,j)_y\,(e_z,\,r,\,s)_z = (e_{yz},k,p)_{yz}$. Hence, $((g,i,j)_y(h,r,s)_z)\lambda = (t,k,p)_{yz}\lambda = ((e_{yz},\,k,\,p)_{yz},\,t)$ while $(g,\,i,\,j)_y\,\lambda(h,\,r,\,s)_z\lambda = ((e_y,\,i,\,j)_y,g)((e_z,r,s)_z,h) = ((e_{yz},k,p)_{yz},gh)$. Using Proposition 3.3 and its notation, $(g,i,j)_y\delta(h,r,s)_z\delta = (t,k,p)_{yz}\delta$. So, $(g,i,j)_y\delta\tau \cdot (h,r,s)_z\delta\tau = (t,k,p)_{yz}\delta\tau$. Thus, $gh = t$. Hence, λ defines an isomorphism of S onto P. Thus, we have established the converse of Theorem 3.4.

Next, we establish the direct part of Theorem 3.4. It is easily checked that S is a semigroup. Let $e \in E_\alpha$. Then, $\{e\} \times M_\alpha$ is a cancellative monoid with identity (e,e_α) where e_α is the identity of M_α. Let $M_e = \{e\} \times M_\alpha$. Thus, $S = \cup(M_e : e \in E)$ and S is the band E of cancellative monoids $(M_e : e \in E)$. Clearly, $E(S) = \{(e,e_\alpha) : e \in E_\alpha,\ \alpha \in Y\}$. Since $e_\alpha e_\beta = e_\alpha\,\pi_{\alpha,\alpha\beta}\,e_\beta\,\pi_{\beta,\alpha\beta} = e_{\alpha\beta}\,e_{\alpha\beta} = e_{\alpha\beta}$, it follows that $E(S)$ is a subsemigroup of S. To complete the proof, we will show that $(e,g)H^*(e,e_y)$ where $e \in E_y$ and $g \in M_y$. We will show that $(e,g)L^*(e,e_y)$. Dually, $(e,g)R^*(e,e_y)$. Suppose $(e,g)(a,b) = (e,g)(c,d)$ where $a \in E_z$, $b \in M_z$, $c \in E_t$ and $d \in M_t$. Thus, $ea = ec$ and $gb = gd$. So, $g\pi_{y,yz}b\pi_{z,yz} = g\pi_{y,yt}d\pi_{t,yt}$ with $yz = yt$. Hence, $b\pi_{y,yz} = d\pi_{t,yt}$. Thus, $e_y\pi_{y,yz}b\pi_{y,yz} = e_y\pi_{y,yt}d\pi_{t,yt}$. Hence, $e_yb = e_yd$. So, $(e,e_y)(a,b) = (e,e_y)(c,d)$. Similarly, $(e,e_y)(a,b) = (e,e_y)(c,d)$ implies $(e,g)(a,b) =$

$(e, g)(c, d)$.

Acknowledgement: The author gratefully acknowledges the support of King Fahd University of Petroleum and Minerals.

References

1. S. Burris, and H.P. Sankappanavor, *A Course in Universal Algebra*, (Springer Verlag, New York, 1981).

2. A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys Amer. Math. Soc. 7, Vol I (AMS, Providence, R.I., 1961).

3. A.H. Clifford, *A Structure Theorem For Orthogroups*, J. Pure and Appl. Algebra 8(1976), 23-50.

4. A. El-Qallali and J.B. Fountain, *Quasi-Adequate Semigroups*, Proc. Roy. Soc. Edinburgh, Series A, 91(1981), 91-99.

5. Abdulsalam El- Qallali, *L^*-Unipotent Semigroups*, J. Pure Appl. Algebra 62(1989), 9-33.

6. J.B. Fountain, *Abundant Semigroups*, Proc. London Math. Soc. (3), 44(1982), 103-129.

7. Freese, R. and McKenzie, R., *Commutator Theory for Congruence Modular Varieties*, London Math. Society Lecture Note no. 125, 1987.

8. Hobby, D. and McKenzie, R., *The Structure of Finite Algebras*, AMS Contemporary Mathematics Series, 1988.

9. J.M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).

10. D.B. McAlister, *One-to-one Partial Right Translations of a Right Cancellative Semigroup*, J. Algebra 43(1976), 231-251.

11. McKenzie, R., *On Minimal, Locally Finite Varieties, with Permuting Congruence Relations*, Berkeley Manuscript, 1976.

12. McKenzie, R., *The Number of Non-isomorphic Models in Quasi-varieties of Semigroups,* Algebra Universalis 16(1983), 195–203.

13. McKenzie, R., McNulty, G., and Taylor, W., *Algebras, Lattices, Varieties*, 1, The Wadsworth and Brooks/Cole Mathematical Series, 1987.

14. Taylor, W., *Some Applications of the Term Condition*, Algebra Universalis 14(1982), 11–24.

15. R.J. Warne, *Orthogroups and Generalizations*, Proc. of the International Symposium on The Semigroup Theory and its Related Fieds, Kyoto, Japan, 1990, 233–241.

16. R.J. Warne, *Super Quasi–Adequate Semigroups*, TAMKANG Journal of Mathematics, 22(1991), 299–312.

17. M. Yamada, *Strictly Inverse Semigroups*, Bull. Shimane Univ. 13(1964), 128-138.

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia