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A Criterion for Algebraically Compact Modules

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Abstract

We establish a characterization of algebraic compactness and use it to obtain some results on quasi–injective modules. We show in particular that a ring is noetherian if and only if pure submodules of quasi–injectives are quasi–injective.

Notation. Let M be an R–module and let α be an infinite cardinal. The submodules $\{m \in M^\alpha$: all coordinates of M are equal$\}$ and $\{m \in M^\alpha$: there exists an ordinal $\sigma_0 < \alpha$ such that $m(\sigma) = 0$ for all $\sigma \geq \sigma_0\}$ are denoted by $\Delta_\alpha M$ and $\sum_\alpha M$ respectively. Clearly $\Delta_\alpha M$ is a direct summand of M^α, $\Delta_\alpha M \cap \sum_\alpha M = 0$, and there is a natural isomorphism $\delta_\alpha : \Delta_\alpha M \to M$. We shall denote by f_α the map $\Delta_\alpha M + \sum_\alpha M \to M$ given by $f_\alpha(n) = \delta_\alpha(m)$ for all $m \in \Delta_\alpha M$ and $f_\alpha(\sum_\alpha M) = 0$. Throughout this note, R is an associative ring with 1 and all modules are left unital.

Definitions. Let M be an R–module.

1. Let M be *pure–injective* if it has the injective property relative to each pure–exact sequences $0 \to A \to B \to C \to 0$ of R–modules.

2. M is *quasi–injective* (respectively *quasi–pure-injective.*) if for each submodule (respectively pure submodule) N of M, every homomorphism $N \to M$ is induced by an endomorphism of M. Clearly every pure–injective is quasi–pure–injective.

3. M is $\prod$–*quasi–injective* (respectively $\prod$–*quasi–pure–injective*) if for each index set I, M^I is quasi–injective (respectively quasi–pure–injective).

4. M is *α–compact*, where α is an infinite cardinal, if every finitely solvable system of α linear equations with constants in M is solvable. If M is α–compact for all cardinals α, M is called *algebraically compact.* It is well–known that M is pure-injective if and only if it is algebraically compact [6].

Proposition 1. *Let M be an R-module, let α be an infinite cardinal and suppose that the map $f_\alpha : \Delta_\alpha M + \sum_\alpha M \to M$ can be extended to a homomorphism $g_\alpha : M^\alpha \to M$. If M is β–compact for all cardinals $\beta < \alpha$, then M is α–compact.*

Proof. Let $\sum_{k\in K} r_{jk}x_k = a_j \quad (a_j \in M, \quad j < \alpha)$ be a system of equations finitely solvable in M. Since M is β–compact for all $\beta < \alpha$, there exists for each ordinal $t < \alpha$ a solution $\{m_k^t\}_{k\in K}$ of the subsystem with $j < t$. For each $k \in K$, let $\mu_k = (m_k^t)_{t<\alpha} \in M^\alpha$ and put $m_k = g_\alpha(\mu_k)$. Then, for any fixed $j < \beta$, we have $\sum_{k\in K} r_{jk}m_k = g_\alpha\left(\sum_{k\in K}\left(r_{jk}m_k^t\right)_{t<\alpha}\right) = g_\alpha\left(b + \delta_\alpha^{-1}(a_j)\right)$ for some $b \in \sum_\alpha M$. So, $\sum_{k\in K} r_{jk}m_k = a_j$, as required.

As a consequence of Proposition 1, we obtain the following criterion for an R–module to be pure–injective.

Theorem 1. *An R–module M is algebraically compact if and only if M has the injective property relative to all pure–exact sequences*

$$0 \to K \to M^{\aleph_0|R|} \to M^{\aleph_0|R|}/K \to 0.$$

Proof. 'Only if' is clear using the equivalence of pure–injectivity and algebraic compactness. Assume now the latter condition and suppose, on the contrary, that M is not algebraically compact. Let α be the smallest cardinal such that M is not α–compact. It follows from [2] that $\alpha \leq \aleph_0|R|$. Now M is β–compact for all $\beta < \alpha$ and clearly $\Delta_\alpha M + \sum_\alpha M$ is a pure submodule of $M^{\aleph_0|R|}$. Therefore, the map $f_\alpha : \Delta_\alpha M + \sum_\alpha M \to M$ defined above can be extended to a homomorphism $g : M^{\aleph_0|R|} \to M$. If we let h denote the canonical embedding $M^\alpha \to M^{\aleph_0|R|}$, then gh extends f_α and by Proposition 1, M is α-compact. This contradiction completes the proof.

Corollary. *Let M be an R-module such that $M^{\aleph_0|R|}$ is flat. Then M is pure-injective if and only if $Ext^1_R(Q, M) = 0$ for all flat R-modules Q.*

Proof. If M is pure-injective and Q is flat then every short exact sequence with third term Q is pure and so $\mathrm{Ext}^1_R(Q, M) = 0$. Conversely, suppose that $\mathrm{Ext}^1_R(Q, M) = 0$ for all flat modules Q. Since $M^{\aleph_0|R|}$ is flat, for every pure-exact sequence $0 \to K \to M^{\aleph_0|R|} \to M^{\aleph_0|R|}/K \to 0$, $M^{\aleph_0|R|}/K$ is also flat. Hence $\mathrm{Ext}^1_R(M^{\aleph_0|R|}/K,\ M) = 0$ and so M is pure-injective by the theorem.

Proposition 2. *For an R-module M, the following statements are equivalent.*

(i) M is $\prod$-quasi-pure-injective.

(ii) $M^{\aleph_0|R|}$ is quasi-pure-injective.

(iii) M is pure-injective.

In particular, if $M^{\aleph_0|R|}$ is quasi-injective, then M is $\prod$-quasi-injective and pure-injective.

Proof. (i) $\Rightarrow$ (ii) is clear. Suppose now that (ii) holds. Let $f : K \to M$, where K is a pure submodule of $M^{\aleph_0|R|}$, be a homomorphism, and denote by $\varphi : M \to M^{\aleph_0|R|}$ and $\psi : M^{\aleph_0|R|} \to M$ any maps such that $\psi\varphi = id_M$. Since M is $\prod$-quasi-pure-injective, φf extends to a map $g : M^{\aleph_0|R|} \to M^{\aleph_0|R|}$. It is now clear that the map $\psi g : M^{\aleph_0|R} \to M$ extends f. By the theorem, M is pure-injective. Finally, if M is pure-injective, then so also is any direct product of copies of M, and so M is $\prod$-quasi-pure-injective. This proves that (iii) $\Rightarrow$ (i).

Proposition 3. *Let M be a quasi–injective R–module.*

(i) If M is finendo, i.e. finitely generated over its endomorphism ring, then M is pure–injective.

(ii) If R is commutative and M is finitely generated then M is pure–injective.

Proof. By [4, Theorem 1.2], M is $\prod$–quasi–injective if and only if M is $R/\mathrm{ann}_R M$–injective. Now use [1, Proposition 19.15] and [1, Theorem 19.17] together with the previous theorem.

Remarks

1. Proposition 3(i) is a generalization of a result of Fuchs [3] stating that if a quasi–injective module is cyclic over its endomorphism ring, then it is pure–injective.

2. Since quasi–injective modules are $\prod$–quasi–injective over artinian rings [5], it follows from Proposition 2 that every quasi–injective R–module, where R is artinian, is pure–injective

It is well–known that a ring is left noetherian if and only if pure submodules of injective modules are injective (or, equivalently, every absolutely pure module is injective). In the following theorem, we prove that this is still true when "injective" is replaced by "quasi–injective".

Theorem 2. *A ring R is left noetherian if and only if pure submodules of quasi–injective left R–modules are quasi–injective.*

Proof. Suppose first that R is left noetherian and let N be a pure submodule of a quasi–injective R–module M. To show that N is quasi–injective we use Fuchs' criterion [2] (see also [5]). Let I be a left R–ideal, let $f : I \rightarrow N$ be a homomorphism such that $\mathrm{ann}\ \nu \subseteq \ker f$ for some $\nu \in N$ and let $\sigma : N \rightarrow M$ be the inclusion map.

Since ann $\nu \subseteq \ker \sigma f$, it follows that there is a map $g : R \to M$ extending σf. Now R/I is finitely presented and N is pure in M; hence there exists a homomorphism $\psi : R/I \to M$ such that $\pi\psi = h$ where $h : R/I \to M/N$ is given by $h(r+I) = m+N$ and π is the natural epimorphism $M \to M/N$. It is easy to check that the map $\varphi : R \to M$ given by $\varphi(1) = g(1) - \psi(1+I)$ is in fact a homomorphism $R \to N$ and that it extends f. This proves that N is quasi–injective. Conversely, assume that pure submodules of quasi–injectives are quasi–injective and let M be an absolutely pure R–module. Clearly, for any non–empty set X, M^X is pure in E^X, where E is the injective hull of M. Since E^X is quasi–injective, M^X is quasi–injective also. By Proposition 2, M is pure–injective and therefore it must be injective.

Finally, let us make the following observation:

Recall that a ring R is said to be a QI–ring (Q–ring) if its quasi–injective modules are injective ($\prod$–quasi–injective). It is known that over a von Neumann regular ring, all modules are absolutely pure. This implies that any von Neumann regular Q–ring R is a QI–ring. If, in addition, the ring R is commutative, the converse is true, since QI–rings are noetherian V–rings and products of injectives are always injective. We conclude therefore that the commutative von Neumann regular Q–rings are precisely the semi–simple artinian ones.

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